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Hook length bias in odd versus distinct partitions

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Abstract. Motivated in part by hook-content formulas for certain restricted partitions in representation theory, we consider the total number of hooks of fixed length in odd versus distinct partitions. We show that there are more hooks of length 2, respectively 3, in all odd partitions of n than in all distinct partitions of n, and make the analogous conjecture for arbitrary hook length $t \geq 2$. To this end, we establish very general linear inequalities for the number of distinct partitions, which is also of independent interest. We also establish additional related partition bias results.

Keywords: hook length, distinct partitions, odd partitions

1 Introduction

Background. Connections between representation theory and the theory of integer partitions are well-known. For example, the irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ may be indexed by partitions of length at most n; moreover, the conjugacy classes of the symmetric group S_n, and therefore the number of non-equivalent irreducible complex representations, may be indexed by the partitions of n. *Hook lengths* of partitions play particularly important roles in establishing these connections. Namely, such irreducible representations can be analyzed via partition *Young tableaux*. The dimension of a representation of S_n (respectively $\mathrm{GL}_n(\mathbb{C})$) corresponding to a particular

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partition is given by a *hook length formula* (respectively a *hook-content formula*). (For more on these topics, see e.g. [17].)

One of many celebrated results in this subject is the Nekrasov-Okounkov formula for arbitrary powers of Euler's infinite product in terms of hook numbers, as well as Han's extension [9], which unifies the Macdonald identities in representation theory and *t-core partition* generating functions. *Restricted* partitions also play important roles in this context. For example, Han and Xiong have established hook-content formulas for distinct partitions in [10, 11]. See also [12, 16] for more in this direction. Here, we further study hook lengths associated to distinct partitions versus odd partitions. Before stating our results, we briefly define some terminology (and refer the reader to [1] for more background on partitions).

Terminology. Recall that a *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_j)$ of *size* $n \in \mathbb{N}_0$ is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j$ called *parts* that add up to n. By convention, the empty partition is the only partition of size 0. We write $|\lambda|$ for the size of λ. We denote by $p(n)$ the number of partitions of n. We define the multiplicity $m_\lambda(i)$ of a part i of a partition λ to be the number of times i appears as a part of λ. We define the length $\ell(\lambda)$ of a partition λ to be the number of parts of λ, and define $\ell_1(\lambda)$ (resp. $\ell_2(\lambda)$) to be the number of parts λ_i of λ with $\lambda_i - \lambda_{i+1} = 1$ (resp. $\lambda_i - \lambda_{i+1} = 2$). We assume $\lambda_k = 0$ if $k > \ell(\lambda)$.

Each partition is naturally equipped with a *Young diagram*, a left-justified vertical array of boxes with rows corresponding to parts (see Figure 1). We abuse notation, and refer to partitions as their Young diagrams, with their parts referred to as rows. The *conjugate* of a partition λ is the partition λ' whose Young diagram has the columns of λ as rows. Each box in a Young diagram of λ may be labeled with a *hook number*, also called *hook length*, which, informally, is the number of boxes in the upside-down-L-shaped portion of the diagram with the box appearing as its corner. More precisely, for a box in the i-th row and j-th column of the Young diagram of a partition λ, its hook length is defined as $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ (see Figure 1).

8	7	4	3	1
6	5	2	1	
3	2			
2	1			

Figure 1: Young diagram with hook lengths for the partition $\lambda = (5, 4, 2, 2)$.

In what follows, we refer to a partition into odd parts as an odd partition and to a partition into distinct parts as a distinct partition. We denote by $\mathcal{O}(n)$, respectively $\mathcal{D}(n)$, the set of odd, respectively distinct, partitions of n. Euler's identity [1, Corollary 1.2] states that $|\mathcal{O}(n)| = |\mathcal{D}(n)|$ for all $n \geq 0$.

Results. Let $a_t(n)$ (respectively $b_t(n)$) be the total number of hooks of length t in all odd (respectively distinct) partitions of n.

For a partition λ, a box in its Young diagram has hook length 1 if and only if it is at the end of a row and there is no box directly below it. Thus, in a partition λ, the number of hooks of length 1 equals the number of different part sizes in λ.

The next result was conjectured by Beck [13] and proved analytically by Andrews [2].

Theorem 1 (Theorem 2, [2]). *The difference between the total number of parts in all distinct partitions of n and the total number of different part sizes in all odd partitions of n equals $c(n)$, the number of partitions of n with exactly one part occurring three times while all other parts occur only once.*

Corollary 2. *For $n \geq 0$, $b_1(n) - a_1(n) = c(n)$.*

Thus, there are at least as many hooks of length 1 in all distinct partitions of n as there are in all odd partitions of n. On the other hand, Euler's identity yields

$$\sum_{t\geq 1} a_t(n) = \sum_{t\geq 1} b_t(n).$$

It is natural to study the relationship between $a_t(n)$ and $b_t(n)$ for any fixed $t \geq 1$. For $t = 1$ the above shows that $b_1(n) \geq a_1(n)$. We conjecture that this bias reverses for $t \geq 2$: eventually (for large enough n), we conjecture that there are at least as many hooks of length t in all odd partitions of n as there are in all distinct partitions of n. More specifically, we make the following conjecture.

Conjecture 3. *For every integer $t \geq 2$, there exists an integer N_t such that for all $n > N_t$, we have $a_t(n) \geq b_t(n)$, and $a_t(n) - b_t(n) \to \infty$ as $n \to \infty$. Moreover, for $4 \leq t \leq 10$, we conjecture the following values of N_t:*

t	2	3	4	5	6	7	8	9	10
N_t	0	7	8	18	16	34	34	56	59

Figure 2: Conjectural values for N_t.

Data supporting this conjecture was obtained by enumerating partitions and not from generating functions; this is because the generating functions for $a_t(n)$ and $b_t(n)$ are difficult to derive explicitly. We are, however, able to write down generating functions for $t = 2$ and $t = 3$, which we use to prove Conjecture 3 for $t = 2$ and $t = 3$.

Theorem 4. *Conjecture 3 is true for $t = 2$ and $t = 3$ with $N_2 = 0$ and $N_3 = 7$, respectively.*

Analogous to Corollary 2, for $t = 2$, we prove that $a_2(n) - b_2(n) = c_2(n)$, where $c_2(n)$ is the number of overpartitions λ of n into odd parts with $m_\lambda(1) \equiv 0, 3 \pmod 4$, and exactly one part greater than 1 overlined. An overpartition of n is a partition of n in which the first occurrence of a part may be overlined [5].

We give complete details for the case $t = 2$ of Theorem 4. The case $t = 3$ is more complicated, and requires tools from combinatorics as well as analytic number theory. For space considerations, we broadly sketch the main idea used in the proof for $t = 3$, and will publish the complete proof in a forthcoming work [3].

2 Hooks of length 2

Let $a_2(m,n)$ (respectively $b_2(m,n)$) be the number of odd (respectively distinct) partitions of n with m hooks of length 2. To prove Theorem 4 for $t = 2$, we first establish the generating functions for the sequences $a_2(n)$ and $b_2(n)$ by finding the bivariate generating functions $\sum_{m,n\geq 0} a_2(m,n)z^m q^n$ and $\sum_{m,n\geq 0} b_2(m,n)z^m q^n$, differentiating with respect to z, and evaluating at $z = 1$. We follow with the analysis of $a_2(n) - b_2(n)$.

Odd partitions. Let λ be an odd partition. The number of hooks of length 2 in λ is equal to the number of different part sizes of λ that are greater than 1 plus the number of different parts of λ that occur at least twice. Thus,

$$F_2(z;q) := \sum_{n,m\geq 0} a_2(m,n)z^m q^n = \left(1 + q + \frac{zq^2}{1-q}\right) \prod_{n=2}^{\infty} \left(1 + zq^{2n-1} + \frac{z^2 q^{2(2n-1)}}{1-q^{2n-1}}\right),$$

and hence

$$\sum_{n\geq 0} a_2(n)q^n = \frac{\partial}{\partial z}\bigg|_{z=1} F_2(z;q) = \frac{1}{(q;q^2)_\infty} \left(q^2 + \sum_{n\geq 2} (q^{2n-1} + q^{2(2n-1)})\right).$$

Here and throughout, the *q-Pochhammer symbol* is defined for $n \in \mathbb{N}_0 \cup \{\infty\}$ by

$$(a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j).$$

Distinct partitions. If λ is a distinct partition, the number of hooks of length 2 in λ equals the number of parts λ_i such that $\lambda_i - \lambda_{i+1} \geq 2$. Thus, the number of hooks of length 2 in λ can be calculated as follows: starting with the Young diagram of λ, remove a staircase of length $\ell(\lambda)$ (i.e. subtract 1 from the smallest part, 2 from the second to last part, etc.) to obtain a partition μ. Then, we count the number of different part sizes in μ. We note that the number of different part sizes in μ is equal to the number of different part sizes in its conjugate μ'. Let $u_n(t,m)$ be the number of partitions of m with t different part sizes and all parts at most n. Then

$$\sum_{m,t\geq 0} u_n(t,m)z^t q^m = \prod_{j=1}^{n} \left(1 + \frac{zq^j}{1-q^j}\right),$$

and

$$G_2(z;q) := \sum_{n,m\geq 0} b_2(m,n) z^m q^n = \sum_{n\geq 1} q^{n(n+1)/2} \prod_{j=1}^{n} \left(1 + \frac{zq^j}{1-q^j}\right).$$

Then, differentiating with respect to z and evaluating at $z=1$, we obtain

$$\sum_{n\geq 0} b_2(n) q^n = \frac{\partial}{\partial z}\Big|_{z=1} G_2(z;q) = \sum_{n\geq 1} q^{n(n+1)/2} \frac{1}{(q;q)_n} \frac{q-q^{n+1}}{1-q}.$$

Using the well-known limiting case of the q-binomial theorem (see e.g. [1, (2.2.6)]) $\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} z^n/(q;q)_n = (-zq;q)_\infty$ with $z=q$, this can be re-written as:

$$\sum_{n\geq 0} b_2(n) q^n = \frac{q}{1-q} \sum_{n\geq 1} \frac{q^{\frac{n^2+n}{2}}}{(q;q)_{n-1}} = \frac{q^2}{1-q} \sum_{n\geq 0} \frac{q^{\frac{n^2+3n}{2}}}{(q;q)_n} = \frac{q^2}{1-q}(-q^2;q)_\infty.$$

Hook bias for $\mathbf{t=2}$. We consider the generating function for $a_2(n) - b_2(n)$. We have

$$\begin{aligned}
\sum_{n\geq 0} (a_2(n) - b_2(n)) q^n &= \frac{1}{(q;q^2)_\infty} \left(q^2 + \sum_{n\geq 2} (q^{2n-1} + q^{2(2n-1)})\right) - \frac{q^2}{1-q}(-q^2;q)_\infty \\
&= (-q;q)_\infty \frac{q^2}{1-q^2} \left(\frac{1+q+q^3}{1+q^2} - 1\right) = (-q;q)_\infty \frac{q^2}{1-q^2} \left(\frac{q-q^2+q^3}{1+q^2}\right) \\
&= q^3 \frac{1+q^3}{1-q^2} (-q^3;q)_\infty = \frac{q^3}{1-q^2}((-q^3;q)_\infty + q^{1+2}(-q^3;q)_\infty). \qquad (2.1)
\end{aligned}$$

In the second line, we used Euler's identity [1, (1.2.5)] $1/(q;q^2)_\infty = (-q;q)_\infty$. Clearly, the expression in (2.1), expanded as a q-series, has non-negative coefficients which can be interpreted combinatorially as follows. The expression $(-q^3;q)_\infty$ is the generating function for the number of distinct partitions of n that do not have 1 and 2 as parts. Glaisher's bijection [7], which splits each even part in a distinct partition into equal odd parts, maps these partitions bijectively to odd partitions λ of n with $m_\lambda(1) \equiv 0 \pmod 4$. Similarly, $q^{1+2}(-q^3;q)_\infty$ is the generating function for the number of distinct partitions of n that have both 1 and 2 as parts. Glaisher's bijection maps these partitions bijectively to odd partitions λ of n with $m_\lambda(1) \equiv 3 \pmod 4$. Moreover, $q^3/(1-q^2) = q^3 + q^5 + \cdots$ is the generating function of the number of partitions of n consisting of a single odd part greater than 1, which we overline. Thus, the right hand side of (2.1) is the generating function for the number of overpartitions of n into odd parts with $m_\lambda(1) \equiv 0, 3 \pmod 4$ and exactly one part overlined, which is greater than 1.

3 Hooks of length 3

In this section, we briefly sketch the main ideas of the proof of Theorem 4 for $t=3$. The complete proof is very involved and not suited for an extended abstract. It will be published in our forthcoming paper [3].

As in Section 2, we define $a_3(m,n)$ (respectively $b_3(m,n)$) to be the number of odd (respectively distinct) partitions of n with m hooks of length 3, and use the same strategy to find the generating functions for the sequences $a_3(n)$ and $b_3(n)$.

Odd partitions. Let λ be an odd partition. Among parts equal to 1, there is a hook of length 3 only in the third to last part equal to 1 (thus, only if the multiplicity of 1 is at least three). For all other parts, there is a hook of length 3 in the second to last and third to last occurrence of the part size (if the multiplicity of the part permits). There is also a hook of length 3 in the last occurrence of the part size if that last occurrence is in row i with $\lambda_i - \lambda_{i+1} \neq 2$. Therefore $a_3(m,n)$ is the number of odd partitions λ of n such that

$$\delta_{m_\lambda(1)\geq 3} + \left(\sum_{u\geq 2} (\delta_{m_\lambda(u)\geq 1} + \delta_{m_\lambda(u)\geq 2} + \delta_{m_\lambda(u)\geq 3}) \right) - \ell_2(\lambda) = m.$$

Here, δ_ρ denotes the Kronecker delta symbol, which evaluates to 1 if property ρ is true, and 0 if not. If $x(k,n)$ is the number of odd partitions λ of n such that

$$\delta_{m_\lambda(1)\geq 3} + \sum_{u\geq 2} (\delta_{m_\lambda(u)\geq 1} + \delta_{m_\lambda(u)\geq 2} + \delta_{m_\lambda(u)\geq 3}) = k,$$

and $y(k,n)$ is the number of odd partitions λ of n with $\ell_2(\lambda) = k$, then

$$a_3(n) = \sum_{m\geq 0} m\, a_3(m,n) = \sum_{k\geq 0} k\, x(k,n) - \sum_{k\geq 0} k\, y(k,n).$$

Let $\mathcal{F}_1(z;q) := \sum_{n,k\geq 0} x(k,n) z^k q^n$ and $\mathcal{F}_2(z;q) := \sum_{n,k\geq 0} y(k,n) z^k q^n$. Then,

$$\sum_{n\geq 0} a_3(n) q^n = \frac{\partial}{\partial z}\Big|_{z=1} \mathcal{F}_1(z;q) - \frac{\partial}{\partial z}\Big|_{z=1} \mathcal{F}_2(z;q).$$

We have

$$\mathcal{F}_1(z;q) = \left(1 + q + q^2 + \frac{zq^3}{1-q}\right) \prod_{n\geq 1} \left(1 + zq^{2n+1} + z^2 q^{2(2n+1)} + \frac{z^3 q^{3(2n+1)}}{1-q^{2n+1}}\right).$$

To find $\mathcal{F}_2(z;q)$, we consider the conjugate of the 2-modular diagram of a partition. In the conjugate diagram, the boxes in the top row are filled with 1 and the boxes in all other rows are filled with 2. We need to count rows with multiplicity 1 among the rows filled with 2 that are less than the first row, which we record in the exponent of z. Thus,

$$\mathcal{F}_2(z;q) = \sum_{n\geq 1} \frac{q^n}{1-q^{2n}} \prod_{j=1}^{n-1} \left(1 + zq^{2j} + \frac{q^{2(2j)}}{1-q^{2j}}\right).$$

Furthermore,

$$\frac{\partial}{\partial z}\Big|_{z=1}\mathcal{F}_1(z;q) = \frac{1}{(q;q^2)_\infty}\left(q^3 + \sum_{n\geq 1}(q^{2n+1} + q^{2(2n+1)} + q^{3(2n+1)})\right)$$
$$= (-q;q)_\infty\left(q^3 + \frac{q^3}{1-q^2} + \frac{q^6}{1-q^4} + \frac{q^9}{1-q^6}\right)$$

and

$$\frac{\partial}{\partial z}\Big|_{z=1}\mathcal{F}_2(z;q) = \sum_{n\geq 1} q^n \frac{1}{(q^2;q^2)_n}\sum_{j=1}^{n-1} q^{2j}(1-q^{2j}).$$

Using the well-known identity $\sum_{n\geq 0} z^n/(q;q)_n = 1/(z;q)_\infty$ (see e.g. [1, (2.2.5)]) and further simplifications similar to those in the calculation of $\frac{\partial}{\partial z}|_{z=1}G_2(z;q)$, we obtain

$$\frac{\partial}{\partial z}\Big|_{z=1}\mathcal{F}_2(z;q) = \frac{1}{(q;q^2)_\infty}\frac{q^4}{1-q^4}.$$

Finally, we have

$$\sum_{n\geq 0} a_3(n)q^n = \frac{\partial}{\partial z}\Big|_{z=1}\mathcal{F}_1(z;q) - \frac{\partial}{\partial z}\Big|_{z=1}\mathcal{F}_2(z;q)$$
$$= (-q;q)_\infty\left(q^3 + \frac{q^3}{1-q^2} + \frac{q^6}{1-q^4} + \frac{q^9}{1-q^6} - \frac{q^4}{1-q^4}\right)$$
$$= (-q^3;q)_\infty\frac{q^3(1+q^3)}{1-q^2} + (-q;q)_\infty\left(\frac{q^6}{1-q^4} + \frac{q^3}{1-q^6}\right). \tag{3.1}$$

Distinct partitions. Let λ be a distinct partition. There is a hook of length 3 in every row $\lambda_i > 1$ except when $\lambda_i - \lambda_{i+1} = 2$. Therefore $b_3(m,n)$ is the number of distinct partitions of n such that $\ell(\lambda) - m_\lambda(1) - \ell_2(\lambda) = m$. If $u(k,n)$ is the number of distinct partitions of n with exactly k parts greater than 1, and $v(k,n)$ is the number of distinct partitions λ of n with $\ell_2(\lambda) = k$, then

$$b_3(n) = \sum_{m\geq 0} m\, b_3(m,n) = \sum_{k\geq 0} k\, u(k,n) - \sum_{k\geq 0} k\, v(k,n).$$

Let $\mathcal{G}_1(z;q) := \sum_{n,k\geq 0} u(k,n)z^k q^n$ and $\mathcal{G}_2(z;q) := \sum_{n,k\geq 0} v(k,n)z^k q^n$. Then,

$$\sum_{n\geq 0} b_3(n)q^n = \frac{\partial}{\partial z}\Big|_{z=1}\mathcal{G}_1(z;q) - \frac{\partial}{\partial z}\Big|_{z=1}\mathcal{G}_2(z;q).$$

We have

$$\mathcal{G}_1(z;q) = (1+q)(-zq^2;q)_\infty.$$

To find $\mathcal{G}_2(z;q)$, let λ be a partition with distinct parts and, as in the case $t=2$, remove a staircase of length $\ell(\lambda)$ from λ to obtain a partition μ with at most $\ell(\lambda)$ parts. Recall that $\ell_1(\mu)$ is the number of parts μ_i of μ such that $\mu_i - \mu_{i+1} = 1$. Note that parts μ_i in μ such that $\mu_i - \mu_{i+1} = 1$ correspond to parts λ_i in λ such that $\lambda_i - \lambda_{i+1} = 2$. Thus, $\ell_1(\mu) = \ell_2(\lambda)$. For a partition ν, denote by $\widetilde{\ell}(\nu)$ the number of parts of ν with multiplicity one. Since $\ell_1(\mu) = \widetilde{\ell}(\mu')$, we have

$$\mathcal{G}_2(z;q) = \sum_{m\geq 1} q^{\frac{m(m+1)}{2}} \prod_{j=1}^{m} \left(1 + zq^j + \frac{q^{2j}}{1-q^j}\right).$$

We compute

$$\frac{\partial}{\partial z}\Big|_{z=1} \mathcal{G}_1(z;q) = (-q;q)_\infty \sum_{m\geq 2} \frac{q^m}{1+q^m}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z}\Big|_{z=1} \mathcal{G}_2(z;q) &= \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} \sum_{j=1}^{n} q^j(1-q^j) = \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} \left(\frac{1-q^{n+1}}{1-q} - \frac{1-q^{2(n+1)}}{1-q^2}\right) \\
&= \frac{q}{1-q^2} \sum_{n\geq 1} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_{n-1}} (1-q^{n+1}) = \frac{q}{1-q^2} \left(\sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}+n+1}}{(q;q)_n} - \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}+2n+3}}{(q;q)_n}\right) \\
&= \frac{q}{1-q^2}\left(q(-q^2;q)_\infty - q^3(-q^3;q)_\infty\right) = \frac{q^2}{1-q^2}(-q^3;q)_\infty.
\end{aligned}$$

For the second to last equality we have again used the q-binomial theorem.

Finally, we have

$$\begin{aligned}
\sum_{n\geq 0} b_3(n)q^n &= \frac{\partial}{\partial z}\Big|_{z=1} \mathcal{G}_1(z;q) - \frac{\partial}{\partial z}\Big|_{z=1} \mathcal{G}_2(z;q) \\
&= (-q;q)_\infty \sum_{m\geq 2} \frac{q^m}{1+q^m} - \frac{q^2}{1-q^2}(-q^3;q)_\infty. \qquad (3.2)
\end{aligned}$$

Hook bias for t = 3. Next we consider the difference $a_3(n) - b_3(n)$. Establishing non-negativity for $n > 7$ is more complex than for $a_2(n) - b_2(n)$ in Section 2, so we provide a sketch of the proof, leaving the full details to our forthcoming paper [3].

Sketch of proof of Theorem 4 for t = 3. We begin by subtracting the generating functions established in (3.1) and (3.2). After a lengthy series of manipulations, calculations, and simplifications, we prove that, to establish the desired non-negativity result, it is sufficient to establish certain linear inequalities for the (restricted) partition numbers

$$q(n) := p(n \mid \text{distinct parts}) \quad \text{and} \quad \rho(n,m) := p(n \mid \text{distinct parts at least } m).$$

An intermediate ingredient we use is the following result of Erdös-Nicolas-Szalay relating $q(n)$ and $\rho(n,m)$.

Theorem 5 (Theorem 1, [6]). *For all n and m satisfying $1 \leq m \leq n$, we have that*

$$\frac{q(n)}{2^{m-1}} \leq \rho(n,m) \leq \frac{q(n+m(m-1)/2)}{2^{m-1}}.$$

To complete our proof of Theorem 4 for $t=3$, we prove the following general theorem on linear inequalities for $q(n)$, which is also of independent interest.

Theorem 6 (Theorem 1.11, [3]). *Suppose $\sum_{k=1}^{r} \alpha_k < \sum_{\ell=1}^{s} \beta_\ell$, where $\{\alpha_k\}_{k=1}^{r}, \{\beta_\ell\}_{\ell=1}^{s} \subset \mathbb{N}$, and $r,s \in \mathbb{N}$. Moreover, let $\{\mu_k\}_{k=1}^{r}, \{\nu_\ell\}_{\ell=1}^{s} \subset \mathbb{N}$, where $\mu_k \neq \mu_j$ and $\nu_j \neq \nu_k$ for $j \neq k$. Then for sufficiently large n, we have that*

$$\sum_{k=1}^{r} \alpha_k q(n+\mu_k) \leq \sum_{\ell=1}^{s} \beta_\ell q(n+\nu_\ell). \tag{3.3}$$

Remark 7. *In [3], we make Theorem 6 effective, and establish a lower bound on n (depending on the sequences given in the hypothesis of the theorem) sufficient to guarantee* (3.3).

Sketch of Proof of Theorem 6. We first show that it suffices to establish

Proposition 8 (Proposition 2.1, [3]). *For $L \in \mathbb{N}$, and $\varepsilon > 0$, there exists an $N_{\varepsilon,L}$ such that for $n > N_{\varepsilon,L}$*

$$0 < \frac{q(n+L)}{q(n)} - 1 < \varepsilon.$$

Continuing our sketch of proof of Theorem 6, we recall the following beautiful exact formula for $q(n)$ as an infinite sum of Kloosterman sums multiplied by Bessel functions due to Hagis [8], reminiscent of the celebrated exact formula for $p(n)$ by Hardy-Ramanujan-Rademacher [15] obtained using the Circle Method.

Theorem 9 (Theorem 4, [8]). *For $n \in \mathbb{N}$, we have that*

$$q(n) = \frac{\pi}{(24n+1)^{1/2}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} k^{-1} \left(\sum_{h \pmod{k}}{}' \chi(h,k) \exp(-2\pi i n h/k) \right) I_1\left(\frac{\pi}{12k}(48n+2)^{1/2} \right).$$

Above, I_1 is the Bessel function of the first order, χ is an explicit exponential function, and the sum on h is taken over $h \pmod{k}$ relatively prime to k. Using Theorem 9, Beckwith-Bessenrodt establish the following asymptotic estimate on $q(n)$, an important ingredient in our proof of Theorem 6.

Theorem 10 (Theorem 2.3, [4])**.** *Let $\mu := \frac{\pi}{6\sqrt{2}}\sqrt{24n+1}$. Then for $n \in \mathbb{N}$, we have that*

$$q(n) = \frac{\pi}{\sqrt{24n+1}} I_1(\mu) + E(n),$$

where

$$|E(n)| \leq \frac{0.9\pi}{\sqrt{24n+1}} \cdot \frac{e^{\mu}}{\mu}\left(1 + 5\mu^2 e^{-\mu}\right).$$

Given the above, a last key ingredient in our proof of Theorem 6 is certain effective estimates on the I_1-Bessel function.

Proposition 11 (Exercise 7.13.2, [14])**.** *Let z be a complex number with $|\arg(z)| \leq \frac{\pi}{2}$ and let $s \in \mathbb{C}$. Then we have*

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 + \delta(z)\right] + i\frac{e^{-z}}{\sqrt{2\pi z}}\left[1 + \gamma(z)\right],$$

where we have the bounds

$$|\gamma(z)| \leq \frac{3}{4|z|}\exp\left(\frac{3}{4|z|}\right), \quad |\delta(z)| \leq \frac{3\pi}{8|z|}\exp\left(\frac{3\pi}{8|z|}\right).$$

Then, a lengthy calculation using Theorem 10 and Proposition 11 proves Proposition 8. We refer the interested reader to [3] for full details of the proof of Theorem 4 for $t = 3$ and Theorem 6 using the ingredients given here.

4 Further bias results

Recall that $\ell_2(\lambda)$ is the number of parts λ_i of λ with $\lambda_i - \lambda_{i+1} = 2$. Note that this partition statistic appeared in our calculations of both $a_3(n)$ and $b_3(n)$. Thus, it is natural to investigate a possible bias in the total number of gaps of size exactly 2 in odd versus distinct partitions. We prove that such a bias exists.

Theorem 12. *For $n \in \mathbb{N}_0$, we have that*

$$\sum_{\lambda \in \mathcal{O}(n)} \ell_2(\lambda) - \sum_{\lambda \in \mathcal{D}(n)} \ell_2(\lambda)$$

is non-negative except for $n = 2$ and $n = 6$ in which cases it equals -1.

Proof. From Section 3,

$$\sum_{\lambda \in \mathcal{D}(n)} \ell_2(\lambda) = (-q^3;q)_\infty \frac{q^2}{1-q^2} \quad \text{and} \quad \sum_{\lambda \in \mathcal{O}(n)} \ell_2(\lambda) = (-q;q)_\infty \frac{q^4}{1-q^4}.$$

Thus, we need to show that, the coefficients of q^n, $n \neq 2, 6$, in

$$H(q) := (-q;q)_\infty \frac{q^4}{1-q^4} - (-q^3;q)_\infty \frac{q^2}{1-q^2}$$

are non-negative. A direct calculation shows that the coefficients of q^2 and q^6 in $H(q)$ are both equal to -1.

After some q-series manipulations, we have that

$$\begin{aligned} H(q) + q^2 + q^6 &= (-q^3;q)_\infty \frac{q^5}{1-q^2} - q^5(-q^4;q)_\infty - q^2(1+q^4) \sum_{m\geq 5} q^m (-q^{m+1};q)_\infty \\ &= (-q^4;q)_\infty \frac{q^7}{1-q} - q^2(1+q^4) \sum_{m\geq 5} q^m (-q^{m+1};q)_\infty \\ &= (-q^4;q)_\infty \sum_{k\geq 7} q^k - (1+q^4) \sum_{k\geq 7} q^k (-q^{k-1};q)_\infty \\ &= (1+q^4) \sum_{k\geq 7} q^k \left((-q^5;q)_\infty - (-q^{k-1};q)_\infty \right). \end{aligned}$$

Moreover, for $k \geq 7$, we have

$$(-q^5;q)_\infty - (-q^{k-1};q)_\infty = \sum_{j\geq 5} q^j (-q^{j+1};q)_\infty - \sum_{j\geq k-1} q^j (-q^{j+1};q)_\infty = \sum_{j=5}^{k-2} q^j (-q^{j+1};q)_\infty.$$

Thus,

$$H(q) + q^2 + q^6 = (1+q^4) \sum_{k\geq 7} q^k \sum_{j=5}^{k-2} q^j (-q^{j+1};q)_\infty, \tag{4.1}$$

which clearly has non-negative coefficients and completes the proof. □

We interpret (4.1) as the generating function for $|\mathcal{H}(n)|$, where $\mathcal{H}(n)$ is the set of partitions λ of n satisfying all of the following conditions:

(i) $m_\lambda(3) \geq 2$, and if $m_\lambda(3) = 2$, then $\lambda_{\ell(\lambda)} < 3$,

(ii) 3 is the only repeated part, and 1 and 2 cannot both occur as parts in λ,

(iii) λ has part greater than 4 and the smallest part s greater than 4 satisfies $5 \leq s \leq \left(\sum_{\lambda_i \leq 3} \lambda_i\right) - 2$.

To see this, write $k \geq 7$ as $k = 3d + r$. $0 \leq r \leq 2$ and interpret q^k as generating d parts equal to 3 and one part equal to r. Moreover, $\sum_{j=5}^{k-2} q^j (-q^{j+1};q)_\infty$ generates non-empty distinct partitions with smallest part between 5 and $k-2$ (inclusive). Then, in analogy to Theorem 1, for $n \neq 2, 6$,

$$\sum_{\lambda \in \mathcal{O}(n)} \ell_2(\lambda) - \sum_{\lambda \in \mathcal{D}(n)} \ell_2(\lambda) = |\mathcal{H}(n)|.$$

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