BUILDING BRIDGES: 6TH EU/US SUMMER SCHOOL & WORKSHOP ON AUTOMORPHIC FORMS AND RELATED TOPICS (BB6) CIRM MARSEILLE, SEPTEMBER 2-13, 2024

EXERCISES: MOCK MODULAR FORMS AND QUANTUM MODULAR FORMS WITH APPLICATIONS

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These problems[3] were selected to complement the lectures and give a wide variety of choice. Please note that there are more problems here than can be completed during the exercise sessions. You are not expected to attempt every problem or finish the problems during the available time, but we hope that there are problems here that you find engaging. In general it is not necessary to work problems in the order presented, although it will be clear by context that some problems are sequential. Each problem is labeled (with $\star, \star\star$) to indicate difficulty level.

1. Modular, Quasimodular, and Mock Modular Forms

Problem 1. ($\star$) (MDG Exercise 4.3.9) Show that $E_4^2 = E_8$. Deduce that for all $n \geq 0$,

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).$$

Problem 2. ($\star\star$) (MDG Exercise 4.4.5) Prove that $\Delta = \frac{E_4^3 - E_6^2}{1728}$ has integral Fourier coefficients.

Problem 3. ($\star\star$) (MDG Exercise 5.4.4) Prove that

$$(E_4^3 - E_6^2)^2 \equiv (E_4E_2 - 1)^2 \equiv -q\frac{dE_2^2}{dq} - 3q\frac{dE_4}{dq} \pmod{7}.$$

[Hint: First show that $E_4^2 \equiv E_2 \pmod{7}$ and $E_6 \equiv 1 \pmod{7}$.]

Problem 4. ($\star\star$) (MDG Exercise 5.4.5) Prove that

$$(E_4^3 - E_6^2)^2 \equiv q^2 \left(\sum_{n=0}^{\infty} p(n)q^n\right) \prod_{n=1}^{\infty}(1 - q^{49n}) \pmod{7},$$

where $p(n)$ is the partition counting function. Deduce that for $n \geq 0$,

$$p(7n+5) \equiv 0 \pmod{7}.$$

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[3]Although these exercises have been lightly edited, some typos may be present.

Problem 5. ($\star\star$) (MDG Exercise 5.5.3) Let $\Delta(\tau)=\sum_{n\geq 1} t(n)q^n$. Prove that

$$t(n)\equiv \sigma_{11}(n) \pmod{691}.$$

[Hint: First show that $E_6^2 = E_{12} - \frac{762048}{691}\Delta$ and $E_{12} = 1+\frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n$.]

Problem 6. ($\star$) Define $\widetilde{E}_k = c_k E_k$ such that $\widetilde{E}_k = c_k + \sum_{n\geq 1}\sigma_{k-1}(n)q^n$. Let a, b be odd integers with $a>b$. Show that the quasimodular form

$$f = (D^a+1)\widetilde{E}_{b+1} - (D^b+1)\widetilde{E}_{a+1}$$

is prime detecting.

Problem 7. ($\star$) Let S be a set of positive integers, and $p_S(n)$ the number of partitions of n with parts in S.

(a) Show that for $|q|<1$,

$$P_S(q) := \sum_{n\geq 0} p_S(n)q^n = \prod_{n\in S}\frac{1}{1-q^n}.$$

(b) Find sets S for which $P_S(q)$ is modular (when $q=e^{2\pi i\tau}$).

Certain problems below (e.g. Problem 8, Problem 10) require some of the theory of q-hypergeometric series [6, 8]. Namely, with $(a_1,a_2,\ldots,a_r;q)_n := \prod_{j=1}^{r}(a_j;q)_n$, the q-hypergeometric series are defined by

$${}_r\phi_s\left(\begin{matrix} a_1, & a_1, & \ldots & a_r \\ b_1, & b_2, & \ldots & b_s\end{matrix}\; q; z\right) := \sum_{n\geq 0}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s,q;q)_n}\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{1+s-r} z^n$$

where $r,s\in\mathbb{N}_0$, $|z|<1, |q|<1, b_j\neq q^{-m}$ for any $m\in\mathbb{N}_0$. The celebrated Watson-Whipple transformation is given by

$$\begin{aligned}
&{}_8\phi_7\left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & g \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/g\end{matrix}\; ; q; \frac{a^2q^2}{bcdeg}\right) \\
(1) \quad &= \frac{(aq;q)_\infty (aq/dg;q)_\infty (aq/eg;q)_\infty (aq/de;q)_\infty}{(aq/d;q)_\infty (aq/e;q)_\infty (aq/g;q)_\infty (aq/deg;q)_\infty}\,{}_4\phi_3\left(\begin{matrix} aq/bc, & d, & e, & g \\ deg/a, & aq/b, & aq/c\end{matrix}\; ; q; q\right).
\end{aligned}$$

Under suitable changes of variables and limiting cases, the Watson-Whipple q-hypergeometric transformation formula leads to the following identity

$$\begin{aligned}
&\sum_{n\geq 0}\frac{(\alpha,\beta,\gamma,\delta,\epsilon;q)_n(1-\alpha q^{2n})q^{n(n+3)/2}}{(\alpha q/\beta,\alpha q/\gamma,\alpha q/\delta,\alpha q/\epsilon,q;q)_n(1-\alpha)}\left(-\frac{\alpha^2}{\beta\gamma\delta\epsilon}\right)^n \\
(2) \quad &= \frac{(\alpha q,\alpha q/(\delta\epsilon);q)_\infty}{(\alpha q/\delta,\alpha q/\epsilon;q)_\infty}\sum_{n\geq 0}\frac{(\delta,\epsilon,\alpha q/(\beta\gamma);q)_n}{(\alpha q/\beta,\alpha q/\gamma,q;q)_n}\left(\frac{\alpha q}{\delta\epsilon}\right)^n.
\end{aligned}$$

Problem 8. ($\star\star$) Prove that

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2}}{1-wq^n} = \frac{(q;q)_\infty^2}{(w;q)_\infty (q/w;q)_\infty},$$

and deduce modular properties of the associated Appell-lerch sums, i.e., in terms of

$$\mu\left(z_1, z_2; \tau\right) := \zeta_1^{\frac{1}{2}} \vartheta^{-1}\left(z_2; \tau\right) \sum_{n\in\mathbb{Z}} \frac{(-1)^n \zeta_2^n q^{\frac{n(n+1)}{2}}}{1-\zeta_1 q^n}, \quad (\zeta_j := e^{2\pi i z_j}).$$

[Hint: Use (limiting) Watson-Whipple q-hypergeometric transformations, i.e. use (2).]

Problem 9. ($\star\star$) Recall that the rank of a partition is defined to be its largest part minus the number of its parts, and $N(n,m) := \#\{\text{partitions of } n \text{ rank } m\}$.

(a) Show that

$$\sum_{n\geq 0} \sum_{m\in\mathbb{Z}} N(n,m) w^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(wq;q)_n (q/w;q)_n}.$$

(Recall that the third order mock theta function $f(q)$ satisfies $f(q) = R(-1;q)$.)

(b) Determine a combinatorial interpretation for the coefficients of the third order mock theta function

$$\omega(q) := \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}.$$

Check your interpretation by computing partitions of a few small integers.

[Hint: First prove (see [6]) that $q\omega(q) = \sum_{n\geq 1} \frac{q^n}{(q;q^2)_n}$.]

Problem 10. ($\star\star$) Prove the mock theta identities (used in Watson's transformation)

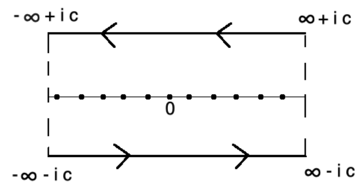
$$f(q) = \frac{2}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n},$$

$$\omega(q) = \frac{1}{(q^2;q^2)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{3n(n+1)}}{1-q^{2n+1}}.$$

[Hint: Use (limiting) Watson-Whipple q-hypergeometric transformations, i.e. use (1), and that the expressions above for f and ω may be re-written as
$\frac{1}{(q;q)_\infty}\left(1+4\sum_{n\geq 1} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}\right)$ and $\frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n q^{3n(n+1)} \frac{1+q^{2n+1}}{1-q^{2n+1}}$, respectively.]

Problem 11. ($\star\star$) Recall that Watson's contour involves integrating the function

$$C_\alpha(z) := \frac{\pi}{\sin(\pi z)} \frac{\exp(-3\alpha z^2/2)}{\cosh(\alpha z/2)}$$



(where $\Re(\alpha) > 0$) around a contour as depicted above (where $c > 0$ is small enough so that the zeros of $\sin(\pi z)$ (which occur at $z \in \mathbb{Z}$) are the only poles of the integrand between the lines forming the contour). Prove that the integral of $C_\alpha(z)$ (as a function of z) over the vertical edges of the contour yields a contribution of 0.

Problem 12. ($\star\star$) Let $a \in (-\frac{1}{2}, \frac{1}{2}), b \in \mathbb{R}, \tau \in \mathbb{H}$, and write $\tau = x + iy, z = u + iv$. Prove that Zwegers' non-holomorphic

$$\mathcal{R}(z;\tau) := \sum_{n \in \frac{1}{2}+\mathbb{Z}} \left(\operatorname{sgn}(n) - E\left(\left(n + \frac{v}{y} \right) \sqrt{2y} \right) \right) (-1)^{n-\frac{1}{2}} \zeta^{-n} q^{-\frac{n^2}{2}}$$

(where $E(z) := 2\int_0^z e^{-\pi t^2} dt$) satisfies

$$-e^{-\pi i a^2 \tau + 2\pi i a (b+\frac{1}{2})} \mathcal{R}(a\tau - b; \tau) = \int_{-\overline{\tau}}^{i\infty} \frac{g_{a+\frac{1}{2}, b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}} dz,$$

where $g_{a,b}(\tau) := \sum_{\nu \in a + \mathbb{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b}$ $(a, b \in \mathbb{R})$ is a modular theta function.

2. Harmonic Maass Forms and Quantum Modular Forms

Problem 1. ($\star\star$) Let $0 < a < c$ be integers. Consider the harmonic Maass form

$$D(a,c;\tau) := q^{4f_c^2 \frac{a}{c}(1-\frac{a}{c})} H(a,c;4f_c^2\tau) + V(a,c;2f_c^2\tau),$$

where $f_c := 2c/\gcd(2c,4)$, and

$$V(a,c;\tau) := -\frac{1}{2}\int_{-\overline{\tau}}^{i\infty} \frac{(-iz)^{-3/2} T(a,c;-1/2z)}{\sqrt{-i(z+\tau)}} dz,$$

$$T(a,c;\tau) := i \sum_{n \in \mathbb{Z}} (n + 1/4) \cosh(2\pi i (n+1/4)(2a/c - 1)) e^{2\pi i \tau \left(n + \frac{1}{4}\right)^2},$$

$$H(a;c;\tau) := \sum_{n \geq 0} \frac{q^{n(n+1)/2}(-q;q)_n}{(q^{a/c};q)_{n+1}(q^{1-a/c};q)_{n+1}}.$$

(a) Prove that $D(a,c;\tau)$'s Fourier expansion is that of a harmonic Maass form, i.e., of the shape

$$M(\tau) = \sum_{n \gg -\infty} c_M^+(n) q^n + \sum_{n<0} c_M^-(n) \Gamma(1-k, -4\pi n y) q^n.$$

(b) Prove that $D(a,c;\tau)$ is annihilated by the weight $1/2$ Laplacian operator

$$\Delta_{\frac{1}{2}} := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{iy}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

[Hint: First (show, and) use the factorization $\Delta_{1/2} = -4y^{3/2}\partial_\tau\sqrt{y}\partial_{\overline{\tau}}$.]

Problem 2. ($\star$) Recall that

$$\xi_{2-k}\colon H_{2-k}(\Gamma_0(N)) \twoheadrightarrow S_k(\Gamma_0(N))$$

where

$$\xi_k := 2iy^k\overline{\frac{\partial}{\partial\overline{\tau}}}.$$

Verify that for $M \in H_{2-k}(\Gamma_0(N))$, $(\xi_{2-k}M)(\tau)$ transforms like a weight k modular form on $\Gamma_0(N)$.

Problem 3. ($\star$) Let $e(z) := e^{2\pi iz}$. For $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, define the Mordell integral

$$h(z;\tau) := \int_{\mathbb{R}} \frac{e(\tau x^2/2)e^{-2\pi zx}}{\cosh(\pi x)}dx.$$

Prove that

$$h(z;\tau)+e(-z)q^{-1/2}h(z+\tau;\tau) = 2e(-z/2)q^{-1/8}.$$

[Hint: write $h(z+\tau;\tau)$ in terms of an integral $\int_{\mathbb{R}+i}$. Use the Residue Theorem.]

Problem 4. ($\star\star$) Prove that $h(z;\tau)+h(z+1;\tau) = \dfrac{2}{\sqrt{-i\tau}}e^{\pi i(z+1/2)^2/\tau}$.

[Hint: The integral $2\int_{\mathbb{R}} e^{\pi i\tau x^2-2\pi x(z+\frac{1}{2})}dx$ can be explicitly evaluated as $2(-i\tau)^{-1/2}e^{\pi i(z+\frac{1}{2})^2/\tau}$.]

Problem 5. ($\star\star$) Prove that $z \mapsto h(z;\tau)$ is the unique holomorphic function satisfying the properties from the previous two problems.

[Hint: Liouville's theorem.]

Problem 6. ($\star$) Prove that Ramanujan's mock theta function $f(q)$ satisfifes $f(\zeta) = O(1)$ at odd ordered roots of unity ζ.

Problem 7. ($\star$) Recall that a strongly unimodal sequence $\{a_j\}_{j=1}^s$ of size n is such that $0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots > a_s > 0$ and $\sum_{j=1}^s a_j = n$, and its rank is $s - 2r + 1$ (the difference between the number of terms after and before the peak of the sequence). We let $u(m,n) := \#\{\text{strongly unimodal sequences of size } n \text{ and rank } m\}$. Show that

$$U(w;q) := \sum_{n\geq 1}\sum_{m\in\mathbb{Z}} u(m,n)w^m q^n = \sum_{n\geq 0}(-wq;q)_n(-w^{-1}q;q)_n q^{n+1}.$$

Problem 8. ($\star$) Verify, as in the F-O-R radial limit theorem, that $U(\zeta_b^a;\zeta_k^h)$ is a convergent, finite, sum, evaluated at pairs of roots of unity (ζ_b^a, ζ_k^h) with $b \mid k$.

For the next problem, we consider other of Ramanujan's mock theta functions

$$\phi(q) := \sum_{n\geq 0}\frac{q^{n^2}}{(-q^2;q^2)_n}, \quad \psi(q) := \sum_{n\geq 1}\frac{q^{n^2}}{(q;q^2)_n},$$

along with the (nearly) modular product

$$c(q) := \frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3(q^4;q^4)_\infty^3}.$$

The next exercises establish the following proposition studied by R. C. Rhoades.

Proposition. *As $q \to \zeta$ radially from within the unit disk, where ζ is a primitive $4k$th root of unity, we have that*

$$\lim_{q\to\zeta}(\phi(q) - c(q)) = -2\sum_{n\geq 0}\zeta^{n+1}(-\zeta^2;\zeta^2)_n = -\psi(\zeta).$$

Moreover, as $q \to \rho$ radially from within the unit disk, where ρ is a primitive odd order root of unity, we have that

$$\lim_{q\to\rho}(\psi(q) - c(q)/2) = -\frac{1}{2}\left(1 + \sum_{n\geq 0}(-1)^n\rho^{2n+1}\left(\rho;\rho^2\right)_n\right).$$

Problem 9. ($\star\star$)

(a) Prove that $\psi(q) = \sum_{n=0}^{\infty} q^{n+1}(-q^2;q^2)_n$.

[Hint: use a q-hypergeometric identity from [6].]

(b) Prove that $\phi(q) + 2\psi(q) = c(q)$.

[Hint: Apply Ramanujan's ${}_1\psi_1$ summation
$${}_1\psi_1(\alpha,\beta;q;z) := \sum_{n\in\mathbb{Z}}\frac{(\alpha;q)_n}{(\beta;q)_n}z^n = \frac{(\beta/\alpha, \alpha z, q/(\alpha z), q;q)_\infty}{(q/\alpha, \beta/(\alpha z), \beta, z;q)_\infty} \ (|\beta/\alpha| < |z| < 1).]$$

(c) Prove for any primitive $4k$th root of unity ζ, we have

$$\lim_{q\to\zeta}\left(\phi(q)-c(q)\right)=-2\sum_{n\geq 0}\zeta^{n+1}(-\zeta^2;\zeta^2)_n.$$

(d) Prove that $\phi(q)=1+\sum_{n\geq 0}(-1)^n q^{2n+1}(q;q^2)_n$.

[Hint: use a q-hypergeometric identity from [6].]

(e) Prove for any primitive odd order root of unity ρ, we have

$$\lim_{q\to\rho}(\psi(q)-c(q)/2)=-\frac{1}{2}\left(1+\sum_{n\geq 0}(-1)^n\rho^{2n+1}(\rho;\rho^2)_n\right).$$

Problem 10. ($\star\star$) Let $f=\sum_{n\geq 1}a(n)q^n$ be a weight $k\in\frac{1}{2}+\mathbb{N}_0$ cusp form with respect to the congruence subgroup $\Gamma_0(4N)$, so that for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(4N)$ and $\tau\in\mathbb{H}$,

$$f(\gamma\tau)=\left(\frac{c}{d}\right)\varepsilon_d^{-2k}(c\tau+d)^k f(\tau),$$

where $\left(\frac{c}{d}\right)$ is the Kronecker symbol and $\varepsilon_d=\begin{cases}1 & \text{if } d\equiv 1 \pmod 4\\ i & \text{if } d\equiv 3 \pmod 4\end{cases}$.

Prove that the non-homomorphic Eichler integral

$$f^*(\overline{\tau})=\frac{(-2\pi i)^{k-1}}{\Gamma(k-1)}\int_{\tau}^{i\infty}f(w)(w-\overline{\tau})^{k-2}dw,$$

satisfies for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(4N)$ and $\tau\in\mathbb{H}$,

$$f^*(\overline{\tau})-\left(\frac{-4}{d}\right)\left(\frac{c}{d}\right)\varepsilon_d^{-2k}(c\overline{\tau}+d)^{k-2}f^*\left(\frac{a\overline{\tau}+b}{c\overline{\tau}+d}\right)=\frac{(-2\pi i)^{k-1}}{\Gamma(k-1)}\int_{-\frac{d}{c}}^{i\infty}f(w)(w-\overline{\tau})^{k-2}dw.$$

(Since the asymptotic expansions of f^* and the homomorphic Eichler integral of f, $\widetilde{f}=\sum_{n\geq 1}n^{1-k}a(n)q^n$ agree at any $\frac{d}{c}\in\mathbb{Q}$, this gives the quantum modularity properties of $\widetilde{f}$ when approaching the real line.)

Problem 11. ($\star$) Verify that

$$J_N(T_{(2,3)};\zeta_N^{-1})=\zeta_N^{-1}F(\zeta_N^{-1}),$$
$$J_N(T^*_{(2,3)};\zeta_N)=\zeta_N^{-1}U(-1;\zeta_N).$$

Problem 12. ($\star\star$)

(a) Show that $F_2(q)$ (defined by Hikami and Lovejoy) has the form

$$F_2(q)=\sum_{n\geq 0}a_n(q)(q;q)_n,$$

where $a_n(q) \in \mathbb{Z}[q]$, and thus is an element of the Habiro ring.

(b) Determine the first couple of coefficients c_n such that

$$F_2(q) = \sum_{n \geq 0} c_n (1-q)^n.$$

References

[1] K. Bringmann, A. Folsom, K. Ono, L. Rolen, *Harmonic Maass Forms and Mock Modular Forms: Theory and Applications,* AMS Colloquium Publications, 64. American Mathematical Society, Providence, RI, 2018. 391pp.

[2] K. Bringmann, K. Ono, and R.C. Rhoades, *Eulerian series as modular forms,* J. Amer. Math. Soc. 21 (2008), no. 4, 1085-1104.

[3] K. Bringmann and L. Rolen, *Half-integral weight Eichler integrals and quantum modular forms,* J. Number Theory (161), 240–254.

[4] J.H. Bruinier and K. Ono *Heegner divisors, L-functions and harmonic weak Maass forms,* Ann. of Math. (2) 172 (2010), no. 3, 2135-2181.

[5] J.H. Bruinier and K. Ono, *Identities and congruences for Ramanujan's $\omega(q)$,* Ramanujan J. 23 (2010), no. 1-3, 151-157.

[6] N. J. Fine, *Basic hypergeometric series and applications*. Mathematical Surveys and Monographs, 27. American Mathematical Society, Providence, RI, 1988.

[7] F. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod 5,7 and 11,* Trans. Amer. Math. Soc. 305 (1988), no. 1, 47-77.

[8] G. Gasper and M. Rahman, *Basic Hypergeometric Series,* Second Edition, Encyclopedia of Mathematics and Its Applications 96, Cambridge University Press, Cambridge, 2004.

[9] K. Hikami and J. Lovejoy, *Torus knots and quantum modular forms*, Res. Math. Sci. (2) iss. 2 (2015). 16pp.

[10] M. Jackson, *On Lerch's transcendant and the basic bilateral hypergeometric series ${}_2\psi_2$,* J. London Math. Soc. 25, (1950). 189-196.

[11] S.Y. Kang, *Mock Jacobi forms in basic hypergeometric series,* Compos. Math. 145 (2009), no. 3, 553-565.

[12] M.R. Murty, M. Dewar, and H. Graves, *Problems in the theory of modular forms*, Institute of Mathematical Sciences Lecture Notes (1), Hindustan, New Delhi, 2015. 291pp.

[13] G. N. Watson, *The final problem: An account of the mock theta functions*, J. London Math. Soc. **2** (2) (1936), 55-80.

[14] S.P. Zwegers, *Mock theta functions,* Ph.D. thesis, Utrecht, 2002.