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Journal of Number Theory 117 (2006) 279–291

JOURNAL OF Number Theory

www.elsevier.com/locate/jnt

Modular forms and Eisenstein's continued fractions

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Received 24 November 2004; revised 7 April 2005

Available online 15 August 2005

Communicated by W. Duke

Abstract

Found in the collected works of Eisenstein are twenty continued fraction expansions. The expansions have since emerged in the literature in various forms, although a complete historical account and self-contained treatment has not been given. We provide one here, motivated by the fact that these expansions give continued fraction expansions for modular forms. Eisenstein himself did not record proofs for his expansions, and we employ only standard methods in the proofs provided here. Our methods illustrate the exact recurrence relations from which the expansions arise, and also methods likely similar to those originally used by Eisenstein to derive them.

Keywords: Modular forms; Continued fractions; q-series; Basic hypergeometric series

1. Introduction

Two particularly striking examples of continued fraction expansions for the fundamental modular forms η and ϑ may be derived from Eisenstein's continued fraction expansions (found in [1–4]). The Dedekind η-function is a fundamental modular form

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0022-314X/$ - see front matter
doi:10.1016/j.jnt.2005.06.001

of weight 1/2 for the full modular group $\Gamma(1)$ defined by

$$\eta(\tau) = q^{1/12} \prod_{n \geqslant 1} (1 - q^{2n}),$$

where we adopt the original notation of Jacobi and define $q = e^{\pi i \tau}$, $\tau \in \mathcal{H}$, the complex upper half-plane. For $\eta(\tau)$ we derive the following continued fraction:

$$\eta(\tau) = \frac{q^{1/12}}{1+}\,\frac{q^2}{1-q^2-}\,\frac{q^2}{1+q^2+}\,\frac{q^6}{1-q^6-}\,\frac{q^4}{1+q^4+}\,\frac{q^{10}}{1-q^{10}-}\,\frac{q^6}{1+q^6+}\,\frac{q^{14}}{1-q^{14}-}\cdots \tag{1}$$

The theta function is a modular form for the group $\Gamma_0(4)$ also of weight 1/2, and is defined by

$$\vartheta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}.$$

We find many continued fraction expansions relating to $\vartheta(q)$ in Eisenstein's work, including the following examples:

$$\vartheta(q) = 1 + \frac{2q}{1-}\,\frac{q^3}{1-}\,\frac{q^5-q^3}{1-}\,\frac{q^7}{1-}\,\frac{q^9-q^5}{1-}\,\frac{q^{11}}{1-}\,\frac{q^{13}-q^7}{1-}\cdots, \tag{2}$$

$$\vartheta^2(q) = 1 + \frac{4q}{q^2+1-}\,\frac{q(q^2+1)^2}{q^4+1+}\,\frac{q^3(q^2-1)^2}{q^6+1-}\,\frac{q^3(q^4+1)^2}{q^8+1+}\,\frac{q^5(q^4-1)^2}{q^{10}+1-}\cdots, \tag{3}$$

where $\vartheta^2(q)$ is a modular form of weight 1. The continued fraction on the right-hand side of (3) is originally given by Eisenstein as an expansion for $2K/\pi$, where K is the complete elliptic integral of the first kind defined by

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

and k the elliptic modulus satisfying $0 \leqslant k \leqslant 1$. (For a discussion of K and some basic properties, see [18] for example.) To write (3), we employ the identity [18, §22.302] relating K to $\vartheta^2(q)$

$$\frac{2K}{\pi} = \vartheta^2(q).$$

We derive some of the examples provided here from more general expansions given by Eisenstein. For example we derive (1) from Eisenstein's expansion

$$\prod_{n\geqslant 0}\frac{1-xq^n}{1-yq^n}=\frac{1}{1+}\,\frac{x-y}{1-q+}\,\frac{qy-x}{1+q+}\,\frac{q^2x-qy}{1-q^3+}\,\frac{q^3y-qx}{1+q^2+}\,\frac{q^4x-q^2y}{1-q^5+}\cdots, \tag{4}$$

where either $|q| < 1$ and $|y| < 1$, or $|q| > 1$ and $|x| < |q|$. From (4) we also derive continued fraction expansions for $\vartheta_4(q)$ and $\vartheta_2(q)$, modular forms of weight 1/2, where $\vartheta_4(q)$ is defined by $\vartheta(-q)$, and $\vartheta_2(q)$ is defined by the sum in ϑ taken over numbers $n + 1/2$, $n \in \mathbb{Z}$. These expansions for $\vartheta_4(q)$ and $\vartheta_2(q)$ are given in Section 4.

Some of the continued fractions originally due to Eisenstein have since emerged in various forms. Eisenstein's continued fraction

$$\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{1}{1-}\,\frac{q}{1-}\,\frac{q^2-q}{1-}\,\frac{q^3}{1-}\,\frac{q^4-q^2}{1-}\,\frac{q^5}{1-}\,\frac{q^6-q^3}{1-}\cdots \tag{5}$$

was proved over 90 years later by Selberg as a consequence of results generalizing the Rogers–Ramanujan identities [17]. Selberg's continued fraction can also be derived from Eisenstein's more general expansion for a partial theta series, given by

$$\sum_{n=0}^{\infty} q^{n^2}x^n=\frac{1}{1-}\,\frac{qx}{1-}\,\frac{(q^3-q)x}{1-}\,\frac{q^5x}{1-}\,\frac{(q^7-q^3)x}{1-}\,\frac{q^9x}{1-}\,\frac{(q^{11}-q^5)x}{1-}\cdots . \tag{6}$$

Eisenstein's continued fractions (5) and (6) may also be derived from more general expansions given by Ramanujan, also due to Rogers and independently discovered by Schur. Ramanathan [16] provides a treatment of these continued fractions from this perspective, where the series in (6) appears as special case of a ratio of limiting values of basic hypergeometric series, and the continued fraction expansion on the right-hand side of (6) is recovered after appropriate substitution. We also offer a direct proof of (6) in Section 3.

In addition to these appearances, Muir provides a treatment of Eisenstein's continued fractions in [14]. He reduces the number of Eisenstein's continued fractions that are independent to five, and derives the others from these (One may also consult [12,13].). Three of these five are (3), (4), and (6), and another is the following continued fraction for a Lambert series

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}=\frac{q}{1-q-}\,\frac{q(1-q)^2}{1-q^2-}\,\frac{q^2(1-q)^2}{1-q^3-}\,\frac{q^2(1-q^2)^2}{1-q^4-}\cdots . \tag{7}$$

This continued fraction was proved independently by Heine [8] using Euler's method. Using the same method, Heine proves another,[1] which can also be derived from (4). The remaining independent Eisenstein continued fraction (originally given in the form as in [14, (V.) p. 136]) is equivalent to

$$\sum_{n=0}^{\infty}(-1)^n q^{n^2} = \frac{1}{1+}\,\frac{q}{1-q+}\,\frac{q^3}{1-q^3+}\,\frac{q^5}{1-q^5+}\,\frac{q^7}{1-q^7+}\cdots \tag{8}$$

and can be found using Euler's method.

Eisenstein himself did not provide proofs for his expansions, and in the third section of this paper we provide direct proofs for (3), (4), and (6) to illustrate the exact recurrence relations from which the expansions arise, and to illustrate methods likely similar to those originally used by Eisenstein to derive them. We will see that (4) arises from a recurrence relation between q-hypergeometric series. Heine did not provide a proof for expansion (4) in his treatment of Eisenstein's continued fractions in [8], yet to establish (4) we will in fact use a variant of the recurrence relation given by Heine [9]

$${}_2\phi_1(a,b;c;q,z) = {}_2\phi_1(a,bq;cq;q,z) + \frac{(1-a)(c-b)z}{(1-c)(1-cq)}\,{}_2\phi_1(aq,bq;cq^2;q,z) \tag{9}$$

to establish his well-known continued fraction for a ratio of q-hypergeometric series. Heine's continued fraction is a q-analogue of Gauss' continued fraction for a ratio of hypergeometric series [7], and many of Eisenstein's continued fractions appear by following methods used by Gauss, whose works Eisenstein began studying in 1842. Before proceeding, we recall the notion of correspondence.

2. Correspondence

Given a continued fraction

$$b_0(z) + \frac{a_1(z)}{b_1(z)+}\,\frac{a_2(z)}{b_2(z)+}\,\frac{a_3(z)}{b_3(z)+}\cdots$$

with polynomial coefficients $a_n(z)$ and $b_n(z)$, one can ask whether there is a formal Laurent series L that corresponds to the continued fraction in the sense that the Laurent expansions of the partial convergents A_n/B_n of the continued fraction agree with L

[1] This appears on the bottom of p. 38 in [3], and as 14. in Section 4.

up to a certain term. Using notation as defined in [10], if we define $\lambda(L)$ to be the exponent of the smallest power of z in L with non-zero coefficient, and $\lambda(0)=\infty$, then a continued fraction is said to correspond to L at 0 if each partial convergent A_n/B_n is meromorphic at the origin and

$$\lim_{n\to\infty}\lambda(L-L(A_n/B_n))=\infty,$$

where $L(A_n/B_n)$ is the Laurent expansion for the partial convergent at $z=0$. In this case, L and $L(A_n/B_n)$ agree up to the term involving z^m, if $m=\lambda(L-L(A_n/B_n))$. We will use the following correspondence theorems, the first given in [10, Theorem 5.5A, p. 160], and the second given in [11, Theorem 12, p. 267].

Theorem 1 (*Jones and Thron*). *Let $\{a_n(z)\}_{n=1}^{\infty}$ and $\{b_n(z)\}_{n=0}^{\infty}$ be sequences of functions meromorphic at the origin with $a_n(z)\neq 0$. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of non-zero formal Laurent series satisfying the recurrence relations*

$$P_n=L(b_n(z))P_{n+1}+L(a_{n+1}(z))P_{n+2}.$$

Then the continued fraction

$$b_0(z)+\frac{a_1(z)}{b_1(z)+}\,\frac{a_2(z)}{b_2(z)+}\,\frac{a_3(z)}{b_3(z)+}\cdots$$

corresponds to the formal Laurent series $L=P_0/P_1$ provided the following conditions are satisfied for $n\geqslant 1$:

$$\lambda(L(b_n))+\lambda(L(b_{n-1}))<\lambda(L(a_n)), \tag{10}$$

$$\lambda(P_n/P_{n+1})+\lambda(L(b_{n-1}))<\lambda(L(a_n)). \tag{11}$$

Theorem 2 (*Lorentzen*). *Let $\{a_n(z)\}_{n=1}^{\infty}$ and $\{b_n(z)\}_{n=1}^{\infty}$ be polynomials with $a_n(z)\neq 0$. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-zero formal Laurent series satisfying the recurrence relations*

$$X_n=b_n(z)X_{n-1}+a_n(z)X_{n-2}.$$

Then the continued fraction

$$\frac{a_1(z)}{b_1(z)+}\,\frac{a_2(z)}{b_2(z)+}\,\frac{a_3(z)}{b_3(z)+}\cdots$$

corresponds to the formal Laurent series $L = -X_0/X_{-1}$ *provided the following conditions are satisfied for* $n \geqslant 1$:

$$\lambda(b_{n-1}(z)) + \lambda(b_n(z)) < \lambda(a_n(z)), \tag{12}$$

$$\lambda(b_n(z)) < \lambda(X_n/X_{n-1}). \tag{13}$$

We remark as in [10] that both correspondence theorems hold provided the conditions (10)–(13) hold for n sufficiently large.

3. Recurrence

In this section we will establish proofs of the remaining independent continued fractions of Eisenstein, and will provide recurrence relations that give (3), (4) and (6).

Proof of (4). We begin by expanding the product in (4) into a sum by the q-binomial theorem [6] given by

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n,$$

where $|z| < 1$, $|q| < 1$. Here we use the q-Pochhammer symbol [6], defined by

$$(a;q)_k = \begin{cases} 1, & k = 0, \\ (1-a)(1-aq)\cdots(1-aq^{k-1}), & k = 1, 2, \ldots, \\ [(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-k})]^{-1}, & k = -1, -2, \ldots, \\ (1-a)(1-aq)(1-aq^2)(1-aq^3)\ldots, & k = \infty. \end{cases}$$

With $z = y$ and $a = x/y$, $y \neq 0$, we have for the product in (4)

$$\prod_{n=0}^{\infty} \frac{1-xq^n}{1-yq^n} = 1 + \frac{(x-y)}{(q-1)} + \frac{(x-y)(xq-y)}{(q-1)(q^2-1)} + \frac{(x-y)(xq-y)(xq^2-y)}{(q-1)(q^2-1)(q^3-1)} + \cdots . \tag{14}$$

The series expansion in (14) also holds for $y = 0$ due to a well-known result of Euler [5], so that we have (14) for all $|y| < 1$, $|q| < 1$. Next we provide a recurrence relation satisfied by a certain family of q-hypergeometric series, which are defined by

$$_2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n. \tag{15}$$

Specifically, we let

$$\begin{aligned} P_0 &= {}_2\phi_1(q, x/y; q; q, yz) \\ &= 1 + \frac{(1-x/y)}{(1-q)} yz + \frac{(1-x/y)(1-qx/y)}{(1-q)(1-q^2)} (yz)^2 + \cdots \\ &= 1 + \frac{(x-y)}{(q-1)} z + \frac{(x-y)(xq-y)}{(q-1)(q^2-1)} z^2 + \cdots, \end{aligned}$$

which is the series in (14) when $z = 1$. We let $P_1 = 1$, and define for $n \geqslant 1$ the functions

$$\begin{aligned} P_{2n}(z) &= {}_2\phi_1(q^n, q^n x/y; q^{2n}; q, yz), \\ P_{2n+1}(z) &= {}_2\phi_1(q^n, q^{n+1} x/y; q^{2n+1}; q, yz) \end{aligned} \tag{16}$$

and for $n \geqslant 0$,

$$\begin{aligned} a_{2n+1}(z) &= -zq^n \frac{(1-q^n)(x-yq^n)}{(1-q^{2n+1})(1-q^{2n})}, \\ a_{2n+2}(z) &= -zq^n \frac{(1-q^{n+1})(y-xq^{n+1})}{(1-q^{2n+2})(1-q^{2n+1})}. \end{aligned}$$

We will prove the following:

Claim. *For* $k \geqslant 0$,

$$P_k = P_{k+1} + a_{k+1}(z) P_{k+2}. \tag{17}$$

One can easily verify that (17) holds for $k = 0$ and 1 using definition (15) of the q-hypergeometric series. For $k \geqslant 2$, we will use the relation

$${}_2\phi_1(a, b; c; q, z) = {}_2\phi_1(a, bq; cq; q, z) + \frac{(1-a)(c-b)}{(1-c)(1-cq)} z \, {}_2\phi_1(aq, bq; cq^2; q, z) \tag{18}$$

originally given by Heine [9] to establish his continued fraction for the ratio ${}_2\phi_1(a, b; c; q, z)/{}_2\phi_1(a, bq; cq; q, z)$. Relation (18) can be verified by equating coefficients of z. To establish (17) for $k = 2n$, $n \geqslant 1$, we let $a = q^n$, $b = q^n x/y$, $c = q^{2n}$, and substitute yz for z in (18). For $k = 2n + 1$, $n \geqslant 1$, we first use the fact that the q-hypergeometric series (15) is unchanged by interchanging a and b. We first interchange a and b in (18), and then substitute cq for c and bq for b. In this transformed identity, we let $a = q^n$, $b = q^n x/y$, $c = q^{2n}$, and substitute yz for z.

We see that conditions (10) and (11) are satisfied, as

$$0+0=\lambda(L(b_n))+\lambda(L(b_{n-1}))<\lambda(L(a_n))=1,$$

$$0+0=\lambda(P_n/P_{n+1})+\lambda(L(b_{n-1}))<\lambda(L(a_n))=1$$

for $n\geqslant 1$, so we can apply the first correspondence theorem given in Section 2. Evaluating at $z=1$ gives the continued fraction expansion

$$\prod_{n\geqslant 0}\frac{1-xq^n}{1-yq^n}=1-\frac{\frac{(y-x)}{(q-1)}}{1-}\,\frac{\frac{(xq-y)}{(q^2-1)}}{1-}\,\frac{\frac{q(yq-x)(q-1)}{(q^3-1)(q^2-1)}}{1-}\,\frac{\frac{q(xq^2-y)(q^2-1)}{(q^4-1)(q^3-1)}}{1-}\cdots. \tag{19}$$

We obtain Eisenstein's continued fraction (4) by taking the reciprocal of both sides of (19), interchanging x and y, and simplifying. □

Proof of (6). We proceed similarly, and show that the family defined by

$$X_{2n}=\sum_{j=0}^{\infty}(q^{2(j+1)};q^2)_n x^{2n+j}q^{-(2n+j)^2+n(2n-1)}, \tag{20}$$

$$X_{2n+1}=\sum_{j=0}^{\infty}(q^{2(j+1)};q^2)_n x^{2n+1+j}q^{-(2n+1+j)^2+n(2n+1)} \tag{21}$$

for $n\geqslant 1$, with

$$X_{-1}=-1,\quad X_0=\sum_{j=0}^{\infty}q^{-j^2}x^j,\quad X_1=\sum_{j=0}^{\infty}q^{-(j+1)^2}x^{j+1},$$

satisfies the recurrence

$$X_n=b_nX_{n-1}+a_nX_{n-2}, \tag{22}$$

where $a_{2n}=-x$, $a_{2n+1}=-x(1-q^{2n})$, $b_{2n}=q^{2n-1}$ and $b_{2n+1}=q^{2n}$ for $n\geqslant 1$, and $a_1=b_1=1$. This will give Eisenstein's original continued fraction for X_0 [2]. We obtain (6) by replacing q by q^{-1}, and simplifying. For $j\geqslant 0$ we denote the coefficient of x^{n+j} in X_n by c_j^n. That is, $X_n=\sum_{j\geqslant 0}c_j^n x^{n+j}$. We let $c_{-1}^n=0$. It is easy to verify (22) for $n=1$. We note that for $n\geqslant 2$, $\lambda(n)=n$, so that by equating coefficients, (22) is equivalent to the relations

$$q^{2n-1}c_{j+1}^{2n-1}-c_{j+1}^{2n-2}=c_j^{2n} \tag{23}$$

and

$$q^{2n} c_{j+1}^{2n} - (1 - q^{2n}) c_{j+1}^{2n-1} = c_j^{2n+1} \tag{24}$$

for $j \geqslant -1$. To establish (23), for $j = -1$ we compute

$$\begin{aligned} q^{2n-1} c_0^{2n-1} - c_0^{2n-2} &= q^{2n-1} (q^2; q^2)_{n-1} q^{-(2n-1)^2 + (n-1)(2n-1)} \\ &\quad - (q^2; q^2)_{n-1} q^{-(2n-2)^2 + (n-1)(2n-3)} \end{aligned}$$

which we see equals zero after simplifying powers of q that appear. For $j \geqslant 0$, we have

$$\begin{aligned} &q^{2n-1} c_{j+1}^{2n-1} - c_{j+1}^{2n-2} \\ &\quad = q^{2n-1} (q^{2(j+2)}; q^2)_{n-1} q^{-(2n+j)^2 + (n-1)(2n-1)} \\ &\qquad - (q^{2(j+2)}; q^2)_{n-1} q^{-(2n-1+j)^2 + (n-1)(2n-3)} \\ &\quad = (q^{2(j+2)}; q^2)_{n-1} (q^{-(2n+j)^2 + n(2n-1)} \\ &\qquad - q^{-(2n+j-1)^2 + (n-1)(2n-3)}) \\ &\quad = (q^{2(j+1)}; q^2)_n q^{-(2n+j)^2 + 2n(n-1)} \\ &\quad = c_j^{2n}. \end{aligned}$$

We establish (24) similarly:

$$\begin{aligned} q^{2n} c_0^{2n} - (1 - q^{2n}) c_0^{2n-1} &= q^{2n} (q^2; q^2)_n q^{-(2n)^2 + n(2n-1)} \\ &\quad - (1 - q^{2n}) (q^2; q^2)_{n-1} q^{-(2n-1)^2 + (n-1)(2n-1)} \\ &= 0, \end{aligned}$$

$$\begin{aligned} q^{2n} c_{j+1}^{2n} - (1 - q^{2n}) c_{j+1}^{2n-1} &= q^{2n} (q^{2(j+2)}; q^2)_n q^{-(2n+1+j)^2 + n(2n-1)} \\ &\quad - (1 - q^{2n}) (q^{2(j+2)}; q^2)_{n-1} q^{-(2n+j)^2 + (n-1)(2n-1)} \\ &= (q^{2(j+2)}; q^2)_{n-1} q^{-(2n+j+1)^2 + n(2n+1)} ((1 - q^{2(j+n+1)}) \\ &\quad - (1 - q^{2n}) q^{2j-2n+2}) \\ &= (q^{2(j+1)}; q^2)_n q^{-(2n+j+1)^2 + n(2n+1)} \\ &= c_j^{2n+1}. \end{aligned}$$

We verify that conditions (12) and (13) are satisfied, as $\lambda(b_n) = 0$, $\lambda(a_{n+1}) = 1$, and $\lambda(X_n/X_{n-1}) = 1$ for $n \geqslant 1$. $\square$

Proof of (3). We prove Eisenstein's continued fraction (3) in the same manner, and show that the family defined by

$$X_{2n-2} = (-1)^{n-1} 4q^{2n^2-2n+1} x^{2n-1} (q^4; q^4)_{n-1} \sum_{j=0}^{\infty} \frac{(qx)^j (q^{2(j+1)}; q^2)_{n-1}}{(-q^{2(j+n)}; q^2)_{n-1} (q^{2(j+2n-1)} + 1)},$$

$$X_{2n-1} = (-1)^{n+1} 4q^{2n^2} (q^{2n} + 1) x^{2n} (q^4; q^4)_{n-1} \times \sum_{j=0}^{\infty} \frac{(qx)^j (q^{2(j+1)}; q^2)_{n-1}}{(-q^{2(j+n+1)}; q^2)_{n-1} (q^{2(j+2n)} + 1)}$$

for $n \geqslant 2$, and

$$X_{-1} = -1, \quad X_0 = 4 \sum_{j=0}^{\infty} \frac{(qx)^{j+1}}{q^{2(j+1)} + 1}, \quad X_1 = 4(q^2+1)(qx)^2 \sum_{j=0}^{\infty} \frac{(qx)^j}{q^{2(j+2)} + 1}$$

satisfies the recurrence relation (22), where we have $a_{2n} = -q^{2n-1}(1+q^{2n})^2$, $b_{2n} = (q^{4n}+1)$, $a_{2n-1} = q^{2n-1}(1-q^{2n-2})^2 x$, $b_{2n-1} = (q^{4n-2}+1)$ for $n \geqslant 1$, and $a_1 = 4qx$. We obtain Eisenstein's continued fraction for (3) by using the series expansion

$$2K/\pi = 1 + 4 \sum_{j=0}^{\infty} \frac{q^{j+1}}{q^{2(j+1)} + 1}$$

(see [19], for example). Let c_j^n be the coefficient of x^{j+n+1} in X_n, where $j \geqslant 0$. That is, $X_n = \sum_{j \geqslant 0} c_j^n x^{j+n+1}$. We let $c_{-1}^n = 0$. For $n = 1$ in (22) we have

$$b_1 X_0 + a_1 X_{-1} = 4(q^2+1) \sum_{j=1}^{\infty} \frac{(qx)^{j+1}}{q^{2(j+1)} + 1} = X_1.$$

We note that for $n \geqslant 0$, $\lambda(X_n) = n + 1$, so that for $n > 1$, (22) is equivalent to the relations

$$(q^{2n}+1)(q^{4n}+1)c_j^{2n-1} - (q^{2n}+1)^2 c_j^{2n-2} = -q^{2n+1}(q^{4n}-1)c_{j-1}^{2n} \tag{25}$$

and

$$(q^{4n+2}+1)c_j^{2n} - (q^{2n}-1)c_j^{2n-1} = q^{2n+1}(q^{2n+2}+1)c_{j-1}^{2n+1} \tag{26}$$

for $j \geqslant 0$, $n \geqslant 1$. To establish (25), we have

$$\begin{aligned}
&(q^{2n}+1)(q^{4n}+1)c_j^{2n-1} - (q^{2n}+1)^2 c_j^{2n-2} \\
&= (q^{2n}+1)(q^{4n}+1)\,\frac{(-1)^{n-1}q^j(q^{2(j+1)};q^2)_{n-1}}{(-q^{2(j+n+1)};q^2)_{n-1}(q^{2(j+2n)}+1)} \\
&\quad -(q^{2n}+1)^2\frac{(-1)^{n-1}q^j(q^{2(j+1)};q^2)_{n-1}}{(-q^{2(j+n)};q^2)_{n-1}(q^{2(j+2n-1)}+1)} \\
&= (q^{2n}+1)\,\frac{(-1)^{n-1}q^j(q^{2(j+1)};q^2)_{n-1}}{(-q^{2(j+n+1)_{n-1}};q^2)_{n-1}}\left(\frac{q^{4n}+1}{q^{2(j+2n)}+1} - \frac{q^{2n}+1}{q^{2(j+n)}+1}\right) \\
&= -q^{2n}(q^{4n}-1)\,\frac{(-1)^n q^j(q^{2j};q^2)_n}{(-q^{2(j+n)};q^2)_n(q^{2(j+2n)}+1)} \\
&= -q^{2n+1}(q^{4n}-1)c_{j-1}^{2n}.
\end{aligned}$$

We note that the product vanishes when $j=0$. We establish (26) similarly:

$$\begin{aligned}
&(q^{4n+2}-1)c_j^{2n} - (q^{2n}-1)c_j^{2n-1} \\
&= (q^{4n+2}+1)\frac{(-1)^n q^j(q^{2(j+1)};q^2)_n}{(-q^{2(j+n+1)};q^2)_n(q^{2(j+2n+1)}+1)} \\
&\quad -(q^{2n}-1)\,\frac{(-1)^{n-1}q^j(q^{2(j+1)};q^2)_{n-1}}{(-q^{2(j+n+1)_{n-1}};q^2)_{n-1}(q^{2(j+2n)}+1)} \\
&= \frac{(-1)^{n-1}q^j(q^{2(j+1)};q^2)_{n-1}}{(-q^{2(j+n+1)_{n-1}};q^2)_{n-1}(q^{2(j+2n)}+1)} \\
&\quad \times\left(\frac{(q^{4n+2}+1)(q^{2(j+n)}-1)}{(q^{2(j+2n+1)}+1)} - (q^{2n}-1)\right) \\
&= q^{2n}(q^{2n+2}+1)\,\frac{(-1)^n(q^{2j};q^2)_n}{(-q^{2(j+n+1)};q^2)_n(q^{2(j+2n+1)}+1)} \\
&= q^{2n+1}(q^{2n+2}+1)c_{j-1}^{2n+1}.
\end{aligned}$$

Again, the product vanishes when $j=0$. Conditions (12) and (13) are satisfied, as $\lambda(b_n)=0$, $\lambda(a_n)=1$, and $\lambda(X_n/X_{n-1})=1$ for $n \geqslant 1$. □

4. Examples

In this section we provide a list of continued fraction expansions for certain modular forms, as well as more general expansions that can be derived from Eisenstein's expansions.

1. $\eta(\tau) = \frac{q^{1/12}}{1+}\,\frac{q^2}{1-q^2-}\,\frac{q^2}{1+q^2+}\,\frac{q^6}{1-q^6-}\,\frac{q^4}{1+q^4+}\,\frac{q^{10}}{1-q^{10}-}\,\frac{q^6}{1+q^6+}\cdots$

2. $\vartheta(q) = 1+\frac{2q}{1-}\,\frac{q^3}{1-}\,\frac{q^5-q^3}{1-}\,\frac{q^7}{1-}\,\frac{q^9-q^5}{1-}\,\frac{q^{11}}{1-}\,\frac{q^{13}-q^7}{1-}\cdots$

3. $\vartheta(q) = 1+\frac{2q}{1-}\,\frac{q^3}{1+q^3-}\,\frac{q^5}{1+q^5-}\,\frac{q^7}{1+q^7-}\cdots$

4. $\vartheta^2(q) = 1+\frac{4q}{q^2+1-}\,\frac{q(q^2+1)^2}{q^4+1+}\,\frac{q^3(q^2-1)^2}{q^6+1-}\,\frac{q^3(q^4+1)^2}{q^8+1+}\,\frac{q^5(q^4-1)^2}{q^{10}+1-}\cdots$

5. $\frac{\vartheta(q)-1}{\vartheta(q)+1} = \frac{q}{1+q-}\,\frac{q^3}{1+q^3-}\,\frac{q^5}{1+q^5-}\,\frac{q^7}{1+q^7-}\cdots$

6. $\vartheta_4(q) = \frac{1}{1+}\,\frac{2q}{1-q-}\,\frac{q}{1+}\,\frac{q^2}{1-q^3-}\,\frac{q^2}{1+}\,\frac{q^3}{1-q^5+}\cdots$

7. $\vartheta_4(q) = 1-\frac{2q}{1+}\,\frac{q^3}{1-q^3+}\,\frac{q^5}{1-q^5+}\,\frac{q^7}{1-q^7+}\cdots$

8. $\frac{1-\vartheta_4(q)}{1+\vartheta_4(q)} = \frac{q}{1-q+}\,\frac{q^3}{1-q^3+}\,\frac{q^5}{1-q^5+}\,\frac{q^7}{1-q^7+}\cdots$

9. $\vartheta_2(q) = \frac{2q^{1/4}}{1-}\,\frac{q^2}{q^2+1-}\,\frac{q^4}{q^4+1-}\,\frac{q^6}{q^6+1-}\,\frac{q^8}{q^8+1-}\cdots$

10. $\prod_{n\geqslant 0}\frac{1-xq^n}{1-yq^n} = \frac{1}{1+}\,\frac{x-y}{1-q+}\,\frac{qy-x}{1+q+}\,\frac{q^2x-qy}{1-q^3+}\,\frac{q^3y-qx}{1+q^2+}\,\frac{q^4x-q^2y}{1-q^5+}\cdots$

11. $\sum_{n=0}^{\infty} q^{n^2}x^n = \frac{1}{1-}\,\frac{qx}{1-}\,\frac{(q^3-q)x}{1-}\,\frac{q^5x}{1-}\,\frac{(q^7-q^3)x}{1-}\,\frac{q^9x}{1-}\,\frac{(q^{11}-q^5)x}{1-}\cdots$

12. $\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} = \frac{q}{1-q}\,\frac{q(1-q)^2}{1-q^2-}\,\frac{q^2(1-q)^2}{1-q^3-}\,\frac{q^2(1-q^2)^2}{1-q^4-}\,\frac{q^3(1-q^2)^2}{1-q^5-}\cdots$

13. $\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{1-}\,\frac{q}{1-}\,\frac{q^2-q}{1-}\,\frac{q^3}{1-}\,\frac{q^4-q^2}{1-}\,\frac{q^5}{1-}\,\frac{q^6-q^3}{1-}\cdots$

14. $\sum_{n=0}^{m-1} q^{n(n+1)/2} = \frac{1}{1-}\,\frac{q}{q+1-}\,\frac{q^2}{q^2+1-}\cdots\frac{q^{m-1}}{q^{m-1}+1}$

15.* $\sum_{n=0}^{2m}\rho^{n^2}x^n = \frac{1-x^{2m+1}}{1-}\,\frac{x}{\rho^{2m}-}\,\frac{(1-\rho^{2m-1})x}{\rho^{2m-1}-}\cdots$

$\cdots\frac{(1-\rho)x}{\rho-}\,\frac{x}{1-}\,\frac{(1-\rho^{2m})x}{\rho^{2m}-}\cdots\frac{(1-\rho^2)x}{\rho^2-}\,\frac{x}{\rho}$

16.* $\sum_{n=0}^{2m-1}\sigma^{n^2}x^n = \frac{1-x^{2m}}{1-}\,\frac{x}{\sigma^{2m-1}-}\,\frac{(1-\sigma^{2m-2})x}{\sigma^{2m-2}-}\,\frac{x}{\sigma^{2m-3}-}\cdots$

$\cdots\frac{(1-\sigma^{2m-4})x}{\sigma^{2m-4}-}\cdots\frac{(1-\sigma^2)x}{\sigma^2-}\,\frac{x}{\sigma}.$

* Here, ρ is a primitive $2m+1$st root of unity, and σ a primitive $2m$th root of unity.

Acknowledgements

The author wishes to thank Professor William Duke for suggesting and motivating the ideas in this work. The author also thanks the referee for very detailed and helpful comments.

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