

Graded dimensions of principal subspaces and modular Andrews–Gordon-type series

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Our results in this paper are threefold: First, we establish the modular properties of the graded dimensions of principal subspaces of level one standard modules for $A_{N-1}^{(1)}$, and of principal subspaces of certain higher level standard modules for $A_{2n}^{(1)}$. Second, we establish the modular properties of families of q -series that appear in identities due to Warnaar and Zudilin, which generalize Macdonald's $A_{2n}^{(2)}$ identities and the Rogers–Ramanujan identities. Third, we formulate a number of conjectures regarding the modularity of series of this type related to A_{N-1} root systems.

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1. Introduction and Statement of Results

Representations of infinite-dimensional Lie algebras have been used to interpret and prove many combinatorial identities. It is well known that interesting identities can be obtained from new interpretations of the characters of standard modules

for affine Lie algebras (see [9, 16–19]). Among these are the well-studied Rogers–Ramanujan identities:

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = \prod_{n \geq 1} \frac{1}{(1 - q^{5n-1})(1 - q^{5n-4})}, \tag{1.1}$$

$$\sum_{n \geq 0} \frac{q^{n^2+n}}{(q; q)_n} = \prod_{n \geq 1} \frac{1}{(1 - q^{5n-2})(1 - q^{5n-3})}, \tag{1.2}$$

where $(a; q)_n = (a)_n := \prod_{j=0}^{n-1} (1 - aq^j)$. On the one hand, these identities encode combinatorial information. For example, by considering relevant generating functions, the first identity (1.1) states that the number of partitions of a non-negative integer n into parts with difference at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 (mod 5). On the other hand, the infinite product expansions given on the right-hand sides together with Jacobi’s triple product identity make it clear that (1.1) and (1.2) are (up to minor adjustment) modular forms when $q = e^{2\pi i\tau}$. The work of Lepowsky and Milne [16] shows that, after multiplication by a certain factor, the product sides of (1.1) and (1.2) (among other identities) are principally specialized characters of irreducible representations of $A_1^{(1)}$. In subsequent work, Lepowsky and Wilson [17, 19] obtained a vertex-operator-theoretic proof of the Rogers–Ramanujan identities by introducing and exploiting classes of twisted vertex operators associated with standard modules for affine Lie algebras. Another astonishing relation among modular functions, Lie theory, finite group theory and vertex operator algebra theory is given by the proof of the McKay–Thompson–Conway–Norton conjectures relating the Monster group to modular functions including $j(\tau)$, by Frenkel, Lepowsky and Meurman [10, 11] and Borcherds [2].

The more general question of when a q -hypergeometric series (which loosely speaking are functions similar in shape to the left-hand side of (1.1) and (1.2)) is modular, continues to be an actively researched area. For example, recent work of the first and third author [3] investigated these questions in relation to the affine Lie superalgebras $sl(m|n)^\wedge$, and recent work of Zagier [30] and Vlasenko–Zwegers [28] examined the modular properties of a general family of q -hypergeometric series as conjectured by Nahm [24].

In this paper we discuss modularity of graded dimensions of certain subspaces, called principal subspaces, of the standard modules for $A_{N-1}^{(1)}$, the untwisted affine Lie algebra associated to A_{N-1} . The principal subspaces were introduced by Feigin and Stoyanovsky [7, 8]. By studying these subspaces in the $A_1^{(1)}$ case, they interpreted the Rogers–Ramanujan and Gordon identities [13]. Explicit formulas of graded dimensions of principal subspaces of level one standard modules for the untwisted affine Lie algebras of types A , D and E have been computed in [6] by the second author with Lepowsky and Milas, by using intertwining operators in vertex operator algebra theory. Such formulas have been studied in the literature,

and have also appeared in the setting of the thermodynamic Bethe Ansatz (see discussion in [6] and references therein).

Our first result establishes the modular properties of various families of graded dimensions of principal subspaces of level k standard modules for $A_{N-1}^{(1)}$. Namely, we prove the modularity of families of graded dimensions in the case $k = 1$ and arbitrary $N \geq 2$ obtained in [6, 12], and when N is odd and $k \geq 1$ as studied in [7, 12, 27]. As is true with the examples discussed above, the families of graded dimensions studied here are of q -hypergeometric type. For example, the second author with Lepowsky and Milas [6] obtained the following expression:

$$\chi_{W(\Lambda_0)}(x_1, \dots, x_{N-1}; q) = \sum_{\mathbf{m}=(m_1, \dots, m_{N-1}) \in \mathbb{N}_0^{N-1}} \frac{q^{\frac{\mathbf{m} \mathbf{C} \mathbf{m}^T}{2}}}{(q)_{m_1} \cdots (q)_{m_{N-1}}} x_1^{m_1} \cdots x_{N-1}^{m_{N-1}}. \tag{1.3}$$

Here, x_1, \dots, x_{N-1} and q are commuting formal variables. For a detailed discussion of these graded dimensions, see Sec. 2.1. Our first theorem below pertains to the graded dimensions (1.3), as well as a number of other families in both cases, N even and N odd.

Theorem 1.1. *In the case of A_{N-1} , the graded dimensions ($q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$)*

$$q^{\varepsilon(\chi)} \cdot \chi(\mathbf{x}(\chi); q)$$

are modular functions on $G(\chi)$ with multiplier $\rho(\chi)$, where $\chi, \mathbf{x}(\chi), \varepsilon(\chi), G(\chi), \rho(\chi)$ and N are as given in (5.1) and (5.2).

Remark 1. As specified in (5.1) and (5.2) pertaining to Theorem 1.1, we let $N = 2n + 1$ or $N = 2n$ with $n \in \mathbb{N}$. We use a similar convention throughout.

Next we turn to the study of related generalized Macdonald-type identities. Macdonald [20] famously proved an affine Weyl denominator formula, which led to many beautiful identities. To describe this, throughout, for a vector $\mathbf{x} := (x_1, x_2, \dots, x_n)$, we let $|\mathbf{x}| := \sum_{j=1}^n x_j$, and $\|\mathbf{x}\| := \sum_{j=1}^n x_j^2$. A special case of Macdonald’s $A_{2n}^{(2)}$ identity [20] gives the following identity for powers of the Dedekind η -function, which is a modular form of weight $1/2$:

$$\eta(\tau)^{2n^2-n} = \sum_{\nu \equiv \rho \pmod{2n+1}} \prod_{1 \leq i < j \leq n} \frac{\nu_i^2 - \nu_j^2}{\rho_i^2 - \rho_j^2} (-1)^{|\nu| - |\rho|} q^{\frac{\|\nu\|^2}{2(2n+1)}}. \tag{1.4}$$

The vector ρ is given in Sec. 5.1. Macdonald’s identity (1.4) reduces to Euler’s pentagonal number theorem in the case $n = 1$, which, when interpreted combinatorially, leads to a beautiful recursion among integer partitions [1]. Euler’s pentagonal number theorem was first embedded into a natural family of specializations of Macdonald’s identities in [15]. More recently, Warnaar and Zudilin [29] established a number of q -series identities that generalize Macdonald’s $A_{2n}^{(2)}$ identities including (1.4), as well as the Rogers–Ramanujan identities (1.1) and (1.2). Our second

theorem establishes the modular properties of various families of these generalized Macdonald series. For more on these series, including (1.4) above, see Sec. 5.3 and Remark 8 therein.

Theorem 1.2. *The generalized Macdonald series ($q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$)*

$$\frac{1}{\eta(\tau)^{\epsilon(n)}} \sum_{\nu \in S_\alpha} \alpha(\nu) \cdot q^{\beta(\nu)}$$

are modular functions on $G(\alpha)$ with multiplier $\rho(\alpha)$, where $\alpha(\nu)$, $\epsilon(n)$, $G(\alpha)$, S_α , and $\beta(\nu)$ are as given in (5.4) and (5.5).

The remainder of the paper is structured as follows. In Sec. 2, we provide background information and establish preliminary results pertaining to representations of Lie algebras and modular forms. In Sec. 3, we prove Theorems 1.1 and 1.2. In Sec. 4, we formulate a number of conjectures (see Conjecture 4.1) regarding the modularity of generalized Rogers–Ramanujan, Macdonald, and Andrews–Gordon series as studied originally by Warnaar and Zudilin [29] in relation to A_{N-1} root systems. The final section of this paper, Sec. 5, provides tables of functions to which Theorem 1.1, Theorem 1.2, and Conjecture 4.1 apply.

2. Preliminaries

In this section, we provide background information and establish preliminary results pertaining to representations of Lie algebras and modular forms.

2.1. Graded dimensions and representations of Lie algebras

Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra of type A of rank $N - 1$, $N \geq 2$, and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let $\{\alpha_1, \dots, \alpha_{N-1}\} \subset \mathfrak{h}^*$ be a set of simple roots of \mathfrak{g} . Denote by Δ the set of roots of \mathfrak{g} and by Δ_+ the set of positive roots. For each root α we fix a root vector x_α . Let $\langle \cdot, \cdot \rangle$ be the rescaled Killing form such that $\langle \alpha, \alpha \rangle = 2$ for $\alpha \in \Delta$. Denote by $\lambda_1, \dots, \lambda_{N-1} \in \mathfrak{h} \simeq \mathfrak{h}^*$ the corresponding fundamental weights of \mathfrak{g} (i.e. $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $i, j = 1, \dots, N - 1$).

Consider the untwisted affine Lie algebra

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where c is a central element and

$$[x \otimes t^m, y \otimes t^n] := [x, y] \otimes t^{m+n} + m \langle x, y \rangle \delta_{m+n,0} c$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Here, $[\cdot, \cdot]$ is the usual Lie bracket. By adjoining a certain degree operator d , also called a derivation, to the Lie algebra $\hat{\mathfrak{g}}$ one obtains the affine Kac–Moody algebra (see [14])

$$\tilde{\mathfrak{g}} := \hat{\mathfrak{g}} \oplus \mathbb{C}d.$$

Denote by $\Lambda_0, \Lambda_1, \dots, \Lambda_{N-1} \in (\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$ the fundamental weights of $\tilde{\mathfrak{g}}$.

Let $\mathfrak{n} \subset \mathfrak{g}$ be the Lie algebra spanned by the root vectors for the positive roots

$$\mathfrak{n} := \prod_{\alpha \in \Delta_+} \mathbb{C}x_\alpha,$$

and denote by $\bar{\mathfrak{n}}$ the subalgebra

$$\bar{\mathfrak{n}} := \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$$

of $\hat{\mathfrak{g}}$. Denote by $L(\Lambda)$ a level k' standard $\hat{\mathfrak{g}}$ -module (also called an integrable highest weight module) with $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \dots + k_{N-1}\Lambda_{N-1}$, where k_0, \dots, k_{N-1} are non-negative integers whose sum is k' . The principal subspace $W(\Lambda)$ of $L(\Lambda)$ was defined by Feigin and Stoyanovsky [7, 8] as follows:

$$W(\Lambda) := U(\bar{\mathfrak{n}}) \cdot v_\Lambda, \tag{2.1}$$

where v_Λ is a highest weight vector of $L(\Lambda)$, and $U(\bar{\mathfrak{n}})$ is the universal enveloping algebra of $\bar{\mathfrak{n}}$. Denote by

$$(m_i^{(a)}) = (m_i^{(a)})_{\substack{1 \leq a \leq N-1 \\ 1 \leq i \leq k'}} \in \text{Mat}_{k' \times (N-1)}(\mathbb{N}_0) \tag{2.2}$$

the $k' \times (N - 1)$ matrices, where $\text{Mat}_{r \times s}(R)$ denotes the set of $r \times s$ matrices over R . Using these matrices, we define

$$M_i^{(a)} = M_{i,k',N}^{(a)} := \begin{cases} m_i^{(a)} + \dots + m_{k'}^{(a)} & \text{if } 1 \leq i \leq k' - 1, \\ m_{k'}^{(a)} & \text{if } i = k'. \end{cases} \tag{2.3}$$

Consider the Lie algebra $\mathfrak{g} = A_{N-1}$ and the level one standard $\hat{\mathfrak{g}}$ -modules $L(\Lambda_0), \dots, L(\Lambda_{N-1})$. Denote by $\chi_{W(\Lambda_i)}(x_1, \dots, x_{N-1}; q)$ the graded dimension (the generating function of the dimensions of the homogeneous subspaces) of the principal subspace $W(\Lambda_i) \subset L(\Lambda_i)$ with respect to certain compatible “weight” and “charge” gradings (see [6]). By using vertex operator algebra theory and Lie algebras techniques, the second author with Lepowsky and Milas obtained in [6] (as given in (1.3) above) explicit formulas for the graded dimensions of $W(\Lambda_i)$ for $i = 0, \dots, N - 1$:

$$\begin{aligned} & \chi_{W(\Lambda_0)}(x_1, \dots, x_{N-1}; q) \\ &= \sum_{(m_1^{(a)}) \in \text{Mat}_{1 \times (N-1)}(\mathbb{N}_0)} \frac{q^{\frac{m_1^{(a)} C m_1^{(a)T}}{2}}}{(q)_{m_1^{(1)}} \cdots (q)_{m_1^{(N-1)}}} x_1^{m_1^{(1)}} \cdots x_{N-1}^{m_1^{(N-1)}}, \end{aligned} \tag{2.4}$$

and for $i = 1, \dots, N - 1$,

$$\begin{aligned} & \chi_{W(\Lambda_i)}(x_1, \dots, x_{N-1}; q) \\ &= p_{W(\Lambda_i)}(q) \sum_{(m_1^{(a)}) \in \text{Mat}_{1 \times (N-1)}(\mathbb{N}_0)} \frac{q^{\frac{m_1^{(a)} C m_1^{(a)T} + 2m_1^{(i)}}{2}}}{(q)_{m_1^{(1)}} \cdots (q)_{m_1^{(N-1)}}} x_1^{m_1^{(1)}} \cdots x_{N-1}^{m_1^{(N-1)}}, \end{aligned} \tag{2.5}$$

where

$$p_{W(\Lambda_i)}(q) = p_{W(\Lambda_i)}(x_1, \dots, x_{N-1}; q) := x_1^{\langle \lambda_1, \lambda_i \rangle} \dots x_{N-1}^{\langle \lambda_{N-1}, \lambda_i \rangle} q^{\frac{1}{2} \langle \lambda_i, \lambda_i \rangle}.$$

Moreover

$$C = C_N := (C_{ab})_{1 \leq a, b \leq N-1}$$

is the Cartan matrix of the Lie algebra A_{N-1} . (For the purposes of this paper, as in Theorems 1.1 and 1.2, we typically take $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$.) In order to avoid the multiplicative factors $p_{W(\Lambda_i)}(q)$, we use the following slightly modified graded dimensions:

$$\chi'_{W(\Lambda_i)}(x_1, \dots, x_{N-1}; q) := (p_{W(\Lambda_i)}(q))^{-1} \chi_{W(\Lambda_i)}(x_1, \dots, x_{N-1}; q) \quad (2.6)$$

for $i = 1, \dots, N - 1$. These formulas were also obtained by Georgiev in [12] by a different method. Similar formulas for graded dimensions of the principal subspaces of the level one standard $\hat{\mathfrak{g}}$ -modules, with \mathfrak{g} of types E and D are given in [6].

Now let $N = 2n + 1$ for integers $n \geq 1$. Consider the Lie algebra $\mathfrak{g} = A_{2n}$ and the principal subspace $W(k' \Lambda_0)$ of the level k' standard $\hat{\mathfrak{g}}$ -module $L(k' \Lambda_0)$, $k' \geq 1$. It was shown in [12] (see also [7, 27]) that the graded dimension of the principal subspace $W(k' \Lambda_0)$ is given by

$$\begin{aligned} & \chi_{W(k' \Lambda_0)}(x_1, \dots, x_{2n}; q) \\ & := \sum_{(m_i^{(a)}) \in \text{Mat}_{k' \times 2n}(\mathbb{N}_0)} \frac{q^{\frac{1}{2} \sum_{a,b=1}^{2n} \sum_{i=1}^{k'} C_{ab} M_i^{(a)} M_i^{(b)}}}{\prod_{a=1}^{2n} \prod_{i=1}^{k'} (q)_{m_i^{(a)}}} \prod_{j=1}^{2n} x_j^{\sum_{i=1}^{k'} M_i^{(j)}}. \end{aligned} \quad (2.7)$$

2.2. Modular forms of half-integral weight

To establish Theorem 1.1, Theorem 1.2, and Conjecture 4.1, we write the functions in question in terms of derivatives of Shimura theta functions of half-integral weight, and the Dedekind η -function, defined for $\tau \in \mathbb{H}$ by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{j \geq 1} (1 - q^j),$$

where $q = e^{2\pi i \tau}$. We first recall the modular transformation properties of this function [25].

Proposition 2.1. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, we have that*

$$\eta(\gamma \tau) = \varrho(\gamma)(c\tau + d)^{\frac{1}{2}} \eta(\tau),$$

where $\varrho(\gamma)$ is a 24th root of unity which can be given explicitly in terms of Dedekind sums [25].

For this paper we further require the following special cases of Shimura’s theta functions [26]

$$\theta(\tau; h, A, B, P_\nu) := \sum_{m \equiv h \pmod{B}} P_\nu(m) \cdot q^{\frac{Am^2}{2B^2}}, \tag{2.8}$$

where $h \in \mathbb{Z}$, $A, B \in \mathbb{N}$, $A|B$, and $B|Ah$, and P_ν is a spherical polynomial of order $\nu \geq 0$. Throughout, we let $e(x) := e^{2\pi ix}$, and $\varepsilon_d = 1$ or i , depending on whether $d \equiv 1$ or $3 \pmod{4}$, respectively.

Proposition 2.2 ([26, Proposition 2.1]). *Assuming the conditions above, we have for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ with $b \equiv 0 \pmod{2}$ and $c \equiv 0 \pmod{2B}$ that*

$$\theta(\gamma\tau; h, A, B, P_\nu) = \left(\frac{A}{d}\right) \left(\frac{2c}{d}\right) \varepsilon_d^{-1} e\left(\frac{abAh^2}{2B^2}\right) (c\tau + d)^{\frac{1+2\nu}{2}} \theta(\tau; ah, A, B, P_\nu). \tag{2.9}$$

In proving Theorems 1.1 and 1.2, we encounter certain q -series which we show in the course of the proof of Proposition 2.3 below may be expressed using Shimura’s theta functions. To be more precise, for $M \in \mathbb{N}$, $h \in \frac{1}{2}\mathbb{Z}$, and $t \in \{0, 1\}$, we define

$$g(\tau; h, M, t) := \sum_{\nu \equiv h \pmod{M}} (-1)^{\frac{(\nu-h)(1-t)}{M}} \nu^t q^{\frac{\nu^2}{2M}}. \tag{2.10}$$

Remark 2. Above and throughout, we abuse notation and allow sums over $\nu \equiv h \pmod{M}$ when h is a half-integer. In this case, by $\nu \equiv h \pmod{M}$ we mean those numbers $\nu \in \mathbb{Q}$ for which M divides $\nu - h$ in \mathbb{Z} as usual.

We establish the modularity of the q -series $g(\tau; h, M, t)$ in the following proposition. The multipliers $\psi(\gamma; h, \mu)$ and groups $\Gamma_{h,u,t}$ under which these functions transform are defined in (2.11) and (2.13) below.

Proposition 2.3. *Let $h \in \frac{1}{2}\mathbb{Z}$, $t \in \{0, 1\}$, and $M \in \mathbb{N}$, excluding the cases $h \in \frac{1}{2} + \mathbb{Z}$ with M even and $t = 0$, and $h \in \mathbb{Z}$ with M odd and $t = 0$. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{h,M,t}$, we have that*

$$g(\gamma\tau; h, M, t) = \psi(\gamma; h, M) (c\tau + d)^{\frac{1}{2}+t} g(\tau; h, M, t).$$

Remark 3. In the cases excluded in the statement of Proposition 2.3, the functions $g(\tau; h, M, t)$ still transform under a suitable subgroup of $\text{SL}_2(\mathbb{Z})$, however we refrain from proving this fact, as these cases are irrelevant to our treatment.

Remark 4. As remarked in [26], if A is even, the restriction that $\gamma \in \Gamma^0(2)$ is unnecessary. We will make use of this fact in the proof of Proposition 2.3.

Before proving Proposition 2.3, we first define the multiplier ψ and subgroup $\Gamma_{h,M,t}$. With notation as above, we define for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ the multiplier

$$\psi(\gamma; h, M) := \left(\frac{M}{d}\right) \left(\frac{2c}{d}\right) \varepsilon_d^{-1} e\left(\frac{abh^2}{2M}\right). \quad (2.11)$$

Using the multiplier ψ , for $\mathbf{x} = (x_1, x_2, \dots, x_n)$, we define

$$\Psi(\gamma; \mathbf{x}, M) := \prod_{j=1}^n \psi(\gamma; x_j, M) = \left(\left(\frac{M}{d}\right) \left(\frac{2c}{d}\right) \varepsilon_d^{-1}\right)^n e\left(\frac{ab\|\mathbf{x}\|^2}{2M}\right), \quad (2.12)$$

where $\|\mathbf{x}\|^2 := \sum_{i=1}^n x_i^2$. We define the subgroup $\Gamma_{h,M,t}$ as

$$\Gamma_{h,M,t} := \begin{cases} \Gamma_{8M,4M} & \text{if } h \in \frac{1}{2} + \mathbb{Z}, t = 0, \text{ and } M \text{ is odd,} \\ \Gamma_{8M,2M} & \text{if } h \in \mathbb{Z}, t = 0, \text{ and } M \text{ is even,} \\ \Gamma_{2M,2M} \cap \Gamma^0(2) & \text{if } h \in \frac{1}{2} + \mathbb{Z}, t = 1, \\ \Gamma_{2M,M} \cap \Gamma^0(2) & \text{if } h \in \mathbb{Z}, t = 1, \end{cases} \quad (2.13)$$

where $\Gamma_{B,C} := \Gamma_0(B) \cap \Gamma_1(C)$.

Using the subgroups $\Gamma_{B,C}$, we also define the groups $G_{k,N}$ and $H_{k,N}$ appearing in Sec. 5:

$$G_{k,N} := \begin{cases} \Gamma_{16k+8N-8,8k+4N-4} & \text{if } N \text{ is even,} \\ \Gamma_{4k+2N-2,2k+N-1} \cap \Gamma^0(2) & \text{if } N \text{ is odd,} \end{cases}$$

$$H_{k,N} := \begin{cases} \Gamma_{8(2k+N-2),2(2k+N-2)} & \text{if } N \text{ is even,} \\ \Gamma_1(2(2k+N-2)) \cap \Gamma^0(2) & \text{if } N \text{ is odd.} \end{cases}$$

Proof of Proposition 2.3. For the proof, we consider four cases: $h \in \frac{1}{2} + \mathbb{Z}$ with $t = 0$ and M odd, $h \in \mathbb{Z}$ with $t = 0$ and M even, $h \in \frac{1}{2} + \mathbb{Z}$ with $t = 1$, and $h \in \mathbb{Z}$ with $t = 1$.

Case 1 ($h \in \frac{1}{2} + \mathbb{Z}$ with $t = 0$, M odd). Since $h \in \frac{1}{2} + \mathbb{Z}$, we may rewrite

$$g(\tau; h, M, 0) = \theta(\tau; 2h, 4M, 4M, 1) - \theta(\tau; 2h + 2M, 4M, 4M, 1). \quad (2.14)$$

Since $A = 4M$ is even, by Remark 4, we do not additionally require $b \equiv 0 \pmod{2}$. Therefore, using (2.9) and (2.14), with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{8M,4M}$, we have

$$g(\gamma\tau; h, M, 0) = \left(\frac{M}{d}\right) \left(\frac{2c}{d}\right) \varepsilon_d^{-1} (c\tau + d)^{\frac{1}{2}} \left(e\left(\frac{abh^2}{2M}\right) \theta(\tau; 2ah, 4M, 4M, 1) \right. \\ \left. - e\left(\frac{ab(h+M)^2}{2M}\right) \theta(\tau; a(2h+2M), 4M, 4M, 1) \right).$$

Using that $a \equiv 1 \pmod{4M}$, M is odd, and $h \in \frac{1}{2} + \mathbb{Z}$, as well as properties of the Jacobi symbol, we find that

$$g(\gamma\tau; h, M, 0) = \psi(\gamma; h, M) (c\tau + d)^{\frac{1}{2}} g(\tau; h, M, 0),$$

as desired. (We point out that one may reduce the level of the subgroup under which $g(\tau; h, M, 0)$ transforms, at the expense of a change in Nebentypus character.)

Case 2 ($h \in \mathbb{Z}$ with $t = 0$ and M even). In this case, we find that

$$g(\tau; h, M, 0) = \theta(2\tau; h, 2M, 2M, 1) - \theta(2\tau; h + M, 2M, 2M, 1).$$

We have

$$g(\gamma\tau; h, M, 0) = \theta(\gamma'(2\tau); h, 2M, 2M, 1) - \theta(\gamma'(2\tau); h + M, 2M, 2M, 1),$$

where, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$, we let $\gamma' := \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$. If we further impose that $\gamma \in \Gamma_{8M, 2M}$, we may apply Proposition 2.2, which results in the desired transformation.

Case 3 ($h \in \frac{1}{2} + \mathbb{Z}$ with $t = 1$). Here we find that

$$g(\tau; h, M, 1) = \frac{1}{2}\theta\left(\frac{\tau}{2}; 2h, 2M, 2M, \text{id}\right).$$

Proceeding as in Case 2, again appealing to Remark 4, we obtain the desired transformation for $\gamma \in \Gamma_{2M, 2M} \cap \Gamma^0(2)$.

Case 4 ($h \in \mathbb{Z}$ with $t = 1$). As in Case 1, we find that

$$g(\tau; h, M, 1) = \theta(\tau; h, M, M, \text{id}).$$

We see that $g(\tau; h, M, 1)$ transforms as a modular form of weight $\frac{3}{2}$ with multiplier as claimed on $\Gamma_{2M, M} \cap \Gamma^0(2)$. □

2.3. Modular determinants

In addition to the half-integral weight modular forms discussed in Sec. 2.2, we make use of Wronskian of (modular) q -series. To be more precise, for $r \in \mathbb{N}$ differentiable functions on \mathbb{H} , f_1, \dots, f_r , their Wronskian $W(f_1, f_2, \dots, f_r)$ is defined by

$$W(f_1, f_2, \dots, f_r)(\tau) := \det \begin{pmatrix} f_1(\tau) & f_2(\tau) & \cdots & f_r(\tau) \\ f_1'(\tau) & f_2'(\tau) & \cdots & f_r'(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(r-1)}(\tau) & f_2^{(r-1)}(\tau) & \cdots & f_r^{(r-1)}(\tau) \end{pmatrix},$$

where

$$f'(\tau) := \frac{1}{2\pi i} \frac{d}{d\tau} f(\tau) = q \frac{df}{dq}.$$

In the following proposition, we show that if the functions f_1, f_2, \dots, f_r are additionally modular forms of the same weight, then their Wronskian is also modular.

Proposition 2.4. *Let f_1, \dots, f_r ($r \geq 1$) be modular forms of weight κ on a congruence subgroup $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ with respective multipliers $v_1 = v_1(\gamma), \dots, v_r = v_r(\gamma)$ (where $\gamma \in \Gamma$). Then the Wronskian $W(f_1, f_2, \dots, f_r)$ is a modular form of weight $r\kappa + r^2 - r$ on Γ with multiplier $\Upsilon = \Upsilon(\gamma) := \prod_{i=1}^r v_i$.*

Proof. While similar results appear in the literature (see [21, 23] for example), for the convenience of the reader, we provide a self-contained proof of Proposition 2.4 here. We proceed by induction on $r \geq 1$, noting first that the base case $r = 1$ is trivial. The truth of Proposition 2.4 in the case $r = 2$ follows after a straightforward calculation, which for brevity we leave to the reader.

For $r > 2$, we first note the following fact: Using the modular transformation properties of $f_i (1 \leq i \leq r)$, it is not difficult to show (see, e.g., [4, p. 54]) that the m th derivative ($m \in \mathbb{N}_0$) of each f_i transforms as follows under $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$:

$$f_i^{(m)}(\gamma\tau) = v_i \sum_{j=1}^{m+1} \binom{m}{j-1} \frac{[\kappa + j - 1]_{m-j+1}}{(2\pi i)^{m-j+1}} c^{m-j+1} (c\tau + d)^{\kappa+m+j-1} f_i^{(j-1)}(\tau), \tag{2.15}$$

where $[a]_\ell := (a + \ell - 1)! / (a - 1)!$ for integers $\ell \geq 0$. Now, we assume that Proposition 2.4 is true for $r - 1$ where $r \geq 3$, and expand the determinant $W(f_1, \dots, f_r)(\tau)$ along the r th row to find

$$W(f_1, \dots, f_r)(\tau) = (-1)^{r-1} \sum_{i=1}^r (-1)^{i-1} f_i^{(r-1)}(\tau) W_i^{(r)}(\tau), \tag{2.16}$$

where $W_i^{(r)}$ denotes $W(f_1, f_2, \dots, f_{i-1}, f_{i+1}, \dots, f_r)$ (i.e. $W_i^{(r)}$ is a Wronskian involving $r - 1$ functions). By induction, we know that $W_i^{(r)}$ is a modular form of weight $(r - 1)\kappa + r^2 - 3r + 2$ on Γ with multiplier Υ/v_i . Using (2.16) and (2.15) and splitting the sum $\sum_{j=1}^r \alpha_j = \alpha_r + \sum_{j=1}^{r-1} \alpha_j$, we find that

$$\begin{aligned} W(f_1, \dots, f_r)(\gamma\tau) &= (c\tau + d)^{\kappa+2r-2} (-1)^{r-1} \sum_{i=1}^r (-1)^{i-1} v_i f_i^{(r-1)}(\tau) W_i^{(r)}(\gamma\tau) \\ &\quad + (1)^{r-1} \sum_{i=1}^r (-1)^{i-1} v_i W_i^{(r)}(\gamma\tau) \\ &\quad \times \sum_{j=1}^{r-1} \binom{r-1}{j-1} \frac{[\kappa + j - 1]_{r-j}}{(2\pi i)^{r-j}} c^{r-j} (c\tau + d)^{\kappa+r+j-2} f_i^{(j-1)}(\tau). \end{aligned} \tag{2.17}$$

By the induction hypothesis (2.17) equals

$$\begin{aligned} &\Upsilon (c\tau + d)^{r\kappa+r^2-r} (-1)^{r-1} \sum_{i=1}^r (-1)^{i-1} f_i^{(r-1)}(\tau) W_i^{(r)}(\tau) \\ &\quad + (-1)^{r-1} \Upsilon \sum_{i=1}^r (-1)^{i-1} W_i^{(r)}(\tau) \\ &\quad \times \sum_{j=1}^{r-1} \binom{r-1}{j-1} \frac{[\kappa + j - 1]_{r-j}}{(2\pi i)^{r-j}} c^{r-j} (c\tau + d)^{r(\kappa-2)+r^2+j} f_i^{(j-1)}(\tau). \end{aligned} \tag{2.18}$$

Finally, using (2.16) once again, and reversing the order of summation, yields that (2.18) equals

$$\begin{aligned} & \Upsilon(c\tau + d)^{r\kappa+r^2-r}W(f_1, \dots, f_r)(\tau) \\ & + (-1)^{r-1}\Upsilon \sum_{j=1}^{r-1} \binom{r-1}{j-1} \frac{[\kappa+j-1]_{r-j}}{(2\pi i)^{r-j}} c^{r-j} (c\tau + d)^{r(\kappa-2)+r^2+j} \\ & \times \sum_{i=1}^r (-1)^{i-1} f_i^{(j-1)}(\tau) W_i^{(r)}(\tau). \end{aligned} \tag{2.19}$$

We now consider the innermost sum (on i) in (2.19) for a fixed j . By expanding the determinant $W_i^{(r)}$ along the j th row, we see

$$\sum_{i=1}^r (-1)^{i-1} f_i^{(j-1)} W_i^{(r)} = (-1)^{j-1} (\Sigma_1 + \Sigma_2). \tag{2.20}$$

Here, letting $M_{bc}^{(i,r)}$ ($b \neq r, i \neq c$) denote the minor of $W(f_1, \dots, f_r)$ obtained by removing column i , row r , row b , and column c , the sums Σ_1 and Σ_2 are defined by

$$\Sigma_1 := \sum_{i=1}^r (-1)^{i-1} f_i^{(j-1)} \sum_{m=1}^{i-1} (-1)^{m-1} f_m^{(j-1)} M_{jm}^{(i,r)}, \tag{2.21}$$

$$\Sigma_2 := \sum_{i=1}^r (-1)^{i-1} f_i^{(j-1)} \sum_{m=i+1}^r (-1)^m f_m^{(j-1)} M_{jm}^{(i,r)}. \tag{2.22}$$

Reversing the order of summation then yields

$$\Sigma_1 = \sum_{i=1}^r \sum_{m=i+1}^r (-1)^{i+m} f_m^{(j-1)} f_i^{(j-1)} M_{ji}^{(m,r)}. \tag{2.23}$$

Combining (2.23) and (2.22), we find that (2.20) equals

$$(-1)^{j-1} \sum_{i=1}^r \sum_{m=i+1}^r (-1)^{i+m} f_i^{(j-1)} f_m^{(j-1)} (M_{ji}^{(m,r)} - M_{jm}^{(i,r)}). \tag{2.24}$$

Now, by definition, $M_{ji}^{(m,r)} = M_{jm}^{(i,r)}$ for any pair (i, m) with $1 \leq i \neq m \leq r$. Thus, for each j , each summand in (2.24) is equal to 0, giving the desired result. \square

Remark 5. One could also deduce Proposition 2.4 by relating $W_{q\frac{d}{dq}}(f_1, \dots, f_r)$ to $W_{\Theta_\kappa}(f_1, \dots, f_r)$, where $\Theta_\kappa := \frac{d}{dq} + \kappa G_2(\tau)$. Here, $G_2(\tau)$ is the Eisenstein series of weight 2, and W_Δ indicates the Wronskian is to be taken with respect to the given (differential) operator Δ . See [22] for more along these lines.

Next we generalize Lemma 2.2 in [21], and show how to write the q -expansion of the Wronskian of r q -series in terms of Vandermonde determinants $V(x_1, \dots, x_r)$,

defined by

$$V(x_1, \dots, x_r) := \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_r \\ x_1^2 & x_2^2 & \cdots & x_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{r-1} & x_2^{r-1} & \cdots & x_r^{r-1} \end{pmatrix}.$$

Proposition 2.5. *Suppose that f_1, \dots, f_r are $r \geq 1$ functions on \mathbb{H} , with compactly convergent q -expansions*

$$f_i(\tau) = \sum_{\nu_i \equiv x_i \pmod{N_i}} a_{\nu_i}^{(i)} q^{\frac{\beta(\nu_i)}{M}},$$

for some $M \in \mathbb{Z}$, $x_i \in \mathbb{Q}$, $N_i \in \mathbb{Z}$, $a_{\nu_i}^{(i)} \in \mathbb{C}$ ($1 \leq i \leq r$), and function β taking values in \mathbb{Q} . Then the Wronskian $W(f_1, \dots, f_r)$ is function on \mathbb{H} with q -expansion given by

$$\begin{aligned} W(f_1, \dots, f_r)(\tau) &= M^{\frac{(1-r)r}{2}} \\ &\times \sum_{\substack{\nu_i \equiv x_i \pmod{N_i} \\ 1 \leq i \leq r}} V(\beta(\nu_1), \dots, \beta(\nu_r)) \prod_{i=1}^r a_{\nu_i}^{(i)} q^{\frac{1}{M}(\beta(\nu_1) + \dots + \beta(\nu_r))}. \end{aligned}$$

Proof. Let $A = (a_{hk})$ ($1 \leq h, k \leq r$) be an $r \times r$ matrix and fix i ($1 \leq i \leq r$). We repeatedly make use of the following fact: Suppose that $a_{hi} = b_{hi} + c_{hi}$. Then $\det(A) = \det(B_i) + \det(C_i)$, where B_i and C_i are the matrices obtained by replacing the i th column of A by (b_{1i}, \dots, b_{ri}) and (c_{1i}, \dots, c_{ri}) , respectively. Iterating this fact, applied to each column $1 \leq i \leq r$, and using that the series f_i are compactly convergent, we find that $W(f_1, \dots, f_r)$ equals

$$\sum_{\substack{\nu_i \equiv x_i \pmod{N_i} \\ 1 \leq i \leq r}} \det \begin{pmatrix} a_{\nu_1}^{(1)} q^{\frac{\beta(\nu_1)}{M}} & a_{\nu_2}^{(2)} q^{\frac{\beta(\nu_2)}{M}} & \cdots & a_{\nu_r}^{(r)} q^{\frac{\beta(\nu_r)}{M}} \\ \left(\frac{\beta(\nu_1)}{M}\right) a_{\nu_1}^{(1)} q^{\frac{\beta(\nu_1)}{M}} & \left(\frac{\beta(\nu_2)}{M}\right) a_{\nu_2}^{(2)} q^{\frac{\beta(\nu_2)}{M}} & \cdots & \left(\frac{\beta(\nu_r)}{M}\right) a_{\nu_r}^{(r)} q^{\frac{\beta(\nu_r)}{M}} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\beta(\nu_1)}{M}\right)^{r-1} a_{\nu_1}^{(1)} q^{\frac{\beta(\nu_1)}{M}} & \left(\frac{\beta(\nu_2)}{M}\right)^{r-1} a_{\nu_2}^{(2)} q^{\frac{\beta(\nu_2)}{M}} & \cdots & \left(\frac{\beta(\nu_r)}{M}\right)^{r-1} a_{\nu_r}^{(r)} q^{\frac{\beta(\nu_r)}{M}} \end{pmatrix}.$$

We next factor out $a_{\nu_i}^{(i)} q^{\frac{\beta(\nu_i)}{M}}$ ($1 \leq i \leq r$) from each column of the determinant, and $(1/M)^{h-1}$ ($1 \leq h \leq r$) from each row, which leaves the expression given on the right-hand side of (2.25) as claimed. \square

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.2. We begin by proving case (i) of Theorem 1.2, as given in (5.4) and (5.5) of Sec. 5. Using the well-known fact that

$$\prod_{1 \leq i < j \leq n} (\nu_i^2 - \nu_j^2) = (-1)^n V(\nu_1^2, \dots, \nu_n^2),$$

we rewrite the series in question as

$$\frac{(-1)^n}{\eta(\tau)^{2n^2-n} \prod_{1 \leq i < j \leq n} (\rho_i^2 - \rho_j^2)} \sum_{\nu} V(\nu_1^2, \dots, \nu_n^2) (-1)^{|\nu| - |\rho + (k-\ell)_n|} q^{\frac{\|\nu\|^2}{2(2k+2n-1)}}, \tag{3.1}$$

where the sum is taken over $\nu_j \equiv \rho_j + k - \ell \pmod{2k + 2n - 1}$.

Since $\prod_{j=1}^n (-1)^{\nu_j - \rho_j - k + \ell} = (-1)^{|\nu| - |\rho + (k-\ell)_n|}$ and $\sum_{j=1}^n \nu_j^2 = \|\nu\|^2$, we are in a position to apply Proposition 2.5 with $a_{\nu_j}^{(j)} := (-1)^{\nu_j - \rho_j - k + \ell}$, $\beta(\nu) := \nu^2$, $x_j := \rho_j + k - \ell$, $N_j := 2k + 2n - 1$, and $M := 2(2k + 2n - 1)$. Thus we see that (3.1) is equal to

$$\frac{(-1)^n (2(2k + 2n - 1))^{\frac{n(n-1)}{2}} W(g_1, g_2, \dots, g_n)(\tau)}{\prod_{1 \leq i < j \leq n} (\rho_i^2 - \rho_j^2) \eta(\tau)^{2n^2-n}},$$

where for $1 \leq j \leq n$, we let $g_j := g(\tau; \rho_j + k - \ell, 2k + 2n - 1, 0)$. The truth of Theorem 1.2 in case (i) of (5.4) and (5.5) now follows from Proposition 2.1, Proposition 2.3, and Proposition 2.4.

The proofs of Theorem 1.2 in the cases (ii)–(iv) follow in a similar manner to that of case (i) proved above, thus for brevity we provide only a detailed sketch of their proofs. We begin with case (ii) and rewrite the series in question as

$$\frac{(-1)^n}{n! \eta(\tau)^{2n^2+n} \prod_{1 \leq i < j \leq n} (\rho_i^{*2} - \rho_j^{*2})} \sum_{\nu} V(\nu_1^2, \dots, \nu_n^2) \prod_{i=1}^n \nu_i \cdot q^{\frac{\|\nu\|^2}{4(k+n)}},$$

where the sum is taken over $\nu_j \equiv \rho_j^* + k - p \pmod{2k + 2n}$. In this case we apply Proposition 2.5 with $a_{\nu_j}^{(j)} = \nu_j$, $\beta(\nu) = \nu^2$, $x_j = \rho_j^* + k - \ell$, $N_j = 2k + 2n$, and $M = 2(2k + 2n)$ ($1 \leq j \leq n$), which is thus related to the Wronskian of the functions $g(\tau; \rho_j^* + k - \ell, 2k + 2n, 1)$. Case (ii) now follows in a similar manner to the proof of case (i) above.

To prove case (iii) as stated in (5.4) and (5.5), we proceed similarly, and encounter a Wronskian of the functions $g(\tau; \tilde{\rho}_j, 2k + 2n - 2, 0)$ ($1 \leq j \leq n$). For case (iv), a Wronskian of the functions $g(\tau; \rho_j, 2k + 2n - 1, 1)$ ($1 \leq j \leq n$) occurs. Cases (iii) and (iv) as stated in (5.4) and (5.5) of Theorem 1.2 now follow in a similar fashion. \square

Proof of Theorem 1.1. We begin by proving case (i) of Theorem 1.1 as given in (5.1) and (5.2) of Sec. 5. By [6, Corollary 5.1],

$$q^{\frac{\|\rho\|^2}{2(2n+3)} - \frac{2n^2-n}{24}} \chi_{W(\Lambda_0)}(1, \dots, 1; q) = q^{\frac{\|\rho\|^2}{2(2n+3)} - \frac{2n^2-n}{24}} \times \sum_{\mathbf{m}=(m_1, \dots, m_{2n-1}) \in \mathbb{N}_0^{2n-1}} \frac{q^{\frac{mCm^t}{2}}}{(q)_{m_1} \cdots (q)_{m_{2n-1}}}, \tag{3.2}$$

where C is the Cartan matrix of A_{2n-1} . By [29, Theorem 1.2], we see that the right-hand side of (3.2) is equal to the function appearing in case (i) as stated in (5.4) and (5.5) of Theorem 1.2 with $\ell = k = 2$. Thus, by Theorem 1.2, we have that $q^{\frac{\|\rho\|^2}{2(2n+3)} - \frac{2n^2-n}{24}} \chi_{W(\Lambda_0)}(1, \dots, 1; q)$ is modular of weight 0 on $G_{2,2n}$ with multiplier $\varrho(\gamma)^{n-2n^2} \Psi(\gamma; \rho, 2n + 3)$ as claimed.

We prove case (ii) of Theorem 1.1 similarly. We begin with the case $N = 2n$ even, $n \in \mathbb{N}$. By [6, Corollary 5.1; 29, Theorem 1.2], we have that $q^{\frac{\|\rho+1_n\|^2}{2(2n+3)} - \frac{2n^2-n}{24}} \chi'_{W(\Lambda_i)}(y_1, \dots, y_{N-1}; q)$ is equal to the function appearing in Theorem 1.2 case (i) as stated in (5.4) and (5.5) with $k = 2$ and $\ell = 1$. Modularity now follows from Theorem 1.2. For $N = 2n + 1$ odd, $n \in \mathbb{N}$, modularity follows by using Theorem 2.3 of [29] and the modularity established in Theorem 1.2 of this paper in case (ii) of (5.4) and (5.5) with $k = 2, \ell = 1$.

Case (iii) of Theorem 1.1 as stated in (5.1) and (5.2) follows in a similar manner by using a result of Feigin–Stoyanovsky [7] and Stoyanovsky [27] (restated as Theorem 2.1 in [29]), and also Theorem 1.2 case (ii) as stated in (5.4) and (5.5) with $\ell = k$ of this paper. \square

Remark 6. We point out that in general, much less is known about identities about specializations of series such as (2.7) when N is even and $k' > 1$ (see [27]).

4. Conjectures

Next we formulate modularity conjectures pertaining to generalized Andrews–Gordon series as studied originally by Warnaar and Zudilin [29], in relation to A_{N-1} root systems. Such identities pertaining to the case N even are less well understood [27], and the conjecture below addresses both cases, N odd and N even.

Conjecture 4.1. *The generalized Andrews–Gordon series ($q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$)*

$$q^{\varepsilon(\chi)} \cdot \chi(q)$$

are modular functions on $G(\chi)$ with multiplier $\rho(\chi)$, where $\chi, G(\chi), \mathcal{E}(\chi)$ and $\rho(\chi)$ are as given in (5.8) and (5.9).

Remark 7. In Theorem 1.1 of this paper, we prove special cases of case (i) of Conjecture 4.1 as given in (5.8) and (5.9). A result in [29] also showed a special case. In general, the truth of [29, Conjectures 1.1, 2.2 and 2.4], when combined with Theorem 1.2 of this paper, would imply the truth of Conjecture 4.1. In particular, as stated in (5.8) and (5.9), we have the following:

- (1) Case (i) of Conjecture 4.1 with N even and $k = 2$ follows from Theorem 1.1.
- (2) Case (i) of Conjecture 4.1 with N odd and $\ell = k$ follows from Theorem 1.1.
- (3) Case (i) of Conjecture 4.1 with N odd, $\ell = 1$, $k = 2$ follows from [29, Theorem 2.3].

5. Tables

5.1. Notation

Before giving the tables of graded dimensions, generalized Macdonald identities, and generalized Andrews–Gordon series to which Theorem 1.1, Theorem 1.2, and Conjecture 4.1 apply (respectively), we first fix some notation. First, for $z \in \mathbb{C}$ and integers $M \geq 1$, we let $\mathbf{z}_M := (z, z, \dots, z) \in \mathbb{C}^M$. We also require some special vectors, defined for a positive integer n , by

$$\begin{aligned} \boldsymbol{\rho} &:= (1/2, 3/2, \dots, n - 1/2) \in (1/2 + \mathbb{Z})^n, \\ \boldsymbol{\rho}^* &:= (1, 2, \dots, n) \in \mathbb{Z}^n, \\ \tilde{\boldsymbol{\rho}} &:= (0, 1, \dots, n - 1) \in \mathbb{Z}^n, \end{aligned}$$

where we abuse notation and omit the dependence on n from the vectors $\boldsymbol{\rho}, \boldsymbol{\rho}^*$, and $\tilde{\boldsymbol{\rho}}$. Using these vectors, for integers $N \geq 2$ we define the $\lfloor \frac{N}{2} \rfloor$ -dimensional vectors

$$\boldsymbol{\rho}_e(N) := \begin{cases} \boldsymbol{\rho} & \text{if } N \text{ is even,} \\ \boldsymbol{\rho}^* & \text{if } N \text{ is odd,} \end{cases} \quad \boldsymbol{\rho}_o(N) := \begin{cases} \tilde{\boldsymbol{\rho}} & \text{if } N \text{ is even,} \\ \boldsymbol{\rho} & \text{if } N \text{ is odd.} \end{cases}$$

5.2. Graded dimensions

Theorem 1.1 pertains to the graded dimensions (1.3), (2.6) and (2.7). Here, we require a special vector $(y_1, \dots, y_i, \dots, y_{N-1})$, defined for each integer $N \geq 2$ and each $1 \leq j \neq i \leq N - 1$ by

$$y_i := \begin{cases} 1 & \text{if } i \text{ is odd,} \\ q^{-2} & \text{if } i \text{ is even,} \end{cases} \quad \text{and} \quad y_j := \begin{cases} q & \text{if } j \text{ is odd,} \\ q^{-1} & \text{if } j \text{ is even.} \end{cases}$$

Data: Theorem 1.1.

Case	Lie algebra	χ	n	N	i	k
(i)	\mathbb{A}_{2n-1}	$\chi_{W(\Lambda_0)}$	$n \in \mathbb{N}$	—	—	—
(ii)	\mathbb{A}_{N-1}	$\chi'_{W(\Lambda_i)}$	—	$N \in 1 + \mathbb{N}$	$1 \leq i \leq N - 1$	—
(iii)	\mathbb{A}_{2n}	$\chi_{W((k-1)\Lambda_0)}$	$n \in \mathbb{N}$	—	—	$k \geq 2$

(5.1)

Case	$\mathfrak{x}(\chi)$	$\varepsilon(\chi)$	$G(\chi)$	$\rho(\chi) = \rho(\chi, \gamma)$
(i)	$\mathbf{1}_{2n-1}$	$\frac{\ \rho\ ^2}{2(2n+3)} - \frac{2n^2-n}{24}$	$G_{2,2n}$	$\varrho(\gamma)^{n-2n^2} \Psi(\gamma; \rho, 2n+3)$
(ii)	$(y_1, \dots, y_i, \dots, y_{N-1})$	$\frac{\ \rho e^{(N)+1}_{\lfloor N/2 \rfloor}\ ^2}{2(N+3)} - \frac{N^2-N}{48}$	$G_{2,N}$	$\varrho(\gamma)^{\frac{N-N^2}{2}} \times \Psi(\gamma; \rho e^{(N)+1}_{\lfloor N/2 \rfloor}, N+3)$
(iii)	$\mathbf{1}_{2n}$	$\frac{\ \rho^*\ ^2}{4(k+n)} - \frac{2n^2+n}{24}$	$G_{k,2n+1}$	$\varrho(\gamma)^{-n-2n^2} \Psi(\gamma; \rho^*, 2k+2n)$

(5.2)

5.3. Generalized Macdonald series

Theorem 1.2 pertains to generalized Macdonald series defined using the following products, as given in [29]:

$$\xi(\nu/\mathfrak{x}) := \prod_{1 \leq i < j \leq n} \frac{\nu_i^2 - \nu_j^2}{x_i^2 - x_j^2} \quad \text{and} \quad \chi(\nu/\mathfrak{x}) := \prod_{i=1}^n \frac{\nu_i}{x_i} \prod_{1 \leq i < j \leq n} \frac{\nu_i^2 - \nu_j^2}{x_i^2 - x_j^2}, \quad (5.3)$$

where $\nu = (\nu_1, \dots, \nu_n)$ and $\mathfrak{x} = (x_1, \dots, x_n)$. (We point out that $\chi(\nu/\mathfrak{x})$ defined in (5.3) is a product, while the functions χ of Theorem 1.1 and Sec. 5.2 are graded dimensions.)

Data: Theorem 1.2.

Case	$\alpha(\nu)$	n	k	ℓ	$\varepsilon(n)$	$\beta(\nu)$	$G(\alpha)$
(i)	$\xi(\nu/\rho)(-1)^{ \nu - \rho + (k-\ell)n}$	$n \in \mathbb{N}$	$k \geq 2$	$\ell \in \{1, k\}$	$2n^2 - n$	$\frac{\ \nu\ ^2}{2(2k+2n-1)}$	$G_{k,2n}$
(ii)	$\chi(\nu/\rho^*)$	$n \in \mathbb{N}$	$k \geq 2$	$\ell \in \{1, k\}$	$2n^2 + n$	$\frac{\ \nu\ ^2}{4(k+n)}$	$G_{k,2n+1}$
(iii)	$\xi(\nu/\tilde{\rho})(-1)^{\frac{(\nu - \tilde{\rho})}{(2k+2n-2)}}$	$n \in \mathbb{N}$	$k \geq 2$	—	$2n^2 - n$	$\frac{\ \nu\ ^2}{2(2k+2n-2)}$	$H_{k,2n}$
(iv)	$\chi(\nu/\rho)$	$n \in \mathbb{N}$	$k \geq 2$	—	$2n^2 + n$	$\frac{\ \nu\ ^2}{2(2k+2n-1)}$	$H_{k,2n+1}$

(5.4)

Case	S_α	$\rho(\alpha) = \rho(\alpha, \gamma)$
(i)	$\{\nu \in \frac{1}{2}\mathbb{Z}^n \mid \nu_i \equiv \rho_i + k - \ell \pmod{2k + 2n - 1}\}$	$q(\gamma)^{n-2n^2} \times \Psi(\gamma; \rho + (\mathbf{k} - \ell)_n, 2k + 2n - 1)$
(ii)	$\{\nu \in \mathbb{Z}^n \mid \nu_i \equiv \rho_i^* + k - \ell \pmod{2k + 2n}\}$	$q(\gamma)^{-2n^2 - n} \times \Psi(\gamma; \rho^* + (\mathbf{k} - \ell)_n, 2k + 2n)$
(iii)	$\{\nu \in \mathbb{Z}^n \mid \nu_i \equiv \tilde{\rho}_i \pmod{2k + 2n - 2}\}$	$q(\gamma)^{n-2n^2} \Psi(\gamma; \tilde{\rho}, 2k + 2n - 2)$
(iv)	$\{\nu \in \frac{1}{2}\mathbb{Z}^n \mid \nu_i \equiv \rho_i \pmod{2k + 2n - 1}\}$	$q(\gamma)^{-n-2n^2} \Psi(\gamma; \rho, 2k + 2n - 1)$

(5.5)

Remark 8. The series in case (i) and case (ii) of Theorem 1.2 are equal to 1 when $k = 1$ by the following celebrated Macdonald identities [20]:

$$\eta(\tau)^{2n^2 - n} = \sum_{\nu} \xi\left(\frac{\nu}{\rho}\right) (-1)^{|\nu| - |\rho|} q^{\frac{\|\nu\|^2}{2(2n+1)}}, \tag{5.6}$$

$$\eta(\tau)^{2n^2 + n} = \sum_{\nu} \chi\left(\frac{\nu}{\rho^*}\right) q^{\frac{\|\nu\|^2}{4n+4}}. \tag{5.7}$$

Note that (5.7) was also proved in [22, Proposition 4.4] by using modular Wronskians.

In addition, when $k = 1$, the series in case (iii) and case (iv) of Theorem 1.2 are equal to the modular function

$$\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{N-1}$$

when $N = 2n$ and $N = 2n + 1$, respectively. This follows from the Macdonald identities [20]

$$\frac{\eta(\tau)^{2n^2 + n - 1}}{\eta(2\tau)^{2n - 1}} = \sum_{\nu} \xi\left(\frac{\nu}{\tilde{\rho}}\right) (-1)^{\frac{|\nu| - |\tilde{\rho}|}{2n}} q^{\frac{\|\nu\|^2}{4n}},$$

$$\frac{\eta(\tau)^{2n^2 + 3n}}{\eta(2\tau)^{2n}} = \sum_{\nu} \chi\left(\frac{\nu}{\rho}\right) q^{\frac{\|\nu\|^2}{2(2n+1)}}.$$

See also [20, 22] for more related to the series in case (iii) with $k = 0$.

5.4. Generalized Andrews–Gordon series

Case (i) of Conjecture 4.1 pertains to the generalized Andrews–Gordon series

$$\chi_{k,\ell}^{(e)}(q) := \sum_{(m_i^{(a)}) \in \text{Mat}_{(k-1) \times (N-1)}(\mathbb{N}_0)} \frac{q^{\frac{1}{2} \sum_{a,b=1}^{N-1} \sum_{i=1}^{k-1} C_{ab} M_i^{(a)} M_i^{(b)} + \sum_{a=1}^{N-1} \sum_{i=\ell}^{k-1} (-1)^a M_i^{(a)}}}{\prod_{a=1}^{N-1} \prod_{i=1}^{k-1} (q)_{m_i^{(a)}}}$$

as studied by Warnaar and Zudilin [29]. In particular, case (i) of Conjecture 4.1 with $N = 3$ and $\ell = 1$ leads to case (ii) of the conjecture regarding the modularity of graded dimensions $\chi'_{W((k-1)\Lambda_j)}(q^{j-3}, q^{2-j}; q)$ when $j = 1, 2$ due to the second author in [5]. The graded dimensions $\chi'_{W((k-1)\Lambda_j)}(q^{j-3}, q^{2-j}; q)$ are obtained by specializing $i = 0$ and replacing k by $k - 1$ in (4.69) of [5]. Case (iii) of Conjecture 4.1 concerns the generalized Andrews–Gordon series

$$\chi_k^{(o)}(q) := \sum_{(m_i^{(a)}) \in \text{Mat}_{(k-1) \times (N-1)}(\mathbb{N}_0)} \frac{q^{\frac{1}{2} \sum_{a,b=1}^{N-1} \sum_{i=1}^{k-1} C_{ab} M_i^{(a)} M_i^{(b)}}}{\prod_{a=1}^{N-1} \left(\prod_{i=1}^{k-2} (q)_{m_i^{(a)}} \right) (q^2; q^2)_{m_{k-1}^{(a)}}}$$

Data: Conjecture 4.1.

Case	Lie algebra	χ	N	k	ℓ	j
(i)	\mathbb{A}_{N-1}	$\chi_{k,\ell}^{(e)}(q)$	$N \geq 2$	$k \geq 2$	$\ell \in \{1, k\}$	–
(ii)	\mathbb{A}_2	$\chi'_{W((k-1)\Lambda_j)}(q^{j-3}, q^{2-j}; q)$	$N = 3$	$k \geq 2$	–	$j \in \{1, 2\}$
(iii)	\mathbb{A}_{N-1}	$\chi_k^{(o)}(q)$	$N \geq 2$	$k \geq 2$	–	–

(5.8)

Case	$G(\chi)$	$\rho(\chi) = \rho(\chi, \gamma)$	$\varepsilon(\chi)$
(i)	$G_{k,N}$	$\varrho(\gamma)^{-\frac{N(N-1)}{2}} \Psi(\gamma; \rho_e(N) + (k-\ell)_n, 2k + N - 1)$	$\frac{\ \rho_e(N) + (k-\ell)_n\ ^2}{2(2k+N-1)} - \frac{N(N-1)}{48}$
(ii)	$G_{k,3}$	$\varrho(\gamma)^{-3} \psi(\gamma; k, 2k + 2)$	$\frac{k^2}{4k+4} - \frac{1}{8}$
(iii)	$H_{k,N}$	$\varrho(\gamma)^{-\frac{N(N-1)}{2}} \Psi(\gamma; \rho_o(N), 2k + N - 2)$	$\frac{\ \rho_o(N)\ ^2}{2(2k+N-2)} - \frac{N(N-1)}{48}$

(5.9)

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