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HARMONIC MAASS FORMS AND MOCK MODULAR FORMS: LECTURE 3 NOTES[1]

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Reference. K. Bringmann, A. Folsom, K. Ono, L. Rolen, *Harmonic Maass Forms and Mock Modular Forms: Theory and Applications,* AMS Colloquium Publications, 64. American Mathematical Society, Providence, RI, 2018. 391pp.
(and references therein).

Review or introduce as needed p1-3: Recall Ramanujan's mock ϑ-function (1920)

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}.$$

Major Question: (Dyson's challenge) how do the mock theta functions fit into the theory of modular forms?

Recall

Theorem (Watson, 1936)**.** *We have that*

$$q^{-\frac{1}{24}} f(q) = 2\sqrt{\frac{2\pi}{\alpha}} q_1^{\frac{4}{3}} \omega\left(q_1^2\right) + 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^{\infty} \frac{\sinh(\alpha t)}{\sinh\left(\frac{3\alpha t}{2}\right)} e^{-\frac{3\alpha t^2}{2}} dt,$$

where $q := e^{-\alpha}, \beta := \pi^2/\alpha, q_1 := e^{-\beta}$ $(\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0)$.

Zwegers (2001) defines the period integeral:

$$G(\tau) := 2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}} dz,$$

where

$$g(\tau) := -\sum_{n=-\infty}^{\infty} (n+\tfrac{1}{6}) q^{\frac{3}{2}(n+\frac{1}{6})^2} = -\tfrac{1}{6}(q^{\frac{1}{24}} - 5q^{\frac{25}{24}} + 7q^{\frac{49}{24}} - \cdots)$$

is a weight 3/2 modular theta function.

[1]Disclaimer. These are unpublished lecture notes of the author, rough in nature, with some abuse of notation, and which may contain typographical errors.

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Remark. The integral converges (even for $\tau \in \mathbb{Q}$) since g is a cusp form (i.e. $g(\tau) = O(q^{\frac{1}{24}})$).

With $F(\tau) := q^{-\frac{1}{24}} f(q)$,

A major theorem:
Theorem (Zwegers, 2001)**.** *The difference*

$$H(\tau) := F(\tau) - G(\tau)$$

transforms as a (component of a) weight $1/2$ *non-holomorphic (vector-valued) modular form. In particular, with* $\boldsymbol{H}(\tau) := (H(\tau), H_1(\tau), H_2(\tau))$ *(and* H_1, H_2 *defined similar to* f*, using* ω*)*

$$\boldsymbol{H}(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} \boldsymbol{H}(\tau), \quad \boldsymbol{H}(\tfrac{-1}{\tau}) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{H}(\tau).$$

Idea: Zwegers "corrects" Watson's transformation for F by constructing the (non-holomorphic) function G with the same exact error to modularity ("error integral"), and subtracting.
One gains modularity at the expense of losing holomorphicity.

$$\underbrace{H(\tau)}_{\substack{\text{✗non-holomorphic} \\ \text{✓transforms like a modular form (no error)}}} := \underbrace{q^{-\frac{1}{24}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}}_{\substack{\text{✓holomorphic} \\ \text{✗error term in modular transformation}}} - \underbrace{2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{g(z)\, dz}{\sqrt{-i(\tau+z)}}}_{\substack{\text{✗non-holomorphic} \\ \text{✗error term in modular transformation}}}$$

The modular decomposition $F - G$ was reminiscent of

Definition (Bruinier-Funke, 2004)**.** A *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma := \Gamma_0(N)$ (where $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$) is a smooth $M : \mathbb{H} \to \mathbb{C}$ satisfying

(1) transformation law: $\forall A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\tau \in \mathbb{H}$,

$$M(A\tau) = \begin{cases} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{-2k} \, (c\tau+d)^k M(\tau), & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}, \\ (c\tau+d)^k M(\tau), & k \in \mathbb{Z}, \end{cases} \qquad \left(\varepsilon_d := \begin{cases} 1, & d \equiv 1 \pmod 4, \\ i, & d \equiv 3 \pmod 4, \end{cases}\right)$$

(2) harmonic: $\Delta_k M = 0$, where $\underbrace{\Delta_k}_{\text{weight } k \text{ Laplacian operator}} := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$(\tau = x + iy)$

(3) M satisfies a suitable growth condition in the cusps. I.e., at $\infty, \exists$ a polynomial $P_M(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$M(\tau) - p_M(\tau) = O(e^{-\epsilon y})$$

as $y \to \infty$ for some $\epsilon > 0$.

Lemma. *Let M be a HMF of weight $k \in \frac{1}{2}\mathbb{Z}\backslash\{1\}$ w.r.t. $\Gamma_0(N)$. Then M has the Fourier expansion (at ∞):*

$$M(\tau) = \sum_{n \gg -\infty} c_M^+(n) q^n + \sum_{n<0} c_M^-(n) \Gamma(1-k, -4\pi n y) q^n.$$

(Similar expansions hold at other cusps.)

Here, the incomplete Gamma function is

$$\Gamma(s, z) := \int_z^\infty e^{-t} t^s \frac{dt}{t},$$

($\Re(s) > 0, z \in \mathbb{C}$; or $s \in \mathbb{C}, z \in \mathbb{H}$; analytically continued in s via a functional equation).

Proof (lemma, sketch). Condition (1) $\Rightarrow$

$$M(\tau) =: \sum_{n \in \mathbb{Z}} a_M(n, y) e(nx).$$

Applying Δ_k (and using cond. (2)), we find that the Fourier coefficients $C(2\pi n y) := a_M(n, y)$ satisfy the differential equation

$$\frac{\partial^2}{\partial w^2} C(w) - C(w) + \frac{k}{w}\left(\frac{\partial}{\partial w} C(w) + C(w)\right) = 0.$$

For $n \neq 0$ there are two linearly independent solutions: e^{-w} and $\Gamma(1-k, -2w)e^{-w}$. The restrictions in summation in the lemma follow from $\Gamma(s, x) \sim x^{s-1} e^{-x}$ as $|x| \to \infty$ and cond. (3). $\square$

Definition. We call

$$M^+(\tau) := \sum_{n \gg -\infty} c_M^+(n) q^n$$

the <u>holomorphic part</u> of the wt. k HMF M, and

$$M^-(\tau) := \sum_{n<0} c_M^-(n) \Gamma(1-k, -4\pi n y) q^n$$

the <u>non-holomorphic part</u> of the wt. k HMF M.

Example. The (normalized) mock theta function $q^{-1} f(q^{24})$ is the holomorphic part of the HMF

$$F(24\tau) - G(24\tau)$$

of weight 1/2 on $\Gamma_0(144)$ (and char. $\chi_{12} := \left(\frac{12}{\cdot}\right)$).

Attributed to Zwegers,

Theorem (Zwegers)**.** *Ramanujan's mock theta functions are* hol. parts of weight* 1/2 *HMFs.*
**That is, if m is one of Ramanujan's mtf's, then for some $\alpha \in \mathbb{Q}$ and $c \in \mathbb{C}$,*

$$m(\tau) = q^\alpha M^+(\tau) + c,$$

where M^+ is the holomorphic part of a weight 1/2 *HMF.*

Following Zagier,

Definition. A mock modular form of weight k is the holomorphc part M^+ of a HMF of weight k for which M^- is nontrivial.

Other examples of HMFs:

- Weakly hol. modular forms (which are not called mock modular as NHP is trivial)
- Non-holomorphic weight 2 Eisenstein series: $$E_2^*(\tau) := \underbrace{E_2(\tau)}_{1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n} - \frac{3}{\pi y}$$ "of manageable growth," i.e., $(E_2^*)^- := -3/(\pi y)$, $[\xi_2(E_2^*) = 3/\pi$, where ξ_k will be defined below].
- Zagier's weight 3/2 Hurwitz class number function:

$$\mathcal{H}(\tau) := -\frac{1}{12} + \sum_{n=1}^{\infty} \underbrace{h(n)}_{\substack{\text{(weighted) number of classes} \\ \text{of + BQF disc } -n}} q^n + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 y\right) q^{-n^2} + \frac{1}{8\pi\sqrt{y}},$$

"of manageable growth." [I.e., $\xi_{3/2}(\mathcal{H}) = -\theta/(16\pi)$.]

- Maass-Poincaré series
- from Dyson's ranks more generally:

  Theorem (Bringmann-Ono, 2010)**.** *If $0 < a < c$, then*

$$q^{-\frac{\ell_c}{24}} \underbrace{R(\zeta_c^a; q^{\ell_c})}_{\text{partition rank generating function}} + \frac{i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta\left(\frac{a}{c};\ell_c u\right)}{\sqrt{-i(\tau+u)}} du$$

  is a harmonic Maass form of weight $\frac{1}{2}$ on Γ_c.

- others (see also Exercises) ...

We can directly relate to spaces of ordinary modular forms via the weight k xi-operator:

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{\tau}}}.$$

Theorem. *We have that*

$$\xi_{2-k} \colon H_{2-k}(\Gamma_0(N)) \twoheadrightarrow S_k(\Gamma_0(N)).$$

Moreover, for a weight $2-k$ harmonic Maass form M of level N we have that

$$\xi_{2-k}(M(\tau)) = \xi_{2-k}(M^{-}(\tau)) = -(4\pi)^{k-1}\sum_{n=1}^{\infty}\overline{c_M^{-}(-n)}n^{k-1}q^n.$$

(Recall, $M^{-}(\tau) = \sum_{n<0} c_M^{-}(n)\Gamma(1-k, -4\pi ny)q^n$.)

Remark. One can show that $\ker(\xi_{2-k}) = \underbrace{M^{!}_{2-k}(\Gamma_0(N))}_{\text{weakly hol. modular forms (poles in cusps)}}$.

Proof (Theorem, sketch). We have

$$\begin{aligned}
\overline{\frac{\partial}{\partial\overline{\tau}}}M &= \overline{\frac{\partial}{\partial\overline{\tau}}}(M^{+}+M^{-}) \\
&= \overline{\frac{\partial}{\partial\overline{\tau}}}M^{-} && (\text{since } \frac{\partial}{\partial\overline{\tau}}M^{+}=0) \\
&= \overline{\frac{\partial}{\partial\overline{\tau}}}(\sum_{n<0} c_M^{-}(n)\Gamma(k-1,-4\pi ny)q^n) \\
&= (\sum_{n<0}\overline{c_M^{-}(n)q^n\frac{\partial}{\partial\overline{\tau}}\Gamma(k-1,-4\pi ny)}) && (\text{since } \frac{\partial}{\partial\overline{\tau}}q^n=0) \\
&= -\sum_{n<0}\overline{c_M^{-}(n)q^n(-4\pi n)\tfrac{i}{2}(-4\pi ny)^{k-2}e^{4\pi ny}}, && (\text{since } \frac{\partial}{\partial\overline{\tau}}=\frac{1}{2}(\partial_x+i\partial_y)),
\end{aligned}$$

using that

$$\frac{\partial}{\partial w}\Gamma(\alpha,w) = -w^{\alpha-1}e^{-w}.$$

Thus,

$$\begin{aligned}
\xi_{2-k}M &= -2iy^{2-k}\cdot\frac{-i}{2}\sum_{n<0}\overline{c_M^{-}(n)}n^{k-1}(-4\pi)^{k-1}y^{k-2}e^{4\pi ny}\overline{q^n} \\
&= -(-4\pi)^{k-1}\sum_{n<0}\overline{c_M^{-}(n)}n^{k-1}q^{-n} \\
&= -(4\pi)^{k-1}\sum_{n>0}\overline{c_M^{-}(-n)}n^{k-1}q^{n}.
\end{aligned}$$

as wanted. The modular transformation property follows from that of M and calculus, and is left as an Exercise. □

Definition. The cusp form associated to the weight $2-k$ HMF M

$$\xi_{2-k}(M(\tau)) = \xi_{2-k}(M^{-}(\tau)) = -(4\pi)^{k-1}\sum_{n=1}^{\infty}\overline{c_M^{-}(-n)}n^{k-1}q^n \in S_k(\Gamma_0(N))$$

is called the **shadow** of the mock modular form M^{+}.

Example. The shadow of the (normalized) third order mock theta function $q^{-1}f(q^{24})$ is (up to constant multiple) the weight 3/2 theta function $g(24\tau) := -\sum_{n=-\infty}^{\infty}(n+\frac{1}{6})q^{36(n+\frac{1}{6})^2}$.

More generally,

Lemma. *Let M be a weight $2-k$ HMF. Suppose the mock modular form M^+ has shadow $\rho(\tau) = \sum_{n=1}^{\infty} c_\rho(n)q^n \in S_k(\Gamma_0(N))$. Then the non-holomorphic part M^- satisfies*

$$M^-(\tau) = 2^{1-k} i \int_{-\overline{\tau}}^{i\infty} \frac{\rho^c(w)}{(-i(w+\tau))^{2-k}} dw,$$

where $\rho^c(\tau) := \overline{\rho(-\overline{\tau})} = \sum_{n=1}^{\infty} \overline{c_\rho(n)} q^n$.

Proof (lemma, sketch). By definition, for $n \in \mathbb{N}$,

$$(1) \qquad i(2\pi n)^{1-k}\Gamma(k-1, 4\pi n y)q^{-n} = \int_{2iy}^{i\infty} \frac{e(n(w-\tau))}{(-iw)^{2-k}} dw = \int_{-\overline{\tau}}^{i\infty} \frac{e(nw)}{(-i(w+\tau))^{2-k}} dw.$$

Applying (1),

$$2^{1-k} i \int_{-\overline{\tau}}^{i\infty} \frac{\sum_{n=1}^{\infty} \overline{c_\rho(n)} e(nw)}{(-i(w+\tau))^{2-k}} dw = 2^{1-k} i \sum_{n=1}^{\infty} \overline{c_\rho(n)} \int_{-\overline{\tau}}^{i\infty} \frac{e(nw)}{(-i(w+\tau))^{2-k}} dw$$
$$= -2^{1-k} \sum_{n=1}^{\infty} \overline{c_\rho(n)} (2\pi n)^{1-k}\Gamma(k-1, 4\pi n y)q^{-n}.$$

Since (by def. of the shadow) we must have

$$c_\rho(n) = -(4\pi)^{k-1}\overline{c_M^-(-n)} n^{k-1},$$

the above becomes

$$\sum_{n=1}^{\infty} c_M^-(-n)\Gamma(k-1, 4\pi n y)q^{-n} = \sum_{n<0} c_M^-(n)\Gamma(k-1, 4\pi |n| y)q^n = M^-(\tau).$$

□

Zwegers' Appell-Lerch sums:

Definition. For $\tau \in \mathbb{H}$ and $z_1, z_2 \in \mathbb{C}\backslash(\mathbb{Z}\tau + \mathbb{Z})$, define **Zwegers' μ-function** by

$$\mu(z_1, z_2; \tau) := \frac{\zeta_1^{\frac{1}{2}}}{\vartheta(z_2; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n \zeta_2^n q^{\frac{n(n+1)}{2}}}{1 - \zeta_1 q^n},$$

where $\zeta_j := e^{2\pi i z_j}$ $(j = 1, 2)$ and the Jacobi theta function is

$$\vartheta(z; \tau) := \sum_{n \in \frac{1}{2}+\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n\left(z + \frac{1}{2}\right)} = -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)\left(1 - \zeta q^{n-1}\right)\left(1 - \zeta^{-1} q^n\right).$$

(Here (and elsewhere when relevant) $\zeta = e^{2\pi i z}$.)

Note. When convenient, we may write functions $a(z; \tau) = a(z)$.

Remark. ϑ is an odd (in z) holomorphic Jacobi form (on $\mathbb{C}\times\mathbb{H}$) of weight and index 1/2, e.g.:

$$\vartheta(z+1) = -\vartheta(z), \qquad \vartheta(z+\tau) = -e^{-\pi i\tau-2\pi iz}\vartheta(z),$$
$$\vartheta(z;\tau+1) = e^{\frac{\pi i}{4}}\vartheta(z;\tau), \quad \vartheta\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{\frac{\pi iz^2}{\tau}}\vartheta(z;\tau).$$

Remark. The function $z_1 \mapsto \mu(z_1,z_2;\tau)$ is meromorphic with simple poles in $\mathbb{Z}\tau+\mathbb{Z}$, with residue $-\frac{1}{2\pi i}\frac{1}{\vartheta(z_2)}$ at $z_1=0$.

The function μ exhibits "mock" Jacobi behavior on $\mathbb{C}\times\mathbb{H}$.
See in particular items ii) in the propositions below.

Proposition. *We have:*

i) $\mu(z_1+1,z_2) = \mu(z_1,z_2+1) = -\mu(z_1,z_2)$,

ii) $$\boxed{\mu(z_1+\tau,z_2;\tau) = -\zeta_1\zeta_2^{-1}q^{\frac{1}{2}}\mu(z_1,z_2;\tau) - \underbrace{i\zeta_1^{\frac{1}{2}}\zeta_2^{-\frac{1}{2}}q^{\frac{3}{8}},}_{\text{elliptic Jacobi error}}}$$

iii) $\mu(z_1+\tau,z_2+\tau) = \mu(z_1,z_2)$,

iv) $\mu(-z_1,-z_2) = \mu(z_1,z_2)$.

Proposition. *We have*

i) $\mu(z_1,z_2;\tau+1) = e^{-\frac{\pi i}{4}}\mu(z_1,z_2;\tau)$.

ii) $$\boxed{\mu\left(\frac{z_1}{\tau},\frac{z_2}{\tau};-\frac{1}{\tau}\right) = -\sqrt{-i\tau}e^{-\frac{\pi i}{\tau}(z_1-z_2)^2}\mu(z_1,z_2;\tau) - \underbrace{\frac{i\sqrt{-i\tau}}{2}e^{-\frac{\pi i}{\tau}(z_1-z_2)^2}h(z_1-z_2;\tau),}_{\text{modular Jacobi error}}}$$

where the Mordell integeral is

$$h(z;\tau) := \int_{\mathbb{R}}\frac{e^{\pi i\tau t^2-2\pi zt}}{\cosh(\pi t)}dt.$$

Remark. The Mordell integral is the unique holomorphic function $z\mapsto h(z;\tau)$ satisfying

i) $h(z+1) = -h(z) + \frac{2}{\sqrt{-i\tau}}e^{\frac{\pi i}{\tau}\left(z+\frac{1}{2}\right)^2}$,

ii) $h(z+\tau) = -\zeta q^{\frac{1}{2}}h(z) + 2\zeta^{\frac{1}{2}}q^{\frac{3}{8}}$.

This is left as an Exercise.

Proof of ii) (sketch). Define

$$f(z_1,z_2;\tau) := \frac{2i}{\sqrt{i\tau}}e^{\frac{\pi i}{\tau}(z_1-z_2)^2}\mu\left(\frac{z_1}{\tau},\frac{z_2}{\tau};-\frac{1}{\tau}\right) + 2i\mu(z_1,z_2;\tau).$$

One can show that f depends only on z_1-z_2. It can be shown that f satisfies h's props. i) and ii). The result follows by uniqueness of h. $\square$

Definition. For $z_1,z_2\in\mathbb{C}$ and $\tau\in\mathbb{H}$, we define the **completed μ-function**

$$\widehat{\mu}(z_1,z_2;\tau) := \mu(z_1,z_2;\tau) + \frac{i}{2}\mathcal{R}(z_1-z_2;\tau),$$

where $\tau = x + iy$, $z = u + iv$, and

$$\mathcal{R}(z;\tau) = \mathcal{R}(z) := \sum_{n \in \frac{1}{2}+\mathbb{Z}} \left(\operatorname{sgn}(n) - E\left(\left(n + \frac{v}{y} \right) \sqrt{2y} \right) \right) (-1)^{n-\frac{1}{2}} \zeta^{-n} q^{-\frac{n^2}{2}}. \tag{2}$$

Here $E(z) := 2\int_0^z e^{-\pi t^2} dt$ and $\operatorname{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$

Remark. For $x \in \mathbb{R}$, $E(x) = \operatorname{sgn}(x)\left(1 - \beta\left(x^2\right)\right)$, where for $y \in \mathbb{R}^+$ $\beta(y) := \int_y^\infty t^{-\frac{1}{2}} e^{-\pi t} dt$. Further, $\beta(u) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \pi u\right)$.

This hints at:

Theorem. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Q}$ such that (α_1, β_1), $(\alpha_2, \beta_2) \notin \mathbb{Z}^2$. Then*

$$\tau \mapsto e^{-\pi i(\alpha_1 - \alpha_2)^2 \tau} \widehat{\mu}\left(\alpha_1 \tau + \beta_1, \alpha_2 \tau + \beta_2; \tau\right)$$

is a harmonic Maass form (for some congruence subgroup) of weight $1/2$. Moreover, we have that the shadow of its holomorphic part is

$$i\sqrt{2} e^{-2\pi i(\alpha_1 - \alpha_2)\left(\beta_1 - \beta_2 + \frac{1}{2}\right)} g_{\alpha_1 - \alpha_2 + \frac{1}{2}, \beta_1 - \beta_2 + \frac{1}{2}}(\tau),$$

where the weight $3/2$ modular theta functions are defined by

$$g_{a,b}(\tau) := \sum_{n \in a + \mathbb{Z}} n e^{2\pi i n b} q^{\frac{n^2}{2}}.$$

Example. Ramanujan's mock theta functions $f(q)$ and $\omega(q)$ satisfy

$$f(q) = 4q^{-\frac{1}{8}} \mu(2\tau + \frac{1}{2}, \tau; 3\tau) + q^{\frac{1}{24}} \frac{\eta^4(3\tau)}{\eta(\tau)\eta^2(6\tau)},$$
$$\omega(q) = -2iq^{-\frac{3}{4}} \mu(3\tau, 2\tau; 6\tau) + q^{-\frac{2}{3}} \frac{\eta^4(6\tau)}{\eta(2\tau)\eta^2(3\tau)}.$$

The other mock theta functions may similarly be expressed in terms of μ and modular forms.

Zwegers more generally shows that $\mu(u, v; \tau)$ (which may be viewed as functions of $(u - v, \tau)$) behave like mock Jacobi forms.

Theorem. *We have*

i) *For $k, \ell, m, n \in \mathbb{Z}$,*

$$\widehat{\mu}\left(z_1 + k\tau + \ell, z_2 + m\tau + n\right) = (-1)^{k+\ell+m+n} q^{\frac{1}{2}(k-m)^2} \zeta_1^{k-m} \zeta_2^{m-k} \widehat{\mu}(z_1, z_2).$$

ii) *If ν_η is the multiplier of η, then for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,*

$$\boxed{\widehat{\mu}\left(\frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d}; \frac{a\tau + b}{c\tau + d} \right) = \nu_\eta(\gamma)^{-3} (c\tau + d)^{\frac{1}{2}} e^{-\frac{\pi i c}{c\tau + d}(z_1 - z_2)^2} \widehat{\mu}(z_1, z_2; \tau).}$$

Modern vs. Historic Definitions of Mock Theta Functions

Recall

Ramanujan's observations:

- The modular forms $\pm b(q)$ (where $b(q) := q^{\frac{1}{24}}\eta^3(\tau)/\eta^2(2\tau)$), together with the 0 function, appear to"cut out" the exponential singularities of $f(q)$.
- That is, as q approaches any even order $2k$ root of unity singularity of $f(q)$, then

$$\boxed{f(q) - (-1)^k b(q) \stackrel{?}{=} O(1)}$$

- That is, asymptotically, towards singularities,

$$\boxed{\text{mock theta} \pm \text{modular form} \stackrel{?}{=} \text{bounded}}$$

Recall, we attribute the following to Ramanujan:

Definition (Ramanujan)**.** A **mock theta function** F of the complex variable q, defined by an Eulerian (q-hypergeometric) series which converges for $|q| < 1$, satisfies

(1) infinitely many roots of unity are exponential singularities,
(2) for every root of unity ζ there is a modular form $\vartheta_\zeta(q)$ such that the difference $F(q) - q^c\vartheta_\zeta(q)$ is bounded as $q \to \zeta$ radially,
(3) there does not exist a single modular form $\vartheta(q)$ such that $F(q) - q^c\vartheta(q)$ is bounded as q approaches any root of unity radially.

"...[no one has] proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition." -B.C. Berndt, 2013

Proof (F-Ono-Rhoades Radial Limit Theorem, sketch).

Definition. The ${}_2\psi_2$ *bilateral q-hypergeometric series* is

$$ {}_2\psi_2\left(\begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix}\; q,\; z\right) := \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n(a_2;q)_n}{(b_1;q)_n(b_2;q)_n} z^n,$$

$(a;q)_{-m} := \prod_{j=1}^{m}(1-aq^{-j})^{-1}, \quad m \in \mathbb{N}.$

We have that

$$ {}_2\psi_2\left(\begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix} q, z\right) = \frac{(\frac{b_2 q}{a_1 a_2 z};q)_\infty(\frac{b_1}{a_2};q)_\infty(a_1 z;q)_\infty(\frac{b_2}{a_1};q)_\infty}{(\frac{q}{a_2};q)_\infty(\frac{b_1 b_2}{a_1 a_2 z};q)_\infty(b_2;q)_\infty(z;q)_\infty} \cdot {}_2\psi_2\left(\begin{matrix} \frac{a_1 a_2 z}{b_2} & a_1 \\ a_1 z & b_1 \end{matrix}\; q,\; \frac{b_2}{a_1}\right)$$

A (limiting Bailey) ${}_2\psi_2$ transformation eventually leads to:

Proposition (Ramanujan, reinterpreted by Y.S. Choi)**.** *Let $q = e^{2\pi i\tau}$, $\alpha = e^{2\pi iu}$, $\beta = e^{2\pi iv}$. We have that*

$$\sum_{n=0}^{\infty} \frac{(\alpha\beta)^n q^{n^2}}{(\alpha q;q)_n(\beta q;q)_n} + \sum_{n=1}^{\infty} q^n(\alpha^{-1};q)_n(\beta^{-1};q)_n =$$
$$iq^{\frac{1}{8}}(1-\alpha)(\beta\alpha^{-1})^{\frac{1}{2}}\boldsymbol{(q\alpha^{-1};q)_\infty\,(\beta^{-1};q)_\infty\,\mu(u,v;\tau)}.$$

The ${}_2\psi_2$ identity implies

$$R\left(\zeta_b^a; q\right) = m(a,b;\tau)\mu\left(-\frac{a}{b}, \frac{a}{b}; \tau\right) - (1-\zeta_b^a)(1-\zeta_b^{-a})U(\zeta_b^a; q)$$

where $m(a,b;\tau) := iq^{\frac{1}{8}}(1-\zeta_b^{-a})\zeta_b^a(\zeta_b^a q; q)_\infty(\zeta_b^{-a}; q)_\infty$.
Using the modular or mock modular transformation properties of m, the Appell-Lerch series μ, and the crank C:

Proposition (F-Ono-Rhoades)**.** *For $z \in \mathbb{R}^+$, as $z \to 0^+$, we have that*

$$m\Big(a, b; \frac{1}{k}(h+iz)\Big)\mu\Big(-\frac{a}{b}, \frac{a}{b}; \frac{1}{k}(h+iz)\Big) \\ = \left(\frac{i}{z}\right)^{\frac{1}{2}} (\psi(\gamma))^{-1} q^{\frac{1}{24}} q_1^{-\frac{1}{24}} (-1)^{ab'} \zeta_{2b}^{ah'-a} \boldsymbol{\zeta_{2b^2}^{-3a^2kh'}} \frac{\zeta_b^a - 1}{1-\zeta_b^{ah'}}(1+O(q_1^\alpha)).$$

Proposition (F-Ono-Rhoades)**.** *For $z \in \mathbb{R}^+$, as $z \to 0^+$, we have that*

$$C\Big(\zeta_b^a; \frac{1}{k}(h+iz)\Big) \\ = \left(\frac{i}{z}\right)^{\frac{1}{2}} (\psi(\gamma))^{-1} q^{\frac{1}{24}} q_1^{-\frac{1}{24}} (-1)^{ab'} \zeta_{2b}^{ah'-a} \boldsymbol{\zeta_{2b^2}^{-a^2kh'}} \frac{\zeta_b^a - 1}{1-\zeta_b^{ah'}}(1+O(q_1^\beta)).$$

□

In the above propositions, $q = e^{(2\pi i/k)(h+iz)}$, $q_1 = e^{(2\pi i/k)(h'+i/z)}$, and $hh' \equiv -1 \pmod{k}$.

• Griffin-Ono-Rolen later generalized this (less explicitly) and show that Ramanujan's mock theta functions, and mock modular forms, satisfy his definition. In particular,

Theorem (G-O-R)**.** *Suppose that $F = F^+ + F^-$ is a HMF of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_1(N)$, where F^- (resp. F^+) is the NHP (resp. HP) of F. If F^- is nonzero and g is any weight k weakly hol. modular form on any $\Gamma_1(N')$, then $F^+ - g$ has exponential singularities as q approaches infinitely many roots of unity ζ.*

Note. HMFs here have principal parts at all cusps.

• Rhoades shows that the modern definition of a mock theta function (roughly: the hol. part of a HMF of weight 1/2 with modular theta function shadow) is not equivalent to Ramanujan's, by constructing two q-series such that either one of them satisfies the modern def. (but not historic) or the other satisfies the historic def. (but not modern).

• However, by the above (Zwegers, FOR, GOR) Ramanujan's mock theta functions satisfy both definitions.

Hardy-Ramanujan-Rademacher-type coefficient formulas:

1952: Dragonette's thesis (under Rademacher) proves Ramanujan's (claimed) asymptotic for the coefficients $\alpha_f(n)$ $(= p_e(n) - p_o(n))$:

Claim (Ramanujan). As $n \to \infty$

$$\alpha_f(n) = (-1)^{n+1} \frac{\sqrt{6}\exp\left(\frac{\pi}{12}\sqrt{24n-1}\right)}{\sqrt{24n-1}} + O\left(\frac{\exp\left(\frac{\pi}{24}\sqrt{24n-1}\right)}{\sqrt{24n-1}}\right).$$

1964: Andrews' thesis (under Rademacher) improves this: for all $\varepsilon > 0$,

$$\alpha_f(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \frac{(-1)^{\left\lfloor\frac{k+1}{2}\right\rfloor} A_{2k}\left(n - \frac{k\left(1+(-1)^k\right)}{4}\right)}{k} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) + O\left(n^{\varepsilon}\right).$$

Here, for $k, n \in \mathbb{N}$, the *Kloosterman sum* $A_k(n)$ is

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{d \pmod{24k} \\ d^2 \equiv -24n+1 \pmod{24k}}} \left(\frac{12}{d}\right) \exp\left(\frac{\pi i d}{6k}\right),$$

and is in terms of the I-Bessel function. ($J_\alpha(x)$ satisfies $x^2\frac{d^2}{dx^2}J + x\frac{d}{dx}J + (x^2 - \alpha^2)J = 0$; and $I_\alpha(x) = i^{-\alpha}J_\alpha(ix)$. Explicitly, $I_\alpha(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+\alpha}}{m!\Gamma(m+\alpha+1)}$; principal branch (from PV of $(x/2)^\alpha$) is analytic in $\mathbb{C} \setminus (-\infty, 0]$. As $k \to \infty$, fixed n, the above IBF $\searrow 0$.)

Recall also the earlier, similar, celebrated Hardy-Ramanujan-Rademacher exact formula

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

Conjecture (Andrews and Dragonette)**.** *If $n \in \mathbb{N}$, then*

$$\alpha_f(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k=1}^{\infty} \frac{(-1)^{\left\lfloor\frac{k+1}{2}\right\rfloor} A_{2k}\left(n - \frac{k\left(1+(-1)^k\right)}{4}\right)}{k} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

Theorem (Bringmann-Ono)**.** *The Andrews-Dragonette Conjecture is true.*

Proof (Andrews-Dragonette Conjecture, sketch). By the above work of Zwegers, $q^{-\frac{1}{24}}f(q)$ is a (vector-valued) mock modular form (hol. part of a v-v HMF). On the other hand, there is a Maass Poincaré series which transforms in exactly the same way and has the same principal parts at cusps, hence their difference is a holomorphic form (of weight 1/2), which is shown to identically equal 0. The exact formula for the coefficients $\alpha_f(n)$ then comes from the Maass-Poincaré series. That is, let

$$\varphi_{s,k}(\tau) := \left(\frac{6}{\pi y}\right)^{\frac{k}{2}} M_{-\frac{k}{2}, s-\frac{1}{2}}\left(\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right),$$

where $M = M$-Whittaker function. (Note. $M_{\kappa,\mu}(x)$ solves $\frac{d^2}{dx^2}M + (-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4}-\mu^2}{x^2})M = 0$.)
For matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ with $c \in \mathbb{N}_0$,

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{cases} e\left(-\frac{b}{24}\right) & \text{if } c = 0, \\ i^{-\frac{1}{2}}(-1)^{\frac{1}{2}(c+ad+1)} e\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right)\omega_{-d,c}^{-1} & \text{if } c > 0. \end{cases}$$

Here $\omega_{a,b} := e\left(\frac{1}{2}s(a,b)\right)$, where the Dedekind sum

$$s(a,b) := \sum_{\mu \pmod{b}} \left(\left(\frac{\mu}{b}\right)\right)\left(\left(\frac{a\mu}{b}\right)\right),$$

with $((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$

Define the Maass Poincaré series $P_k(s;\tau)$ by

$$P_k(s;\tau) := \frac{2}{\sqrt{\pi}} \sum_{M \in \Gamma_\infty \backslash \Gamma_0(2)} \chi(M)^{-1}(c\tau + d)^{-k}\varphi_{s,k}(M\tau).$$

One can show: $P_k(1 - k/2;\tau)$ is absolutely convergent for $k < 1/2$ and annihilated by Δ_k.

The function $P_{\frac{1}{2}}(s;\tau)$ can be analytically continued by its Fourier expansion to $s = 3/4$. (This requires an interpretation of $A_{2k}(n)$ as sums of quadratic forms of fixed discriminant, and extends an argument of Hooley (in B-O's 2006 paper) on the equidistribution of CM points.)

One checks that $P_{\frac{1}{2}}\left(\frac{3}{4};\tau\right)$ has the same principal parts at cusps and the same modular transformation properties as $q^{-1/24}f(q)$. The proof continues as outlined. $\square$