On a conjecture of B. Berndt and B. Kim

Kathrin Bringmann · Amanda Folsom

Abstract We prove a recent conjecture of B. Berndt and B. Kim regarding the positivity of the coefficients in the asymptotic expansion of a class of partial theta functions. This generalizes results found in Ramanujan's second notebook and recent work of Galway and Stanley.

1 Introduction

In his second notebook [1, p. 324], Ramanujan claims an asymptotic expansion for the partial theta function

$$2\sum_{n=0}^{\infty}(-1)^n q^{n^2+n} \sim 1 + t + t^2 + 2t^3 + 5t^4 + \cdots \qquad (1.1)$$

with $q = \frac{1-t}{1+t}$ as $t \to 0^+$. It is not clear, just given the left-hand side of (1.1), that the coefficients of its asymptotic expansion (in t) are always positive integers, nor that one should expect this. Galway [3] proved this curious fact to be true. Stanley [5], answering a question of Galway, then gave a nice combinatorial interpretation

K. Bringmann
Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany
e-mail: kbringma@math.uni-koeln.de

A. Folsom (✉)
Mathematics Department, Yale University, P.O. Box 208283, New Haven, CT 06520-8283, USA
e-mail: amanda.folsom@yale.edu

of the coefficients in the asymptotic expansion (1.1) as the number of fixed-point-free alternating involutions in the symmetric group S_{2n}, providing a second proof that the coefficients are positive integers.

Berndt and Kim [2] study more general partial theta functions

$$f_b(t) := 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2 + bn}, \tag{1.2}$$

where b is real. They show that, similarly to (1.1), $f_b(t)$ admits an asymptotic expansion of the form

$$f_b(t) \sim \sum_{n=0}^{\infty} a_n t^n, \tag{1.3}$$

where the a_n are given explicitly in terms of Euler numbers and Hermite polynomials [2, Theorem 1.1]. For the purposes of this paper, we do not require the explicit shape of the a_n, so we do not state it here. (We point out a small typo in (1.3) of [2]: the exponent of $(1-t)/(1+t)$ should read $(1-2b)/4$ rather than $(2b-1)/4$, as the authors correctly state in (2.9) of [2].)

In analogy to the results established by Galway and Stanley pertaining to the coefficients of (1.1), Berndt and Kim prove that the coefficients a_n of the generalized partial theta functions defined in (1.2) are integers if $b \in \mathbb{N}$ [2, Theorem 2.5] and make the following conjecture regarding their positivity.

Conjecture (Berndt–Kim [2]) *For any positive integer b, for sufficiently large n, the coefficients a_n in the asymptotic expansion (1.3) have the same sign.*

The purpose of this note is to prove this conjecture of Berndt and Kim.

Theorem 1 *The Berndt–Kim conjecture is true. More precisely, if $b \equiv 1, 2 \pmod 4$, then $a_n > 0$ for $n \gg 0$, and if $b \equiv 0, 3 \pmod 4$, then $a_n < 0$ for $n \gg 0$.*

2 Proof of Theorem 1

To prove the theorem, we first observe that for integers $b \geq 2$,

$$f_{b+1}(t) = -q^{-b}(f_{b-1}(t) - 2). \tag{2.1}$$

We proceed by induction on b to prove Theorem 1. The case $b = 1$ follows from [3] as mentioned in Sect. 1. To prove the case $b = 2$, we employ the fact that the q-series in this case is essentially a modular form. To be more precise, we have that

$$f_2(t) = -q^{-1}\left(g\left(\frac{i\theta}{2\pi}\right) - 1\right),$$

where we have adopted the notation $\theta = \log(\frac{1+t}{1-t})$ from [2]. The function $g(\tau)$, $\tau \in \mathbb{H}$, is the modular form defined by

$$g(\tau) := \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi i n^2 \tau} = \frac{\eta(\tau)^2}{\eta(2\tau)},$$

where $\eta(\tau) := e^{\frac{2\pi i \tau}{24}} \prod_{n \geq 1}(1 - e^{2\pi i n \tau})$ is Dedekind's eta-function, a well-known modular form of weight $1/2$. Employing the modular transformation of η (see, e.g., [4])

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

we obtain that

$$g\left(\frac{i\theta}{2\pi}\right) \to 0$$

as $\theta \to 0^+$ and thus as $t \to 0^+$. Therefore,

$$f_2(t) \sim q^{-1} = \frac{1+t}{1-t},$$

which clearly has positive coefficients in its t-expansion.

We now assume that Theorem 1 holds for some $b - 1 \geq 1$ and prove that it also holds for $b + 1$. (The first two inductive cases $b - 1 = 1, 2$ are proven above.) We use (2.1) and split

$$q^{-b} = (1+t)^b(1-t)^{-b}.$$

Since the t-expansion of $(1+t)^b$ is finite and contains only positive coefficients, it suffices to prove that the t-coefficients in the asymptotic expansion of

$$-(1-t)^{-b}(f_{b-1}(t) - 2)$$

eventually all have the same sign. We will address the fact that the sign depends on the residue class of $b \pmod 4$ as stated in Theorem 1 later.

It is easy to show that

$$(1-t)^{-b} = \sum_{j=0}^{\infty} \binom{b-1+j}{b-1} t^j.$$

By induction, we may assume without loss of generality that

$$f_{b-1}(t) - 2 \sim \sum_{n \geq 0} \alpha_n t^n$$

as $t \to 0^+$, where $\alpha_n > 0$ for $n \geq L$, for some $L \geq 0$. Set

$$\beta_n := \binom{b-1+n}{b-1}.$$

It suffices to show that

$$\sum_{0 \le n \le m} \alpha_n \beta_{m-n} \tag{2.2}$$

is positive for $m \gg 0$. We break the sum (2.2) into two parts

$$\Sigma_1 := \sum_{0 \le n \le L-1} \alpha_n \beta_{m-n}, \qquad \Sigma_2 := \sum_{L \le n \le m} \alpha_n \beta_{m-n},$$

where we assume that $m \gg 0$ is sufficiently large to ensure $m \ge L$.

Since L is independent of m and the β_j's are monotonically decreasing (in j, for fixed b), we may bound

$$\Sigma_1 \ll \beta_m \ll m^{b-1}$$

as $m \to \infty$. Similarly, Σ_2 may be estimated from below by

$$\Sigma_2 \gg \sum_{L \le n \le m} \beta_{m-n} = \sum_{0 \le n \le m-L} \beta_n \gg \sum_{1 \le n \le m} n^{b-1} \gg \int_0^m x^{b-1} dx \gg m^b$$

as $m \to \infty$. Since clearly $m^b \gg m^{b-1}$, the positivity of Σ_2 dominates, and we have that the coefficients in the asymptotic t-expansion of $f_{b+1}(t)$ have, for sufficiently large m, the same sign as claimed.

The more precise claim that the coefficients in the asymptotic t-expansion for $f_b(t)$ are eventually positive for $b \equiv 1, 2 \pmod 4$ and eventually negative for $b \equiv 0, 3 \pmod 4$ follows easily by induction using (2.1) and the previously established facts that the coefficients in the asymptotic t-expansion of $f_1(t)$ and $f_2(t)$ are eventually positive.

References

1. Berndt, B.: Ramanujan's Notebooks, Part V. Springer, New York (1998)
2. Berndt, B., Kim, B.: Asymptotic expansions of certain partial theta functions. Proc. Am. Math. Soc. **139**, 3779–3788 (2011)
3. Galway, W.: An asymptotic expansion of Ramanujan. In: Number Theory (Ottawa, ON, 1996). CRM Proc. Lecture Notes, vol. 19, pp. 107–110. Amer. Math. Soc, Providence (1999)
4. Rademacher, H.: Topics in Analytic Number Theory. Die Grundlehren der math. Wiss., vol. 169. Springer, Berlin (1973)
5. Stanley, R.: A survey of alternating permutations. Contemp. Math. **531**, 165–196 (2010)