BUILDING BRIDGES: 6TH EU/US SUMMER SCHOOL & WORKSHOP ON AUTOMORPHIC FORMS AND RELATED TOPICS (BB6) CIRM MARSEILLE, SEPTEMBER 2-13, 2024

MOCK THETA FUNCTIONS: LECTURE 2 NOTES[1]

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Reference. K. Bringmann, A. Folsom, K. Ono, L. Rolen, *Harmonic Maass Forms and Mock Modular Forms: Theory and Applications,* AMS Colloquium Publications, 64. American Mathematical Society, Providence, RI, 2018. 391pp.
(and references therein).

Recall that Dyson's rank of a partition rank(π) := largest part of π − #parts of π. Let

$$N(m,n) := p(n\colon \text{rank } m).$$

Lemma. *We have that*

$$R(w;q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}.$$

Recall the q-Pochhammer symbol $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j),\ n \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. Exercise. □

Observe that with $w = 1$,

$$R(1;q) = \sum_{n=0}^{\infty}\left(\sum_{m=-\infty}^{\infty} N(m,n)\right) q^n = \sum_{n=0}^{\infty} p(n)q^n.$$

Hence

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty} = \frac{q^{\frac{1}{24}}}{\eta(\tau)}, \quad \text{a modular form*}.$$

*up to mult. by $q^{-1/24}$ (with $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$).

With $w = -1$ we find

$$R(-1;q) = \sum_{n=0}^{\infty} (p_e(n) - p_o(n))\, q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}.$$

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[1]Disclaimer. These are unpublished lecture notes of the author, rough in nature, with some abuse of notation, and which may contain typographical errors.

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Here $p_{e/o}(n) := p(n\colon \text{even/odd rank})$.

Question: is $R(-1;q)$ also modular?

S. Ramanujan (1887-1920) in his last letter to G.H. Hardy (1920) defined 17 q-hypergeometric series:

> *...I discovered very interesting funcitons recently which I call 'Mock' ϑ-functions. Unlike the 'False' θ-functions (studied partially by Prof. Rogers...) they enter into mathematics as beautifully as the ordinary theta functions.*

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \quad \text{as above, independent of later rank interpretation; order 3}$$

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \quad \text{order 3}$$

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} \quad \text{order 5}$$

$$\vdots$$

Note. Ramanujan did not define his notion of "order." In hindsight we believe it is connected to certain invariants of modular forms.

Other modular "Eulerian" (q-hypergeoemtric) series (not in the Last Letter):

Partition identity (above):

$$q^{-\frac{1}{24}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{q^{-\frac{1}{24}}}{(q;q)_\infty} = \eta^{-1}(\tau),$$

which by the RHS is (nearly) modular (and by LHS is the generating function for $p(n)$),

Rogers-Ramanujan identities:

$$q^{-\frac{1}{60}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{q^{-\frac{1}{60}}}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

$$q^{\frac{11}{60}} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{q^{\frac{11}{60}}}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

which by the RHS are modular functions (weight 0). These are famously equivalent to the (combinatorial) R-R identities $p(n\colon \text{parts } 1,4 \pmod 5) = p(n\colon \text{superdistinct parts})$, and similar from the second identity

Ramanujan:

> *... Suppose there is a function in the Eulerian form and suppose that all or an infinity of points are exponential singularities, and also suppose that at these points the asymptotic form of the function closes neatly ... The question is: Is the function taken the sum of two functions one of which is an ordinary*

theta function and the other a (trivial) function which is $O(1)$ *at* all *the points* $e^{2m\pi i/n}$*? The answer is* it is not necessarily so. *When it is not so I call the function Mock* ϑ*-function. I have not proved rigorously that* it is not necessarily so. *But I have constructed a number of examples...*

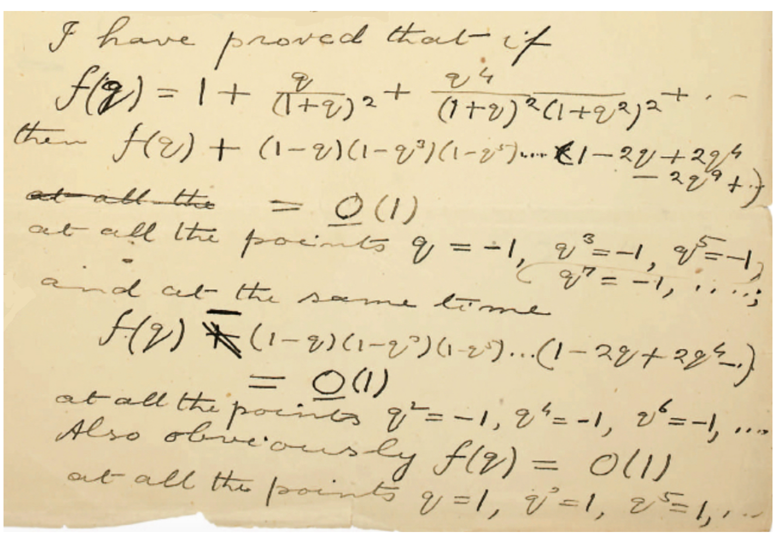
Paraphrased: Suppose an Eulerian series has infinitely many exponential singularities with suitable asymptotic behavior. Is it the sum of a modular theta function plus another which is bounded at roots of unity? Not necessarily, and these are called Mock Theta Functions. I [Ramanujan] haven't proved this but have an example.

Note. We believe "by closes neatly" he means an asymptotic like the following for the first Rogers-Ramanujan function:
With $q = e^{-\alpha}$, as $\alpha \to 0^+$,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \sqrt{\frac{2}{5-\sqrt{5}}} \exp\left(\frac{\pi^2}{15\alpha} - \frac{\alpha}{60}\right) + o(1)$$

(and similar at other singularities). The point is apparently that the series in α in the exponential must be polynomial (finite terms) as opposed to infinite. Note also that $o(1)$ may be replaced by $O(1)$.

Example (from Ramanujan's Last Letter):


I have proved that if
$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$
then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$
~~at all the~~ $= O(1)$
at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1, \ldots$;
and at the same time
$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots)$
$= O(1)$
at all the points $q^2=-1$, $q^4=-1$, $q^6=-1, \ldots$
Also obviously $f(q) = O(1)$
at all the points $q=1$, $q^3=1$, $q^5=1, \ldots$

I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then

$$f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots) = O(1)$$

at all the points $q = -1, q^3 = -1, q^5 = -1, q^7 = -1, \ldots,$ and at the same time

$$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots) = O(1)$$

at all the points $q^2 = -1, q^4 = -1, q^6 = -1, \cdots$ Also obviously $f(q) = O(1)$ at all the points $q = 1, q^3 = 1, q^5 = 1, \ldots$ And so $f(q)$ is a mock ϑ-function.

Ramanujan's example rephrased: Let $b(q) := q^{\frac{1}{24}}\eta^3(\tau)/\eta^2(2\tau)(q = e^{2\pi i\tau})$.

Note. As appearing above, $\displaystyle\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}$.

- The modular forms $\pm b(q)$, together with the 0 function, appear to"cut out" the exponential singularities of $f(q)$.
- That is, as q approaches any even order $2k$ root of unity singularity of $f(q)$, then

$$\boxed{f(q) - (-1)^k b(q) \stackrel{?}{=} O(1)}$$

- That is, asymptotically, towards singularities,

$$\boxed{\text{mock theta} \pm \text{modular form} \stackrel{?}{=} \text{bounded}}$$

Note. $f(\zeta) = O(1)$ for odd ordered roots of unity ζ. Proof: Exercise.

We attribute the following to Ramanujan:

Definition (Ramanujan)**.** A **mock theta function** F of the complex variable q, defined by an Eulerian (q-hypergeometric) series which converges for $|q| < 1$, satisfies

(1) infinitely many roots of unity are exponential singularities,
(2) for every root of unity ζ there is a modular form $\vartheta_\zeta(q)$ such that the difference $F(q) - q^c\vartheta_\zeta(q)$ is bounded as $q \to \zeta$ radially,
(3) there does not exist a single modular form $\vartheta(q)$ such that $F(q) - q^c\vartheta(q)$ is bounded as q approaches any root of unity radially.

Towards the above,

Theorem (Watson, 1936)**.** *We have that*

$$q^{-\frac{1}{24}} f(q) = 2\sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{4}{3}}\omega\left(q_1^2\right) + 4\sqrt{\frac{3\alpha}{2\pi}}\int_0^\infty \frac{\sinh(\alpha t)}{\sinh\left(\frac{3\alpha t}{2}\right)} e^{-\frac{3\alpha t^2}{2}}\, dt,$$

where $q := e^{-\alpha}, \beta := \pi^2/\alpha, q_1 := e^{-\beta}$ $(\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0)$.

Remark. With $\alpha = -2\pi i\tau$ (and $\Im(\tau) > 0$ so that $\Re(\alpha) > 0$), and viewing

$$f(q) \text{``}=\text{''} f(\tau), \quad (q = e(\tau)), \qquad \omega(q_1^2) \text{``}=\text{''} \omega\left(\tfrac{-1}{2\tau}\right), \quad (q_1^2 = e\left(\tfrac{-1}{2\tau}\right)),$$

Watson's transformation shows a <u>modular-like</u> transformation of weight $1/2$ under $\tau \mapsto \frac{-1}{2\tau}$, with an <u>error</u> (integral).

Note. We may use the notation $e(u) := e^{2\pi i u}$ (as above) throughout.

This heads towards Ramanujan's (radial limit as $q = -e^{-\alpha} \to -1$):

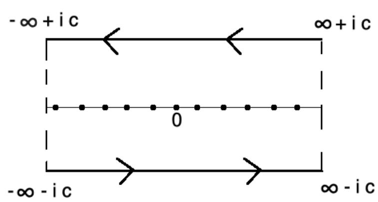
$$\lim_{\alpha \to 0^+} f(q) + \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\pi^2}{24\alpha} - \frac{\alpha}{24}\right) = 4.$$

Proof (Watson's theorem, sketch). We have that (Exercise)

$$f(q) = \frac{2}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n},$$

$$\omega(q) = \frac{1}{(q^2;q^2)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{3n(n+1)}}{1-q^{2n+1}}.$$

The idea is to use the Residue Theorem creatively / carefully:
(1) find a function such that when integrated around a boundary yields $f(q)$ as its residue



sum on one hand:

$$\frac{1}{2\pi i}\left(\int_{-\infty - ic}^{\infty - ic} + \int_{\infty + ic}^{-\infty + ic}\right) \frac{\pi}{\sin(\pi z)} \frac{\exp(-3\alpha z^2/2)}{\cosh(\alpha z/2)} dz \overset{\text{residue theorem}}{=} \sum_{m \in \mathbb{Z}} \operatorname{Res}(\text{integrand}, z = m)$$

$$= \pi \sum_{m \in \mathbb{Z}} \exp(-3\alpha m^2/2) \underbrace{\frac{2e^{-\alpha m/2}}{1+e^{-\alpha m}}}_{\frac{1}{\cosh(\alpha m/2)}} \lim_{z \to m} (z-m) \underbrace{\frac{2ie^{\pi i z}}{e^{2\pi i z} - 1}}_{\frac{1}{\sin(\pi z)}}$$

$$= 2 \sum_{m \in \mathbb{Z}} \frac{(-1)^m q^{\frac{m(3m+1)}{2}}}{1+q^m} \qquad (q = e^{-\alpha})$$

$$= f(q)(q;q)_\infty,$$

where $c > 0$ is small enough so that the zeros of $\sin(\pi z)$ (which occur at $z = m, m \in \mathbb{Z}$) are the only poles of the integrand between the lines forming the contour.
Exercise: the integral over the vertical edges of the contour yields a contribution of 0.

(2) compute the two integrals above directly on the other hand, using that $\frac{1}{\sin(\pi z)} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi i z}$, use residue theorem again and shift contour (through saddle points in the quadratic exponential appearing $\exp((2m+1)\pi i z - 3\alpha z^2/2)$). Eventually (with the known transformation properties of $\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty$) this yields a modular-like transformation law (sim. to modular theta functions). Each integral (eventually) realizes (part of) $\omega(q_1^2)$ (as a residue sum) and the "error" integral. $\square$

Ramanujan's claim revisited:

F-Ono-Rhoades: As $q \to -1$, we have

$$f(-0.994) \sim -1 \cdot 10^{31},\ f(-0.996) \sim -1 \cdot 10^{46},\ f(-0.998) \sim -6 \cdot 10^{90} \dots$$

and

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\dots$	$3.969\dots$	$3.976\dots$	$3.984\dots$	$3.992\dots$

suggesting

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

As $q \to i$, we have

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$2\cdot 10^6 - 4.6\cdot 10^6 i$	$2\cdot 10^8 - 4\cdot 10^8 i$	$1.0\cdot 10^{12} - 2\cdot 10^{12} i$
$f(q)-b(q)$	$\sim 0.05+3.85i$	$\sim 0.04+3.89i$	$\sim 0.03+3.92i$

suggesting

$$\lim_{q\to i}(f(q)-b(q)) = 4i.$$

What are the $O(1)$ constants in

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = O(1)?$$

Where do they come from?

Recall that $f(q) = R(-1;q)$. We prove more generally (letting $\zeta_N := e^{2\pi i/N}$)

Theorem (F-Ono-Rhoades)**.** *Let $1 \le a < b$, $1 \le h < k$ with $\gcd(a,b) = \gcd(h,k) = 1$, $b|k$ and $h' \in \mathbb{Z}$ with $hh' \equiv -1 \pmod{k}$. Then, as $q \to \zeta_h^k$ radially within the unit disc, we have that*

$$\lim_{q\to\zeta_h^k}\left(R\left(\zeta_b^a;q\right) - \zeta_{b^2}^{-a^2h'k} C\left(\zeta_b^a;q\right)\right) = -\left(1-\zeta_b^a\right)\left(1-\zeta_b^{-a}\right) U\left(\zeta_b^a;\zeta_k^h\right).$$

Here

$$U(w;q) := \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} u(m,n) w^m q^n \overset{\text{[Exercise]}}{=} \sum_{n=0}^{\infty}(-wq;q)_n(-w^{-1}q;q)_n q^{n+1},$$

where $u(m,n) := \#\{$strongly unimodal sequences of size n and rank $m\}$ (Ex. 1+2+6+4+2 is a s.u.s. of size 15 and rank 0), and

$$C(w;q) = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} M(m,n) w^m q^n = \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-wq^n)(1-w^{-1}q^n)},$$

where $M(m,n) := p(n\colon \text{crank } m)$.

Note. $U(\zeta_b^a;\zeta_k^h)$ is a convergent, finite, sum (Exercise).

Formally: A s.u.s. $\{a_j\}_{j=1}^s$ of size n is such that $0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots > a_s > 0$ and $\sum_{j=1}^s a_j = n$, and its rank is $s-2r+1$, the difference between number of terms after and before the peak.

The crank of partition is its largest part if it has no 1s, and is the difference between (the number of parts larger than the number of 1s) and (the number of 1s) otherwise.

Corollary (F-Ono-Rhoades). Towards even order $2k$ roots of unity ζ, we have (Ramanujan's claim, and more explicitly)

$$\lim_{q \to \zeta}(f(q) - (-1)^k b(q)) = -4U(-1;\zeta), \text{ a convergent, finite sum.}$$

(This is the case $a/b = 1/2$ of the theorem above.)

Proof: uses the theory of harmonic Maass forms, and incorporates quantum modular forms (as in next two lectures).

Major Question: How do the mock theta functions fit into the theory of modular forms?

• The above starts to indicate some of the place of the mock theta functions in the theory modular forms, "asymptotically."
• However, the F-O-R asymptotic results and other newer results towards how they fit in "modularly" were not proved until recently using a more overarching theory of harmonic Maass forms (which we will introduce shortly).

Watson's transformation shows a modular-like transformation of weight 1/2 under $\tau \mapsto \frac{-1}{2\tau}$. Roughly:

$$f(\tau) = (*)\tau^{-\frac{1}{2}}\omega\left(\tfrac{-1}{2\tau}\right) + \text{ Error Integral} \quad \longleftrightarrow \quad \begin{cases} f(\frac{-1}{\tau}) & = (*)\tau^{\frac{1}{2}}\omega\left(\frac{\tau}{2}\right) + \text{ Error Integral,} \\ \omega\left(\frac{-1}{2\tau}\right) & = (*)\tau^{\frac{1}{2}} f(\tau) \ + \text{ Error Integral.} \end{cases}$$

F. Dyson (1987):
"...Somehow it should be possible to build [the mock theta functions] into a coherent group-theoretical structure, analogous to the structure of modular forms...This remains a challenge for the future..."

Decades later,
Zwegers (2001) defines the period integeral:

$$G(\tau) := 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}}dz,$$

where

$$g(\tau) := -\sum_{n=-\infty}^{\infty}(n+\tfrac{1}{6})q^{\frac{3}{2}(n+\frac{1}{6})^2} = -\tfrac{1}{6}(q^{\frac{1}{24}} - 5q^{\frac{25}{24}} + 7q^{\frac{49}{24}} - \cdots)$$

is a weight 3/2 modular theta function (more below).

Remark. The integral converges (even for $\tau \in \mathbb{Q}$) since g is a cusp form (i.e. $g(\tau) = O(q^{\frac{1}{24}})$).

With $F(\tau) := q^{-\frac{1}{24}} f(q)$,

A major theorem:

Theorem (Zwegers, 2001)**.** *The difference*

$$H(\tau) := F(\tau) - G(\tau)$$

transforms as a (component of a) weight $1/2$ *non-holomorphic (vector-valued) modular form. In particular, with* $\boldsymbol{H}(\tau) := (H(\tau), H_1(\tau), H_2(\tau))^T$ *(and* H_1, H_2 *defined similar to* H *using* ω *below)*

$$\boldsymbol{H}(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} \boldsymbol{H}(\tau), \quad \boldsymbol{H}(\tfrac{-1}{\tau}) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{H}(\tau).$$

Idea: Zwegers "corrects" Watson's transformation for F by constructing the (non-holomorphic) function G with the same exact error to modularity ("error integral"), and subtracting.

One gains modularity at the expense of losing holomorphicity.

$$\underbrace{H(\tau)}_{\substack{\times\text{non-holomorphic} \\ \checkmark\text{transforms like a modular form (no error)}}} \quad := \quad \underbrace{q^{-\frac{1}{24}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}}_{\substack{\checkmark\text{holomorphic} \\ X\text{error term in modular transformation}}} \quad - \quad \underbrace{2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{g(z)\ dz}{\sqrt{-i(\tau+z)}}}_{\substack{\times\text{non-holomorphic} \\ X\text{error term in modular transformation}}}$$

To make this more precise, let

$$\mathbf{F}(\tau) = (F(\tau),\ F_1(\tau),\ F_2(\tau))^T := \left(q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}} \omega\left(q^{\frac{1}{2}}\right),\ 2q^{\frac{1}{3}} \omega\left(-q^{\frac{1}{2}}\right) \right)^T,$$

$$\mathbf{G}(\tau) := 2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{(g(w),\ g_0(w),\ -g_2(w))^T}{\sqrt{-i(w+\tau)}}\ dw =: (G(\tau), G_1(\tau), G_2(\tau))^T,$$
$$\mathbf{H}(\tau) := \mathbf{F}(\tau) - \mathbf{G}(\tau) =: (H(\tau), H_1(\tau), H_2(\tau))^T$$

(so that e.g. $H_1(\tau) = F_1(\tau) - G_1(\tau)$ where $G_1(\tau) = 2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{g_0(w)}{\sqrt{-i(w+\tau)}}\ dw$), with

$$g_0(\tau) := \sum_{n\in\mathbb{Z}} (-1)^n \left(n+\tfrac{1}{3}\right) q^{\frac{3}{2}\left(n+\frac{1}{3}\right)^2}, \quad g_2(\tau) := \sum_{n\in\mathbb{Z}} \left(n+\tfrac{1}{3}\right) q^{\frac{3}{2}\left(n+\frac{1}{3}\right)^2}.$$

Note. Notation (indexing of functions) here is slightly different than in Zwegers.

Proof (Zwegers' theorem, sketch).

$$
\begin{aligned}
G(\tfrac{-1}{\tau}) &= 2i\sqrt{3}\int_{1/\overline{\tau}}^{i\infty} \frac{g(z)}{\sqrt{-i(z-\frac{1}{\tau})}}dz \\
&\overset{z\mapsto -1/u}{=} 2i\sqrt{3}\int_{-\overline{\tau}}^{0} \frac{g(-1/u)u^{-2}}{\sqrt{-i(-\frac{1}{u}-\frac{1}{\tau})}}du \\
&\overset{\text{modularity of } g}{=} 2i\sqrt{3}\sqrt{-i\tau}\int_{-\overline{\tau}}^{0} \frac{g_0(u)}{\sqrt{-i(u+\tau)}}du \\
&= 2i\sqrt{3}\sqrt{-i\tau}\left(\int_{-\overline{\tau}}^{i\infty} - \int_{0}^{i\infty}\right) \frac{g_0(u)}{\sqrt{-i(u+\tau)}}du,
\end{aligned}
$$

using that

$$
(g_0(-1/\tau), g(-1/\tau), g_2(-1/\tau))^T = -(-i\tau)^{\frac{3}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} (g_0(\tau), g(\tau), g_2(\tau))^T.
$$

I.e.,

$$
G(\tfrac{-1}{\tau}) - \sqrt{-i\tau}G_1(\tau) = -\sqrt{-i\tau}\cdot 2i\sqrt{3}\int_0^{i\infty} \frac{g_0(u)}{\sqrt{-i(u+\tau)}}du. \tag{1}
$$

OTOH let

$$
j_0(\tau) := \int_0^{\infty} e^{3\pi i\tau t^2}\frac{\sin(2\pi\tau t)}{\sin(3\pi\tau t)}dt
$$

as in Watson, which is equivalent to

$$
F(\tfrac{-1}{\tau}) - \sqrt{-i\tau}F_1(\tau) = \sqrt{-i\tau}4\sqrt{3}j_0(\tfrac{-1}{\tau})). \tag{2}
$$

Mittag-Leffler partial fraction theory (long calculation) on $\sinh(2\pi v)/\sinh(3\pi v)$ and the integral identity

$$
\int_{-\infty}^{\infty} \frac{e^{-\pi t w^2}}{w-ir}dw = \pi i r\int_0^{\infty} \frac{e^{-\pi r^2 w}}{\sqrt{w+t}}dw
$$

which holds for $r\in\mathbb{R}\backslash\{0\}$ eventually leads to the identity (also using modularity of $g(\tau)$)

$$
4\sqrt{3}j_0(\tfrac{-1}{\tau}) = -2i\sqrt{3}\int_0^{i\infty} \frac{g_0(u)}{\sqrt{-i(u+\tau)}}du, \tag{3}
$$

Subtracting (1) from (2) using (3) reveals that

$$
H(\tfrac{-1}{\tau}) = \sqrt{-i\tau}H_1(\tau)
$$

(where $H_1(\tau) = F_1(\tau) - G_1(\tau)$). □

Recall: Zwegers "corrects" Watson's transformation for F by constructing the (non-holomorphic) function G with the same exact error to modularity ("error integral"), and subtracting.

One gains modularity at the expense of losing holomorphicity.

$$\underbrace{H(\tau)}_{\substack{\text{✗non-holomorphic}\\ \text{✓transforms like a modular form (no error)}}} := \underbrace{q^{-\frac{1}{24}}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}}_{\substack{\text{✓holomorphic}\\ \text{✗error term in modular transformation}}} - \underbrace{2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty}\frac{g(z)\ dz}{\sqrt{-i(\tau+z)}}}_{\substack{\text{✗non-holomorphic}\\ \text{✗error term in modular transformation}}}$$

This modular decomposition $H = F - G$ was reminiscent of

Definition (Bruinier-Funke, 2004)**.** A *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma := \Gamma_0(N)$ (where $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$) is a smooth $M : \mathbb{H} \to \mathbb{C}$ satisfying

(1) transformation law: $\forall A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\tau \in \mathbb{H}$,

$$M(A\tau) = \begin{cases} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{-2k}\ (c\tau+d)^k M(\tau), & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}, \\ (c\tau+d)^k M(\tau), & k \in \mathbb{Z}, \end{cases} \qquad \left(\varepsilon_d := \begin{cases} 1, & d \equiv 1 \pmod 4, \\ i, & d \equiv 3 \pmod 4, \end{cases}\right)$$

(2) harmonic: $\Delta_k M = 0$, where $\underbrace{\Delta_k}_{\text{weight } k \text{ Laplacian operator}} := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$(\tau = x + iy)$

(3) M satisfies a suitable growth condition in the cusps. I.e., at $\infty, \exists$ a polynomial $P_M(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$M(\tau) - p_M(\tau) = O(e^{-\epsilon y})$$

as $y \to \infty$ for some $\epsilon > 0$.

Lemma. *Let M be a HMF of weight $k \in \frac{1}{2}\mathbb{Z}\backslash\{1\}$ w.r.t. $\Gamma_0(N)$. Then M has the Fourier expansion (at ∞):*

$$M(\tau) = \sum_{n \gg -\infty} c_M^+(n) q^n + \sum_{n<0} c_M^-(n)\Gamma(1-k, -4\pi n y) q^n.$$

(Similar expansions hold at other cusps.)

Here, the incomplete Gamma function is

$$\Gamma(s,z) := \int_z^{\infty} e^{-t} t^s \frac{dt}{t},$$

$(\Re(s) > 0, z \in \mathbb{C}$; or $s \in \mathbb{C}, z \in \mathbb{H}$; analytically continued in s via a functional equation).

Proof (sketch). Condition (1) $\Rightarrow$

$$M(\tau) =: \sum_{n\in\mathbb{Z}} a_M(n,y) e\left(nx\right).$$

Applying Δ_k (and using cond. (2)), we find that the Fourier coefficients $C(2\pi n y) := a_M(n,y)$ satisfy the differential equation

$$\frac{\partial^2}{\partial w^2}C(w) - C(w) + \frac{k}{w}\left(\frac{\partial}{\partial w}C(w) + C(w)\right) = 0.$$

For $n \neq 0$ there are two linearly independent solutions: e^{-w} and $\Gamma(1-k,-2w)e^{-w}$. The restrictions in summation in the lemma follow from $\Gamma(s,x) \sim x^{s-1}e^{-x}$ as $|x| \to \infty$ and cond. (3). $\square$

Definition. We call

$$M^+(\tau) := \sum_{n \gg -\infty} c_M^+(n) q^n$$

the holomorphic part of the wt. k HMF M, and

$$M^-(\tau) := \sum_{n<0} c_M^-(n)\Gamma(1-k,-4\pi ny)q^n$$

the non-holomorphic part of the wt. k HMF M.

Example. The (normalized) mock theta function $q^{-1}f(q^{24})$ is the holomorphic part of the HMF

$$H(24\tau) := F(24\tau) - G(24\tau)$$

of weight 1/2 on $\Gamma_0(144)$ (and char. $\chi_{12} := \left(\frac{12}{\cdot}\right)$).

Attributed to Zwegers,

Theorem (Zwegers)**.** *Ramanujan's mock theta functions are* hol. parts of weight 1/2 HMFs. *That is, if m is one of Ramanujan's mtf's, then for some $c \in \mathbb{Q}$ and $d \in \mathbb{C}$,*

$$m(\tau) = q^c M^+(\tau) + d,$$

where M^+ is the holomorphic part of a weight 1/2 HMF.

Following Zagier,

Definition. A mock modular form of weight k is the holomorphc part M^+ of a HMF of weight k for which M^- is nontrivial.