

Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters

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Motivated by recent developments in the representation theory of vertex algebras and conformal field theory, we prove several asymptotic results for partial and false theta functions arising from Jacobi forms, as the modular variable τ tends to 0 along the imaginary axis, and the elliptic variable z is unrestricted in the complex plane. We observe that these functions exhibit Stokes' phenomenon—the asymptotic behavior of these functions sharply differs depending on where the elliptic variable z is located within the complex plane. We apply our results to study the asymptotic expansions of regularized characters and quantum dimensions of the $(1, p)$ -singlet W -algebra modules important in logarithmic conformal field theory. This, in particular, recovers and extends several results from the work of T. Creutzig *et al.* [Int. Math. Res. Not. (2016); e-print [arXiv:1411.3282](https://arxiv.org/abs/1411.3282)] pertaining to regularized quantum dimensions. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4973634>]

I. INTRODUCTION

Partial and false theta functions have recently appeared in several areas of number theory, quantum topology, and representation theory. In this section, we briefly discuss recent advances in these areas related to the main objects of study. We also outline our main results.

A. Asymptotics of Jacobi partial and false theta functions

The (Jacobi) *partial theta functions* are defined by

$$F_{d,\ell}(z; \tau) := \sum_{n \geq 0} \zeta^{\ell n + d} q^{(\ell n + d)^2}, \quad (1.1)$$

for $d \in \mathbb{Q}$ and $\ell \in \mathbb{N}$. Here and throughout, we let $\zeta := e^{2\pi iz}$ and $q := e^{2\pi i\tau}$, with $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, the complex upper half-plane. These functions are aptly named; we find weight $1/2$ modular Jacobi theta functions (up to suitable changes of variables) if the sum in (1.1) is extended to be over the full lattice of integers. Despite their non-modularity, functions like $F_{d,\ell}$ and their specializations have a rich history. In particular, they are known to play fundamental roles within the theory of q -hypergeometric series and integer partitions in number theory, dating back to the time of Rogers and Ramanujan, and continuing into the present day.^{1,3,6,17} More recently, we have begun to understand partial theta functions within the theory of modular forms, as they are connected to mock modular forms and also quantum modular forms.^{18,23,28} Specializations of the partial theta functions $F_{d,\ell}$ are also intimately related to Eichler integrals of modular forms.^{10,27,28} Outside of number theory, partial theta functions appear in connection to topological invariants of 3 manifolds,^{21,22,27} as

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generating functions for colored Jones polynomials for alternating knots,²⁰ and in the representation theory of vertex algebras, the last of which we elaborate upon below.

In all of the above aspects, it is important to understand the asymptotic properties of partial theta functions. For example, in his second notebook [Ref. 4 [p. 324]], Ramanujan claimed an asymptotic expansion for the partial theta function

$$2 \sum_{n \geq 0} (-1)^n q^{n^2+n} = 1 + T + T^2 + 2T^3 + 5T^4 + \dots$$

with $q =: \frac{1-T}{1+T}$, as $T \rightarrow 0^+$, combinatorial properties and generalizations of which have been studied in Refs. 5, 19, and 25.

Similar expansions have been important within the theory of quantum modular forms, for example, due to the work of Zagier,²⁷ the Eichler integrals of weight k cusp forms $g(\tau) = \sum_{n \geq 1} a_g(n)q^n$, close relatives to partial theta functions, are known to satisfy ($N \in \mathbb{N}_0$) (see also Ref. 10),

$$\sum_{n \geq 1} a_g(n)n^{1-k}q^n = \sum_{n=0}^N \frac{(-x)^n}{n!} L_g \left(e^{\frac{2\pi i \ell}{m}}; k-1-n \right) + O(x^{N+1}) \tag{1.2}$$

with $\tau = \frac{\ell}{m} + \frac{ix}{2\pi}$, $\frac{\ell}{m} \in \mathbb{Q}$, as $x \rightarrow 0^+$, where the asymptotic coefficients in (1.2) are given in terms of twisted L -values. The asymptotic properties of the partial theta functions

$$\phi_m^{(a)}(\tau) := m \sum_{n \geq 0} \chi_{2m}^{(a)}(n)q^{\frac{n^2}{4m}},$$

where $\chi_{2m}^{(a)}$ are certain characters, were similarly studied in Refs. 16, 21, 22, and 27, in connection to topological invariants of 3 manifolds, representation theory, and quantum modular forms.

The typical situation involves partial theta functions obtained by specializing the variable z in (1.1) to be a point in $\mathbb{Q}\tau + \mathbb{Q}$. This produces a one-variable function in τ , and the asymptotics of these resulting functions are studied as $\tau \rightarrow 0$. Our first set of results, which are both of independent interest and of interest in representation theory as we discuss below, gives asymptotic expansions for the two-variable Jacobi partial theta functions $F_{d,\ell}(z; it)$ for any $z \in \mathbb{C}$ as $t \rightarrow 0^+$. These functions exhibit Stokes' phenomenon, in that their asymptotic properties sharply differ depending on where in the complex plane z lies. To describe this, we write $z = (z_0 + j)/\ell$, where $j \in \mathbb{Z}$, $z_0 = x_0 + iy_0$, with $x_0, y_0 \in \mathbb{R}$ and $-1/2 < x_0 \leq 1/2$. Our results are also phrased using the differential operator $\mathcal{D}_z := \frac{1}{2\pi i} \frac{\partial}{\partial z}$ and the Bernoulli polynomials $B_n(x)$. We note that some special cases of asymptotic expansions of $F_{d,\ell}$ have already been investigated in the literature. For instance, the authors in Ref. 7 obtained its asymptotic expansion formula in the $|z| < \frac{1}{4\ell}$ region for $d \in \mathbb{Q}^+$.

Theorem 1.1. *We have the following behavior as $t \rightarrow 0^+$ for any $N \in \mathbb{N}_0$.*

(i) *If $\text{Im}(z) > 0$ or ($\text{Im}(z) < 0$ and $|x_0| > |y_0|$), or $z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z}$, then*

$$F_{d,\ell}(z; it) = \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1-\zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

(ii) *If $\text{Im}(z) < 0$ and $|x_0| \leq |y_0|$, then*

$$F_{d,\ell}(z; it) = (2\ell^2 t)^{-\frac{1}{2}} e^{\frac{2\pi i j d}{\ell} - \frac{\pi z_0^2}{2\ell^2 t}} \sum_{\substack{n \in \mathbb{Z} \\ |n+x_0| \leq |y_0|}} e^{-\frac{\pi n^2}{2\ell^2 t} - \frac{\pi z_0 n}{\ell^2 t} - \frac{2\pi i n d}{\ell}} + \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1-\zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

(iii) *If $z \in \frac{1}{\ell}\mathbb{Z}$, then*

$$F_{d,\ell}(z; it) = \frac{1}{2} (2\ell^2 t)^{-\frac{1}{2}} \zeta^d - \zeta^d \sum_{a=0}^N \frac{B_{2a+1}(\frac{d}{\ell})}{2a+1} \frac{(-2\pi \ell^2 t)^a}{a!} + O(t^{N+1}).$$

As a corollary, we deduce the asymptotic behavior of the Jacobi partial theta functions $F_{d,\ell}$, which are again dependent on the location of $z \in \mathbb{C}$.

Corollary 1.2. We have the following behavior, as $t \rightarrow 0^+$.

(i) If $\text{Im}(z) > 0$ or $(\text{Im}(z) < 0 \text{ and } |x_0| > |y_0|)$ or $(z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z})$, then

$$F_{d,\ell}(z; it) \sim \frac{\zeta^d}{1 - \zeta^\ell}.$$

(ii) If $\text{Im}(z) < 0$, $|x_0| \leq |y_0|$, and $x_0 \neq 1/2$, then

$$F_{d,\ell}(z; it) \sim (2\ell^2 t)^{-\frac{1}{2}} e^{\frac{2\pi i j d}{\ell} - \frac{\pi z_0^2}{2\ell^2 t}}.$$

(iii) If $\text{Im}(z) < 0$, $|x_0| \leq |y_0|$, and $x_0 = 1/2$, then

$$F_{d,\ell}(z; it) \sim 2(2\ell^2 t)^{-\frac{1}{2}} \cos\left(\frac{\pi}{\ell} \left(d + \frac{y_0}{2\ell t}\right)\right) e^{-\frac{\pi}{8\ell^2 t} + \frac{\pi y_0^2}{2\ell^2 t} + \frac{\pi i(2j+1)d}{\ell}}.$$

(iv) If $z \in \frac{1}{\ell}\mathbb{Z}$, then

$$F_{d,\ell}(z; it) \sim \frac{\zeta^d}{2\ell(2t)^{\frac{1}{2}}}.$$

False theta functions, similar to partial theta functions, are usually defined as sums over a full lattice, but sign changes are made in the summands so that modularity properties are lost. These functions also enjoy a rich history within the theory of q -series and integer partitions, and are often intertwined with partial theta functions.² A typical example of a false theta function is

$$\sum_{n \in \mathbb{Z}} \text{sgn}(n) q^{(\ell n + d)^2},$$

where $\text{sgn}(x) := 1$ for $x \geq 0$, and $\text{sgn}(x) := -1$ for $x < 0$, which may also be viewed as a difference of two partial theta functions. In this paper, we are particularly interested in the following difference of two Jacobi partial theta functions

$$G_{d,\ell}(z; \tau) := F_{d,\ell}(z; \tau) - F_{-d,\ell}(z; \tau).$$

Observe that $G_{d,\ell}(0; \tau) + q^{d^2}$ is a false theta function as above. In Section V (Theorem 5.1 and Corollary 5.2), we establish asymptotic expansions and asymptotic behavior of the functions $G_{d,\ell}$ analogous to Theorem 1.1 and Corollary 1.2. All of these results are of independent interest and also have consequences in representation theory, which we will now discuss.

B. Partial and false theta functions in representation theory

Partial and false theta functions have recently appeared in the representation theory of vertex algebras in the study of *regularized* characters of $(1, p)$ -singlet W -algebra^{8,12,13,16} ($p \geq 2$). Previously, in Ref. 23, it was observed that the usual characters $\text{ch}[M_{r,s}](\tau)$ of atypical modules $M_{r,s}$ ($M_{1,1}$ the singlet vertex algebra) can be written as quotients of differences of two partial theta series and the Dedekind η -function. These characters are interesting from several different standpoints. For example, they admit elegant representations as multi-hypergeometric q -series and are also quantum modular forms²⁸ with quantum set \mathbb{Q} (for details see Ref. 8). In Ref. 13, motivated by developments surrounding the Verlinde formula in conformal field theory, atypical and typical characters are regularized by using a new complex parameter ε , which can be also viewed as the $U(1)$ -charge in physics. The resulting expression, denoted by $\text{ch}[M_{r,s}^\varepsilon](\tau)$ ($r \in \mathbb{Z}$, $1 \leq s \leq p - 1$), has in the numerator a difference of two partial theta functions discussed earlier. This numerator, denoted in Section VI by $C_{r,s}(\varepsilon; \tau)$, gives rise to a false theta function. Indeed, if specialized at $\varepsilon = 0$, $C_{1,s}$ is a false theta function as above, while for $r \neq 1$, $C_{r,s}$ can be written as the sum of a false theta function and a finite q -series.⁸ Interestingly, the regularized characters admit a certain

modular-type transformation formula if $\varepsilon \notin i\mathbb{R}$.¹³ This fact was instrumental for proving the Verlinde formula of characters,^{13,16} which is conjecturally isomorphic to the Grothendieck ring of the category of modules for the singlet vertex algebra. It is known that ordinary fusion ring admits one-dimensional representation coming from quantum dimensions; thus in Ref. 13, the regularized quantum dimension of M was defined as

$$\text{qdim}(M^\varepsilon) := \lim_{t \rightarrow 0^+} \frac{\text{ch}[M^\varepsilon](it)}{\text{ch}[V^\varepsilon](it)}, \tag{1.3}$$

now depending on ε . Without going into details, we only mention that regularized quantum dimensions define a representation of the Verlinde algebra of characters. Very recently, in Ref. 16, regularized quantum dimensions of irreducible modules for the singlet algebras were computed on a certain subset of the ε -plane. It was observed that quantum dimension have peculiar properties in different regions on the ε -plane, roughly corresponding to $\text{Re}(\varepsilon) > 0$ and $\text{Re}(\varepsilon) < 0$.

In this paper, we extend and generalize several results from Ref. 16. First, in Theorem 6.2, we determine explicit (full) asymptotic expansions of all regularized irreducible characters simply as corollaries to the more general asymptotic formulas for the Jacobi partial and false theta functions (Theorem 1.1, Corollary 1.2, Theorem 5.1, and Corollary 5.2). These results immediately imply several properties observed earlier in Ref. 16, including Stokes' phenomenon. Moreover, we extend known formulas for regularized quantum dimensions in Ref. 16 to the whole ε -plane including the imaginary axis. To describe this, we write $\varepsilon = (\varepsilon_0 + ik)/\sqrt{2p}$, with $k \in \mathbb{Z}$, $\varepsilon_0 = u_0 + iv_0$ with $u_0, v_0 \in \mathbb{R}$, and $-1/2 < v_0 \leq 1/2$.

Our next result gives formulas for regularized quantum dimensions—as defined in (1.3)—of the $(1, p)$ -singlet algebra modules, both typical and atypical. We point out that explicit formulas for $\text{ch}[F_\lambda^\varepsilon](\tau)$ and $\text{ch}[M_{r,s}^\varepsilon](\tau)$ are given in (6.1) and (6.2), respectively. As in Theorem 1.1 and Corollary 1.2, Theorem 1.3 exhibits Stokes' phenomenon.

Theorem 1.3. *Assume the notation and hypotheses above.*

- (i) *If one of the following is true: $\text{Re}(\varepsilon) < 0$ or $(\text{Re}(\varepsilon) > 0$ and $|v_0| > |u_0|$) or $(\text{Re}(\varepsilon) > 0, p|k, |u_0| < 1 - |v_0|, \text{ and } v_0 \neq 1/2)$ or $(u_0 = 0 \text{ and } v_0 \neq 0)$, then*

$$\text{qdim}[M_{r,s}^\varepsilon] = e^{\pi\varepsilon\sqrt{2p}(1-r)} \frac{\sinh\left(\frac{\sqrt{2}\pi s\varepsilon}{\sqrt{p}}\right)}{\sinh\left(\frac{\sqrt{2}\pi\varepsilon}{\sqrt{p}}\right)}, \tag{1.4}$$

$$\text{qdim}[F_\lambda^\varepsilon] = e^{2\pi\varepsilon\left(\lambda - \sqrt{\frac{p}{2}} + \sqrt{\frac{1}{2p}}\right)} \frac{\sinh\left(\frac{\sqrt{2p}\pi\varepsilon}{\sqrt{p}}\right)}{\sinh\left(\frac{\sqrt{2}\pi\varepsilon}{\sqrt{p}}\right)}. \tag{1.5}$$

- (ii) *If $\text{Re}(\varepsilon) > 0, p \nmid k, |v_0| \leq |u_0|$, and $v_0 \neq 1/2$, then*

$$\text{qdim}[M_{r,s}^\varepsilon] = (-1)^{k(r+1)} \frac{\sin\left(\frac{\pi ks}{p}\right)}{\sin\left(\frac{\pi k}{p}\right)}, \quad \text{qdim}[F_\lambda^\varepsilon] = 0.$$

- (iii) *If $\text{Re}(\varepsilon) > 0, p|k, 1 - |u_0| \leq |v_0| \leq |u_0|$, and $v_0 \neq 1/2$, then*

$$\text{qdim}[M_{r,s}^\varepsilon] = (-1)^{(r+1)(k+1) + \frac{k(s+1)}{p}} \frac{\sin\left(\frac{\pi s}{p}\right)}{\sin\left(\frac{\pi}{p}\right)}, \quad \text{qdim}[F_\lambda^\varepsilon] = 0.$$

- (iv) *If $\text{Re}(\varepsilon) > 0, v_0 = 1/2$, and $|u_0| \geq 1/2$, then*

$$\text{qdim}[F_\lambda^\varepsilon] = 0,$$

and $\text{qdim}[M_{r,s}^\varepsilon]$ exists if and only if

$$\tan\left(\frac{\pi s}{2p} + \frac{\pi r}{2}\right) \tan\left(\frac{\pi}{2p}\right) = \tan\left(\frac{\pi(2k+1)s}{2p} + \frac{\pi r}{2}\right) \tan\left(\frac{\pi(2k+1)}{2p}\right).$$

If this condition is satisfied, then

$$\text{qdim}[M_{r,s}] = \frac{(-1)^{(r+1)k+1} \sin\left(\frac{\pi s}{2p} + \frac{\pi r}{2}\right) \cos\left(\frac{\pi}{2p}(2k+1)s + \frac{\pi r}{2}\right)}{\cos\left(\frac{\pi}{2p}\right) \sin\left(\frac{\pi}{2p}(2k+1)\right)}.$$

(v) If $\varepsilon_0 = 0$ and $p \nmid ks$, then

$$\text{qdim}[M_{r,s}^\varepsilon] = (-1)^{k(r+1)} \frac{\sin\left(\frac{\pi ks}{p}\right)}{\sin\left(\frac{\pi k}{p}\right)}, \quad \text{qdim}[F_\lambda^\varepsilon] = 0.$$

(vi) If $\varepsilon_0 = 0$, $p \mid ks$, then

(a) if $k \notin p\mathbb{Z}$, then

$$\text{qdim}[M_{r,s}^\varepsilon] = 0, \quad \text{qdim}[F_\lambda^\varepsilon] = 0;$$

(b) if $k \in p(2\mathbb{Z})$, then

$$\text{qdim}[M_{r,s}^\varepsilon] = s, \quad \text{qdim}[F_\lambda^\varepsilon] = e^{\frac{2\pi i k \lambda}{\sqrt{2p}}} p;$$

(c) if $k \in p(2\mathbb{Z} + 1)$, then

$$\text{qdim}[M_{r,s}^\varepsilon] = (-1)^{(s+1)+p(r+1)} s, \quad \text{qdim}[F_\lambda^\varepsilon] = e^{\frac{2\pi i k \lambda}{\sqrt{2p}}} p.$$

Remarks.

- (i) Parts (i) (not including the $u_0 = 0, v_0 \neq 0$ region), (ii), and (iii) of Theorem 1.3 were proven in Ref. 16 by using a different method.
- (ii) Part (vi) of Theorem 1.3 for $k = 0$ was given in Ref. 13.
- (iii) In part (iv), there are several choices of parameters, some of which are dependent on k , which yield a solution. For instance, if r is odd and $s = 1$, then the quantum dimension of $M_{r,s}$ is 1, and if r is even and $s = p - 1$, then the quantum dimension is -1 .
- (iv) Quantum dimensions in Theorem 1.3 can be interpreted as modified Hopf link invariants associated with the unrolled quantum group $\bar{U}_\mu^H(\mathfrak{sl}_2)$, $\mu = e^{\pi i/p}$.^{11,15}

This paper is structured as follows. In Section II, we introduce basic functions and prove several technical lemmas. In Sections III and IV, we determine asymptotic properties of the partial theta functions $F_{d,\ell}$ and $G_{d,\ell}$ for $\text{Im}(z) \neq 0$ and $\text{Im}(z) = 0$, respectively. In particular, their asymptotic behavior is stated in Corollaries 3.4, 3.5, and 4.7. In Section V, we prove Theorem 1.1 and Corollary 1.2, and establish the analogous results for the functions $G_{d,\ell}$ in Theorem 5.1 and Corollary 5.2. Finally, in Section VI, we prove Theorem 1.3 and establish related asymptotic expansions in Theorem 6.2 and Corollary 6.3.

II. AUXILIARY FUNCTIONS

In this section, we provide some preliminary results on certain functions required to prove our main results. Namely, we establish the asymptotic properties of some functions defined using the error function, we give some transformation properties for various theta functions, and we establish expansions for some special functions.

A. Properties of error functions

The error function erf is defined, for $w \in \mathbb{C}$, by

$$\text{erf}(w) := \frac{2}{\sqrt{\pi}} \int_0^w e^{-t^2} dt.$$

We establish asymptotic properties of certain functions defined using erf below; in doing so, we make use of the following expansions [Ref. 26, 7.6.2, 7.12 (i)] of erf and the complementary error

function $\operatorname{erfc}(w) := 1 - \operatorname{erf}(w)$, the first of which converges for any $w \in \mathbb{C}$, and the second of which holds for any $N \in \mathbb{N}_0$, as $w \rightarrow 0$, for $|\operatorname{Arg}(w)| < 3\pi/4$,

$$\operatorname{erf}(w) = e^{-w^2} \sum_{n \geq 1} \frac{w^{2n-1}}{\Gamma(n + \frac{1}{2})}, \quad (2.1)$$

$$\operatorname{erfc}(w) = \frac{e^{-w^2}}{\sqrt{\pi}} \sum_{m=0}^N \frac{(-1)^m (\frac{1}{2})_m}{w^{2m+1}} + O(w^{-2N-3}). \quad (2.2)$$

Lemma 2.1. For $w \in \mathbb{C}$, we have that

$$\sum_{n \geq 0} \frac{(2w)^n \Gamma(\frac{n+1}{2})}{n!} = \sqrt{\pi} e^{w^2} (1 + \operatorname{erf}(w)). \quad (2.3)$$

Proof. The statement follows by splitting the sum in (2.3) into even and odd terms, using the Taylor expansion for e^{w^2} , as well as (2.1). \square

Next, we define the function

$$F(t, w) := \frac{w}{\sqrt{t}} e^{\frac{w^2}{t}} \left(1 + \operatorname{erf}\left(\frac{w}{\sqrt{t}}\right) \right),$$

and determine its asymptotic behavior in Lemma 2.2.

Lemma 2.2. We have the following asymptotic behavior, as $t \rightarrow 0^+$.

(i) If $|\operatorname{Arg}(w)| \leq \pi/4$, then

$$F(t, w) - \frac{2w}{\sqrt{t}} e^{\frac{w^2}{t}} = -\frac{1}{\sqrt{\pi}} + O(t).$$

(ii) If $|\operatorname{Arg}(w)| > \pi/4$, then

$$F(t, w) = -\frac{1}{\sqrt{\pi}} \left(1 - \frac{t}{2w^2} \right) + O(t^2).$$

Proof. (i) If $|\operatorname{Arg}(w)| \leq \pi/4$, then the claim follows directly from (2.2).

(ii) If $|\operatorname{Arg}(w)| > \pi/4$, then $|\operatorname{Arg}(-w)| < 3\pi/4$ and the claim follows similarly. \square

Remark. The change in the behavior of the asymptotic expansion of $F(t, w)$ across the boundary $|\operatorname{Arg}(w)| = \pi/4$ is an example of Stokes' phenomenon. The lines $\operatorname{Arg}(w) = \pm\pi/4$ are called *anti-Stokes* lines.

B. Jacobi and partial theta functions

In this section, we provide a transformation property for the Jacobi theta function

$$\Theta(z; \tau) := \sum_{n \in \mathbb{Z}} (-1)^n \zeta^n q^{n^2}, \quad (2.4)$$

and also establish a shifting property of the partial theta functions $F_{d,\ell}$.

The function Θ satisfies the following well-known modular transformation property (see Ref. 24, Chapter 1), which we make use of,

$$\Theta(z; \tau) = (-2i\tau)^{-\frac{1}{2}} \sum_{n \in 1+2\mathbb{Z}} e^{-\frac{\pi i}{8\tau}(n+2z)^2}. \quad (2.5)$$

A direct calculation yields the following shifting property for the functions $F_{d,\ell}$.

Lemma 2.3. For $m \in \mathbb{Z}$, we have that

$$F_{d+m\ell,\ell}(z; \tau) = F_{d,\ell}(z; \tau) - \sum_{a \geq 0} \mathcal{D}_z^{2a} \left(\frac{\zeta^d (1 - \zeta^{\ell m})}{1 - \zeta^\ell} \right) \frac{(2\pi i \tau)^a}{a!}.$$

In Section III, we require the expansions of certain derivative functions similar to those appearing above. Using the Bernoulli polynomial generating function (see 24.2.3 in Ref. 26), we establish the following lemma.

Lemma 2.4. For $z \in \mathbb{C} \setminus \{0\}$, and $a \in \mathbb{N}_0$, we have that

$$\mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) = -\ell^{2a} \left(\frac{(2a)!}{(2\pi i \ell z)^{2a+1}} + \sum_{b \geq 0} \frac{(2\pi i \ell z)^b B_{2a+b+1} \left(\frac{d}{\ell} \right)}{b!(2a+b+1)} \right). \tag{2.6}$$

Moreover, for $a \in \mathbb{N}_0$, we have that

$$\lim_{z \rightarrow 0} \left(\mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) + \frac{(2a)!}{\ell(2\pi i z)^{2a+1}} \right) = -\ell^{2a} \frac{B_{2a+1} \left(\frac{d}{\ell} \right)}{2a+1}. \tag{2.7}$$

III. ASYMPTOTIC EXPANSIONS OF $F_{d,\ell}$ AND $G_{d,\ell}$ IF $\text{Im}(z) \neq 0$

In this section, we establish the asymptotic expansions of the partial theta functions $F_{d,\ell}$ and the functions $G_{d,\ell}$, if $\text{Im}(z) \neq 0$, making use of some of the results established in Section II. In Section III A, we consider the case $\text{Im}(z) > 0$, and in Section III B, we treat the case $\text{Im}(z) < 0$. Recall that we write $z = (z_0 + j)/\ell$, where $j \in \mathbb{Z}$, $z_0 = x_0 + iy_0$, with $x_0, y_0 \in \mathbb{R}$ and $-1/2 < x_0 \leq 1/2$.

A. $\text{Im}(z) > 0$

In Lemma 3.1, we establish the asymptotic expansions of the functions $F_{d,\ell}$ and $G_{d,\ell}$ in the case $\text{Im}(z) > 0$.

Lemma 3.1. We have, for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ with $\text{Im}(z) > 0$,

$$\begin{aligned} F_{d,\ell}(z; \tau) &= \sum_{a \geq 0} \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) \frac{(2\pi i \tau)^a}{a!}, \\ G_{d,\ell}(z; \tau) &= 2i \sum_{a \geq 0} \mathcal{D}_z^{2a} \left(\frac{\sin(2\pi dz)}{1 - \zeta^\ell} \right) \frac{(2\pi i \tau)^a}{a!}. \end{aligned}$$

Proof. Using the definition of the functions $F_{d,\ell}$, we have

$$F_{d,\ell}(z; \tau) = \sum_{a \geq 0} \frac{(2\pi i \tau)^a}{a!} \sum_{n \geq 0} (\ell n + d)^{2a} \zeta^{\ell n + d} = \sum_{a \geq 0} \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) \frac{(2\pi i \tau)^a}{a!}. \tag{3.1}$$

This establishes the claimed result for the functions $F_{d,\ell}$. The statement for $G_{d,\ell}$ then follows from (3.1) and the definition of the functions $G_{d,\ell}$. \square

From Lemma 3.1 we deduce the following asymptotic behavior of the functions $F_{d,\ell}$ and $G_{d,\ell}$.

Corollary 3.2. For $\text{Im}(z) > 0$, as $t \rightarrow 0^+$, we have that

$$F_{d,\ell}(z; it) \sim \frac{\zeta^d}{1 - \zeta^\ell}, \quad G_{d,\ell}(z; it) \sim \frac{2i \sin(2\pi dz)}{1 - \zeta^\ell}.$$

B. $\text{Im}(z) < 0$

In Lemma 3.3, we establish the asymptotic expansions of the functions $F_{d,\ell}$ and $G_{d,\ell}$ if $\text{Im}(z) < 0$.

Lemma 3.3. For $z \in \mathbb{C}$ with $\text{Im}(z) < 0$, and $N \in \mathbb{N}_0$, as $t \rightarrow 0^+$, we have that

$$F_{d,\ell}(z; it) = (2\ell^2 t)^{-\frac{1}{2}} e^{\frac{2\pi i j d}{\ell} - \frac{\pi z_0^2}{2\ell^2 t}} \sum_{\substack{n \in \mathbb{Z} \\ |n+x_0| \leq |y_0|}} e^{-\frac{\pi n^2}{2\ell^2 t} - \frac{\pi z_0 n}{\ell^2 t} - \frac{2\pi i n d}{\ell}} + \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}), \tag{3.2}$$

$$G_{d,\ell}(z; it) = 2i(2\ell^2 t)^{-\frac{1}{2}} e^{-\frac{\pi z_0^2}{2\ell^2 t}} \sum_{\substack{n \in \mathbb{Z} \\ |n+x_0| \leq |y_0|}} \sin\left(\frac{2\pi d}{\ell}(j-n)\right) e^{-\frac{\pi n^2}{2\ell^2 t} - \frac{\pi z_0 n}{\ell^2 t}} + 2i \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\sin(2\pi d z)}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}). \tag{3.3}$$

Proof. With

$$\mathcal{M}_{d,\ell}(z; \tau) := \sum_{n \in \mathbb{Z}} \zeta^{\ell n+d} q^{(\ell n+d)^2},$$

we have that

$$F_{d,\ell}(z; \tau) = \mathcal{M}_{d,\ell}(z; \tau) - F_{\ell-d,\ell}(-z; \tau). \tag{3.4}$$

We analyze the two functions in (3.4) separately.

Since $\text{Im}(-z) > 0$, Lemma 3.1 yields that

$$-F_{\ell-d,\ell}(-z; \tau) = \sum_{a \geq 0} \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) \frac{(2\pi i \tau)^a}{a!}. \tag{3.5}$$

Next, we write the function $\mathcal{M}_{d,\ell}$ in terms of the theta function in (2.4) and use (2.5) to obtain

$$\mathcal{M}_{d,\ell}(z; it) = (2\ell^2 t)^{-\frac{1}{2}} e^{\frac{2\pi i j d}{\ell} - \frac{\pi z_0^2}{2\ell^2 t}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi n^2}{2\ell^2 t} - \frac{\pi z_0 n}{\ell^2 t} - \frac{2\pi i n d}{\ell}}. \tag{3.6}$$

Inserting (3.5) and (3.6) into (3.4), we have shown that

$$F_{d,\ell}(z; it) = \sum_{a \geq 0} \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + (2\ell^2 t)^{-\frac{1}{2}} e^{\frac{2\pi i j d}{\ell} - \frac{\pi z_0^2}{2\ell^2 t}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi n^2}{2\ell^2 t} - \frac{\pi z_0 n}{\ell^2 t} - \frac{2\pi i n d}{\ell}}.$$

The claimed expansion for $F_{d,\ell}$ now follows by ignoring the exponentially small terms.

To prove the result for $G_{d,\ell}$, we use the result just established for $F_{d,\ell}$ together with the definition of $G_{d,\ell}$. □

Using Lemma 3.3, we establish the asymptotic behavior of the partial theta functions $F_{d,\ell}$ for $\text{Im}(z) < 0$ in Corollary 3.4. Unlike the previous case in which $\text{Im}(z) > 0$, a more careful analysis is required.

Corollary 3.4. We have the following behavior for $\text{Im}(z) < 0$, as $t \rightarrow 0^+$.

(i) If $|x_0| > |y_0|$, then

$$F_{d,\ell}(z; it) \sim \frac{\zeta^d}{1 - \zeta^\ell}.$$

(ii) If $|x_0| \leq |y_0|$ and $x_0 \neq 1/2$, then

$$F_{d,\ell}(z; it) \sim (2\ell^2 t)^{-\frac{1}{2}} e^{\frac{2\pi i j d}{\ell} - \frac{\pi z_0^2}{2\ell^2 t}}.$$

(iii) If $x_0 = 1/2$ and $|y_0| \geq 1/2$, then

$$F_{d,\ell}(z; it) \sim 2(2\ell^2 t)^{-\frac{1}{2}} \cos\left(\frac{\pi}{\ell} \left(d + \frac{y_0}{2\ell t}\right)\right) e^{-\frac{\pi}{8\ell^2 t} + \frac{\pi y_0^2}{2\ell^2 t} + \frac{\pi i(2j+1)d}{\ell}}.$$

Proof. One can see that the dominant term in the sum in (3.2) occurs for $n = 0$, and additionally for $n = -1$ if $x_0 = 1/2$. Moreover, we obtain a contribution exactly if $|x_0| > |y_0|$. This yields (i).

We next assume $|x_0| \leq |y_0|$. One directly obtains (ii) if $x_0 \neq 1/2$. If $x_0 = 1/2$, we simplify the exponent of the $n = -1$ term in the sum in (3.2) to give the claim (iii). \square

Remark. Lemma 2.2 together with results in Section IV (Lemma 4.1 and Corollary 4.2) gives another method for determining asymptotic behaviors and expansions of the functions $F_{d,\ell}$ in a certain subset of $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$. This provides another “explanation” of the anti-Stokes lines $|u_0| = |v_0|$.

Finally, we establish the asymptotic main terms of the functions $G_{d,\ell}$ if $\text{Im}(z) < 0$.

Corollary 3.5. We have the following behavior for $\text{Im}(z) < 0$, as $t \rightarrow 0^+$:

- (i) If $|x_0| > |y_0|$ or $(|x_0| \leq |y_0|, \ell \mid 2d, \text{ and } x_0 \neq 1/2)$ or $(\ell \nmid 2dj \text{ but } \ell \nmid 2d, |y_0| < 1 - |x_0|, \text{ and } x_0 \neq 1/2)$, then

$$G_{d,\ell}(z; it) \sim \frac{2i \sin(2\pi dz)}{1 - \zeta^\ell}.$$

- (ii) If $\ell \nmid 2dj, |x_0| \leq |y_0|, \text{ and } x_0 \neq 1/2$, then

$$G_{d,\ell}(z; it) \sim 2i(2\ell^2 t)^{-\frac{1}{2}} \sin\left(\frac{2\pi jd}{\ell}\right) e^{-\frac{\pi z_0^2}{2\ell^2 t}}.$$

- (iii) If $\ell \mid 2dj \text{ but } \ell \nmid 2d, 1 - |y_0| \leq |x_0| \leq |y_0|, \text{ and } x_0 \neq 1/2$, then

$$G_{d,\ell}(z; it) \sim 2i(-1)^{\frac{2dj}{\ell}} (2\ell^2 t)^{-\frac{1}{2}} e^{-\frac{\pi z_0^2}{2\ell^2 t} - \frac{\pi}{2\ell^2 t} + \frac{\pi \text{sgn}(x_0) z_0}{\ell^2 t}} \text{sgn}(x_0) \sin\left(\frac{2\pi d}{\ell}\right).$$

- (iv) If $x_0 = 1/2 \text{ and } |y_0| \geq 1/2$, then

$$G_{d,\ell}(z; it) \sim 2(2\ell^2 t)^{-\frac{1}{2}} e^{-\frac{\pi}{8\ell^2 t} + \frac{\pi y_0^2}{2\ell^2 t}} \sum_{\pm} \pm \cos\left(\frac{\pi}{\ell} \left(d \pm \frac{y_0}{2\ell t}\right)\right) e^{\pm \frac{\pi i(2j+1)d}{\ell}}.$$

Proof. The only thing that has to be considered is that by applying case (ii) of Corollary 3.4 to the two functions $F_{d,\ell}$ and $F_{-d,\ell}$ defining $G_{d,\ell}$, the overall contribution could be 0.

If $\ell \nmid 2dj$ this is not the case. If $\ell \mid 2d$, then we use Lemma 3.3, and find that the entire contribution arising from the first sum on the right-hand side of (3.3) vanishes. If $\ell \mid 2dj \text{ but } \ell \nmid 2d$, then we again use the expansion in Lemma 3.3 and look at the next highest term, which occurs for $n = -\text{sgn}(x_0)$. \square

IV. ASYMPTOTIC EXPANSIONS OF $F_{d,\ell}$ AND $G_{d,\ell}$ IF $\text{Im}(z) = 0$

We are left to consider the asymptotic properties of the partial theta functions $F_{d,\ell}$ and the functions $G_{d,\ell}$ if $\text{Im}(z) = 0$. For this, we begin by establishing a slight generalization of Theorem 4.1 of Ref. 7, namely, Lemma 4.1, which holds for any $z \in \mathbb{C}$ satisfying $|z| < 1/(4\ell)$. From this we ultimately deduce asymptotic expansions of the functions $F_{d,\ell}$ and $G_{d,\ell}$ for real z with $|x_0| < 1/4$ in Corollary 4.3. In Corollary 4.6 we are able to remove the restriction that $|x_0| < 1/4$ and provide asymptotic expansions for any $z \in \mathbb{R}$, after establishing some technical lemmas.

Lemma 4.1. For $|z| < 1/(4\ell), \tau \in \mathbb{H}, N \in \mathbb{N}_0, d \in \mathbb{Q}, \text{ and } \ell \in \mathbb{N}$, we have that

$$F_{d,\ell}(z; \tau) = \sum_{b \geq 0} \frac{(2\pi i \ell z)^b}{b!} \left(\frac{\Gamma\left(\frac{b+1}{2}\right)}{2(-2\pi i \ell^2 \tau)^{\frac{b+1}{2}}} - \sum_{a=0}^N \frac{(2\pi i \ell^2 \tau)^a}{a!} \frac{B_{2a+b+1}\left(\frac{d}{\ell}\right)}{2a+b+1} \right) + O(|\tau|^{N+1}).$$

Remark. For $d > 0$, Lemma 4.1 is given in Ref. 7 as Theorem 4.1.

Proof. Choose any $r \in \mathbb{N}$ such that $d' := r\ell + d > 0$. By Lemma 2.3, we have

$$F_{d,\ell}(z; \tau) = F_{d',\ell}(z; \tau) + \sum_{a \geq 0} \mathcal{D}_z^{2a} \left(\frac{\zeta^d (1 - \zeta^{\ell r})}{1 - \zeta^\ell} \right) \frac{(2\pi i \tau)^a}{a!}. \tag{4.1}$$

For the first term in (4.1), we may use the asymptotic expansion in Lemma 4.1 with d' instead of d , as it is known to be true from Ref. 7. For the second term, we apply Lemma 2.4 twice, giving the claim. \square

We apply Lemma 4.1 to determine the asymptotic expansion of the partial theta functions $F_{d,\ell}$.

Corollary 4.2. For $|z| < 1/(4\ell)$ and $N \in \mathbb{N}_0$, as $t \rightarrow 0^+$, we have that

$$F_{d,\ell}(z; it) = \frac{1}{2\ell(2t)^{\frac{1}{2}}} e^{-\frac{\pi z^2}{2t}} \left(1 + \operatorname{erf} \left(\frac{\sqrt{\pi} iz}{\sqrt{2t}} \right) \right) + \sum_{a=0}^N \left(\mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) + \frac{(2a)!}{\ell(2\pi iz)^{2a+1}} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

Proof. Using Lemma 2.1, the first term in Lemma 4.1 for $\tau = it$ evaluates as the first summand in the corollary. The second term in Lemma 4.1 is

$$- \sum_{a=0}^N \frac{(-2\pi \ell^2 t)^a}{a!} \sum_{b \geq 0} \frac{(2\pi i \ell z)^b}{b!} \frac{B_{2a+b+1}(\frac{d}{\ell})}{2a + b + 1}.$$

The claim follows by applying Lemma 2.4. \square

We next turn to the question of establishing the asymptotic properties of the functions $F_{d,\ell}$ and $G_{d,\ell}$ if $z \in \mathbb{R}$. We first do so in Corollary 4.3 if $|x_0| < 1/4$.

Corollary 4.3. For $z = (x_0 + j)/\ell \in \mathbb{R}$, where $j \in \mathbb{Z}, \ell \in \mathbb{N}$, and $|x_0| < 1/4$, for any $N \in \mathbb{N}_0$, as $t \rightarrow 0^+$, we have

$$F_{d,\ell}(z; it) = \frac{e^{\frac{2\pi i d j}{\ell}}}{2\ell(2t)^{\frac{1}{2}}} e^{-\frac{\pi x_0^2}{2\ell^2 t}} \left(1 + \operatorname{erf} \left(\frac{\sqrt{\pi} i x_0}{\ell \sqrt{2t}} \right) \right) + \sum_{a=0}^N \left(\mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) + e^{\frac{2\pi i d j}{\ell}} \frac{(2a)! \ell^{2a}}{(2\pi i x_0)^{2a+1}} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}),$$

$$G_{d,\ell}(z; it) = \frac{i \sin \left(\frac{2\pi d j}{\ell} \right)}{\ell(2t)^{\frac{1}{2}}} e^{-\frac{\pi x_0^2}{2\ell^2 t}} \left(1 + \operatorname{erf} \left(\frac{\sqrt{\pi} i x_0}{\ell \sqrt{2t}} \right) \right) + 2i \sum_{a=0}^N \left(\mathcal{D}_z^{2a} \left(\frac{\sin(2\pi d z)}{1 - \zeta^\ell} \right) + \frac{\sin \left(\frac{2\pi d j}{\ell} \right) (2a)! \ell^{2a}}{(2\pi i x_0)^{2a+1}} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

Proof. We have

$$F_{d,\ell}(z; \tau) = e^{\frac{2\pi i d j}{\ell}} \sum_{n \geq 0} e^{2\pi i \frac{x_0}{\ell} (\ell n + d)} q^{(\ell n + d)^2} = e^{\frac{2\pi i d j}{\ell}} F_{d,\ell} \left(\frac{x_0}{\ell}; \tau \right).$$

The expansions for the functions $F_{d,\ell}$ and $G_{d,\ell}$ then follow from Corollary 4.2. \square

In order to remove the restriction in Corollary 4.3 that $|x_0| < 1/4$, we begin by establishing Lemma 4.4, which holds for certain real values of z .

Lemma 4.4. If $z = \frac{w_0}{\ell} + \frac{c}{h\ell} \in \mathbb{R}$, $h \geq 2$, $\gcd(c, h) = 1$, and $|w_0| < \frac{1}{4h}$, then we have for any $N \in \mathbb{N}_0$, as $t \rightarrow 0^+$,

$$F_{d,\ell}(z; it) = \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\zeta^d}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}),$$

$$G_{d,\ell}(z; it) = 2i \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\sin(2\pi dz)}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

Proof. We may write

$$F_{d,\ell}(z; \tau) = e^{\frac{2\pi idc}{h\ell}} \sum_{j=0}^{h-1} e^{\frac{2\pi icj}{h}} F_{\ell j+d, \ell h} \left(\frac{w_0}{\ell}; \tau \right). \tag{4.2}$$

By (4.2), Corollary 4.2, and the fact that $\sum_{j=0}^{h-1} e^{\frac{2\pi icj}{h}} = 0$, we obtain

$$\begin{aligned} F_{d,\ell}(z; it) &= -e^{\frac{2\pi idc}{h\ell}} \sum_{j=0}^{h-1} e^{\frac{2\pi icj}{h}} \sum_{a=0}^N \left(\mathcal{D}_{\frac{w_0}{\ell}}^{2a} \left(\frac{e^{2\pi i(\ell j+d)\frac{w_0}{\ell}}}{e^{2\pi i\ell h\frac{w_0}{\ell}} - 1} \right) - \frac{(2a)!}{\ell h (2\pi i\frac{w_0}{\ell})^{2a+1}} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}) \\ &= -\sum_{a=0}^N \mathcal{D}_z^{2a} \left(\sum_{j=0}^{h-1} e^{2\pi i\left(\frac{dc}{h\ell} + \frac{cj}{h}\right)} \frac{e^{2\pi i(\ell j+d)\frac{w_0}{\ell}}}{e^{2\pi ihw_0} - 1} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}) \\ &= -\sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{e^{\frac{2\pi idc}{h\ell} + \frac{2\pi idw_0}{\ell}}}{e^{2\pi ihw_0} - 1} \sum_{j=0}^{h-1} e^{2\pi i\left(\frac{c}{h} + w_0\right)j} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}) \\ &= -\sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{e^{2\pi id\left(\frac{c}{h\ell} + \frac{w_0}{\ell}\right)}}{e^{2\pi i\ell\left(\frac{c}{h\ell} + \frac{w_0}{\ell}\right)} - 1} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}). \end{aligned}$$

From this, we may conclude the claim. □

In order to fully remove the restriction that $|x_0| < 1/4$ in Corollary 4.3, in addition to Lemma 4.4, we need the following technical result.

Lemma 4.5. We have $\Omega = \mathbb{R} \setminus \mathbb{Z}$, where

$$\Omega := \left\{ w_0 + \frac{c}{h} : \frac{c}{h} \in \mathbb{Q} \setminus \{0\}, \gcd(c, h) = 1, h \geq 2, w_0 \in \mathbb{R}, |w_0| < \frac{1}{4h} \right\}.$$

Proof. A short calculation reveals that the sets

$$D_{\frac{c}{h}} := \left\{ w_0 \in \mathbb{R} : \left| w_0 - \frac{c}{h} \right| < \frac{1}{4h} \right\},$$

where $c/h \in \mathbb{Q} \setminus \{0\}$, cover the interval $(0, 1)$, and thus, $\mathbb{R} \setminus \mathbb{Z} \subseteq \Omega$. By considering the cases $0 < w_0 < \frac{1}{4h}$ and $-\frac{1}{4h} < w_0 < 0$ separately, it is not difficult to see that Ω contains no integer. □

We are now able to establish the general asymptotic expansions of the functions $F_{d,\ell}$ and $G_{d,\ell}$ if $\text{Im}(z) = 0$.

Corollary 4.6. For $z \in \mathbb{R}$, as $t \rightarrow 0^+$, for any $N \in \mathbb{N}_0$, the following are true.

(i) If $z \in \frac{1}{\ell}\mathbb{Z}$, then we have that

$$F_{d,\ell}(z; it) = \frac{e^{2\pi idz}}{2\ell(2t)^{\frac{1}{2}}} - e^{2\pi idz} \sum_{a=0}^N \frac{B_{2a+1}\left(\frac{d}{\ell}\right) (-2\pi\ell^2 t)^a}{(2a+1)a!} + O(t^{N+1}),$$

$$G_{d,\ell}(z; it) = \frac{i \sin(2\pi dz)}{\ell(2t)^{\frac{1}{2}}} - \sum_{a=0}^N \left(e^{2\pi idz} B_{2b+1}\left(\frac{d}{\ell}\right) - e^{-2\pi idz} B_{2a+1}\left(-\frac{d}{\ell}\right) \right) \frac{(-2\pi\ell^2 t)^a}{a!(2a+1)} + O(t^{N+1}).$$

(ii) If $z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z}$, then we have that

$$F_{d,\ell}(z; it) = \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{e^{2\pi i dz}}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}),$$

$$G_{d,\ell}(z; it) = 2i \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\sin(2\pi dz)}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

Proof. (i) In this case, we may apply Corollary 4.3 with $x_0 = 0$. We first note that $\text{erf}(0) = 0$. Next, in order to evaluate the sum on b which appears, we use (2.7) with $z \mapsto x_0, \ell \mapsto 1$ and $d \mapsto d/\ell$. This gives the expression for $F_{d,\ell}$ in case (i) of Corollary 4.6. The expression for $G_{d,\ell}$ immediately follows.

(ii) In this case, we have that $z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z}$, hence $\ell z \in \mathbb{R} \setminus \mathbb{Z}$, thus, we may apply Lemma 4.5 to ℓz , to deduce that we may write $z = w_0/\ell + c/(h\ell)$ for some $c \in \mathbb{Z}, h \in \mathbb{N}$ such that $\gcd(c, h) = 1$ and $h \geq 2$, and some $w_0 \in \mathbb{R}$ satisfying $|w_0| \leq 1/(4h)$. We then apply Lemma 4.4 to obtain the expressions for $F_{d,\ell}$ and $G_{d,\ell}$ given in (ii) of Corollary 4.6. \square

From Corollary 4.6, we deduce the following asymptotic behavior if $\text{Im}(z) = 0$.

Corollary 4.7. As $t \rightarrow 0^+$, the following are true.

(i) If $z \in \frac{1}{\ell}\mathbb{Z}$, we have that

$$F_{d,\ell}(z; it) \sim \frac{\zeta^d}{2\ell(2t)^{\frac{1}{2}}}.$$

(ii) If $z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z}$, we have that

$$F_{d,\ell}(z; it) \sim \frac{\zeta^d}{1 - \zeta^\ell}.$$

(iii) If $z \in \frac{1}{\ell}\mathbb{Z}$ and $2dz \notin \mathbb{Z}$, we have that

$$G_{d,\ell}(z; it) \sim \frac{i \sin(2\pi dz)}{\ell(2t)^{\frac{1}{2}}}.$$

(iv) If $z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z}$ and $2dz \notin \mathbb{Z}$, we have that

$$G_{d,\ell}(z; it) \sim \frac{2i \sin(2\pi dz)}{1 - \zeta^\ell}.$$

(v) If $z \in \frac{1}{\ell}\mathbb{Z}$ and $2dz \in \mathbb{Z}$, we have that

$$G_{d,\ell}(z; it) \sim (-1)^{2dz+1} \frac{2d}{\ell}.$$

(vi) If $z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z}$ and $2dz \in \mathbb{Z}$, we have that

$$G_{d,\ell}(z; it) \sim (-1)^{2dz+1} \frac{8\pi\ell d \zeta^\ell}{(1 - \zeta^\ell)^2} t.$$

Remark. The functions given on the right-hand sides of the displayed asymptotics in parts (v) and (vi) are equal to zero if and only if $d = 0$, in which case the functions $G_{0,\ell}$ appearing on the left-hand sides are identically equal to zero.

V. ASYMPTOTIC BEHAVIOR OF $F_{d,\ell}$ AND $G_{d,\ell}$

We are now able to prove Theorem 1.1 and Corollary 1.2, using results established in Secs II–IV. In particular, Theorem 1.1 follows from Lemmas 3.1 and 3.3, and Corollary 4.6. Corollary 1.2 can be concluded from Corollaries 3.2, 3.4, and 4.7. We establish results analogous to Theorem 1.1 and

Corollary 1.2 for the functions $G_{d,\ell}$ in Theorem 5.1 and Corollary 5.2. These follow in exactly the same way, with the exception of using Corollary 3.5 instead of Corollary 3.4. As we have seen previously, this behavior depends on where in \mathbb{C} the Jacobi variable z is located. We begin with Theorem 5.1, which gives asymptotic expansions for the functions $G_{d,\ell}$ as $t \rightarrow 0^+$.

Theorem 5.1. *We have the following behavior, as $t \rightarrow 0^+$, for any $N \in \mathbb{N}_0$.*

(i) *If $\text{Im}(z) > 0$ or $(\text{Im}(z) < 0 \text{ and } |x_0| > |y_0|)$, or $(z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z})$, then*

$$G_{d,\ell}(z; it) = 2i \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\sin(2\pi dz)}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

(ii) *If $\text{Im}(z) < 0$ and $|x_0| \leq |y_0|$, then*

$$G_{d,\ell}(z; it) = 2i(2\ell^2 t)^{-\frac{1}{2}} e^{-\frac{\pi z_0^2}{2\ell^2 t}} \sum_{\substack{n \in \mathbb{Z} \\ |n+x_0| \leq |y_0|}} e^{-\frac{\pi n^2}{2\ell^2 t} - \frac{\pi z_0 n}{\ell t}} \sin\left(\frac{2\pi d}{\ell}(j-n)\right) + 2i \sum_{a=0}^N \mathcal{D}_z^{2a} \left(\frac{\sin(2\pi dz)}{1 - \zeta^\ell} \right) \frac{(-2\pi t)^a}{a!} + O(t^{N+1}).$$

(iii) *If $z \in \frac{1}{\ell}\mathbb{Z}$, then*

$$G_{d,\ell}(z; it) = \frac{i \sin(2\pi dz)}{\ell(2t)^{\frac{1}{2}}} - \sum_{a=0}^N \left(\zeta^d B_{2a+1}\left(\frac{d}{\ell}\right) - \zeta^{-d} B_{2a+1}\left(-\frac{d}{\ell}\right) \right) \frac{(-2\pi \ell^2 t)^a}{a!(2a+1)} + O(t^{N+1}).$$

Next we give the asymptotic behavior of the functions $G_{d,\ell}$ as $t \rightarrow 0^+$. There are seven different cases in this result, dependent on the location of z in \mathbb{C} .

Remark. By using the previous theorem, we can now easily compute the asymptotic expansion of

$$\tilde{G}_{d,\ell}(z; \tau) := G_{d,\ell}(z; \tau) + \zeta^\ell q^{d^2}.$$

This can be used to recover several asymptotic formulas previously obtained in Refs. 5 and 8 and other papers for certain special values of z , d , and ℓ .

Corollary 5.2. *We have the following behavior as $t \rightarrow 0^+$.*

(i) *If $\text{Im}(z) > 0$ or $(\text{Im}(z) < 0 \text{ and } |x_0| > |y_0|)$ or $(\text{Im}(z) < 0, |x_0| \leq |y_0|, \ell \nmid 2d, \text{ and } x_0 \neq 1/2)$ or $(\text{Im}(z) < 0, \ell \mid 2dj \text{ but } \ell \nmid 2d, |y_0| < 1 - |x_0|, \text{ and } x_0 \neq 1/2)$ or $(z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z} \text{ and } 2dz \notin \mathbb{Z})$, then*

$$G_{d,\ell}(z; it) \sim \frac{2i \sin(2\pi dz)}{1 - \zeta^\ell}.$$

(ii) *If $\text{Im}(z) < 0, |x_0| \leq |y_0|, x_0 \neq 1/2$, and $\ell \nmid 2dj$, then*

$$G_{d,\ell}(z; it) \sim 2i(2\ell^2 t)^{-\frac{1}{2}} \sin\left(\frac{2\pi jd}{\ell}\right) e^{-\frac{\pi z_0^2}{2\ell^2 t}}.$$

(iii) *If $\text{Im}(z) < 0, \ell \mid 2dj \text{ but } \ell \nmid 2d$ and $1 - |y_0| \leq |x_0| \leq |y_0|$ and $x_0 \neq 1/2$, then*

$$G_{d,\ell}(z; it) \sim 2i(-1)^{\frac{2dj}{\ell}} (2\ell^2 t)^{-\frac{1}{2}} e^{-\frac{\pi z_0^2}{2\ell^2 t} - \frac{\pi}{2\ell^2 t} + \frac{\pi \text{sgn}(x_0) z_0}{\ell^2 t}} \text{sgn}(x_0) \sin\left(\frac{2\pi d}{\ell}\right).$$

(iv) *If $\text{Im}(z) < 0, x_0 = 1/2$ and $|y_0| \geq 1/2$, then*

$$G_{d,\ell}(z; it) \sim 2(2\ell^2 t)^{-\frac{1}{2}} e^{-\frac{\pi}{8\ell^2 t} + \frac{\pi y_0^2}{2\ell^2 t}} \sum_{\pm} \pm \cos\left(\frac{\pi}{\ell} \left(d \pm \frac{y_0}{2\ell t}\right)\right) e^{\pm \frac{\pi i(2j+1)d}{\ell}}.$$

(v) If $z \in \frac{1}{\ell}\mathbb{Z}$ with $2dz \notin \mathbb{Z}$, then we have

$$G_{d,\ell}(z; it) \sim i(2\ell^2 t)^{-\frac{1}{2}} \sin(2\pi dz).$$

(vi) If $z \in \frac{1}{\ell}\mathbb{Z}$ and $2dz \in \mathbb{Z}$, then

$$G_{d,\ell}(z; it) \sim (-1)^{2dz+1} \frac{2d}{\ell}.$$

(vii) If $z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z}$ and $2dz \in \mathbb{Z}$, then

$$G_{d,\ell}(z; it) \sim (-1)^{2dz+1} \frac{8\pi\ell d \zeta^\ell}{(1 - \zeta^\ell)^2} t.$$

VI. REGULARIZED CHARACTERS OF SINGLET ALGEBRA MODULES

In this section, we apply the results from Secs. I–V to study asymptotic properties of characters of the $(1, p)$ -singlet vertex operator algebra, $p \in \mathbb{N}_{\geq 2}$, and their quantum dimensions. We do not recall the definition of the $(1, p)$ -singlet vertex algebra here; instead we refer the reader to Ref. 13. In this paper, we are only interested in regularized characters of irreducible modules, whose explicit formulae we recall next.

In vertex algebra theory, it is customary to use $\text{ch}[M](\tau)$ to denote the character (or modified graded dimension) of a V -module M . By definition,

$$\text{ch}[M](\tau) = \text{tr}_M q^{L(0) - \frac{c}{24}},$$

where $L(0)$ is the degree operator (acting semisimply) of M and $c \in \mathbb{C}$ is the central charge. A regularized character of M is simply a function depending on a complex variable ε , denoted by $\text{ch}[M^\varepsilon](\tau)$, such that

$$\lim_{\varepsilon \rightarrow 0} \text{ch}[M^\varepsilon](\tau) = \text{ch}[M](\tau).$$

Although there are many different ways to introduce a regularization, for the singlet vertex algebra this can be done canonically via resolutions in terms of Fock modules (again for details see Ref. 13). One of the upshots of the regularization in Ref. 13 is a one-to-one correspondence among irreducible modules and their regularized characters.

As we already mentioned in the Introduction, the $(1, p)$ -singlet vertex algebra admits two types of irreducible characters: atypical and typical. Here we are only interested in regularized characters. Typical (regularized) characters are given by

$$\text{ch}[F_\lambda^\varepsilon](\tau) := \frac{e^{2\pi\varepsilon(\lambda - \frac{\alpha_0}{2})} q^{\frac{1}{2}(\lambda - \frac{\alpha_0}{2})^2}}{\eta(\tau)}, \tag{6.1}$$

where $\lambda \in \mathbb{C}$, $\alpha_0 := \sqrt{2p} - \sqrt{2/p}$, and $\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is Dedekind’s η -function. On the other hand, atypical characters are given by¹³

$$\begin{aligned} &\text{ch}[M_{r,s}^\varepsilon](\tau) \\ &= \frac{1}{\eta(\tau)} \sum_{n \geq 0} \left(e^{\frac{2\pi\varepsilon}{\sqrt{2p}}(2pn - s - pr + 2p)} q^{\frac{1}{4p}(2pn - s - pr + 2p)^2} - e^{\frac{2\pi\varepsilon}{\sqrt{2p}}(2pn + s - pr + 2p)} q^{\frac{1}{4p}(2pn + s - pr + 2p)^2} \right), \end{aligned} \tag{6.2}$$

where $r \in \mathbb{Z}$ and $1 \leq s \leq p - 1$. We stress that in particular, $M_{1,1}$ is the $(1, p)$ -singlet vertex algebra.¹³ We note that Ref. 16 used a slightly different ε -parametrization; $\text{ch}[M_{r,s}^\varepsilon]$ in Ref. 13 is precisely $\text{ch}[M_{r,s}^{-\varepsilon}]$ in Ref. 16. We normalize the atypical characters $\text{ch}[M_{r,s}^\varepsilon]$ by defining the functions

$$C_{r,s}(\varepsilon; \tau) := \eta(\tau) \text{ch}[M_{r,s}^\varepsilon](\tau).$$

It is not difficult to see that we may decompose the normalized characters $C_{r,s}$ as a difference of functions defined using the Jacobi partial theta functions $F_{d,t}$ as follows:

$$C_{r,s}(\varepsilon; \tau) = F_{2p-s-pr,2p} \left(\frac{-i\varepsilon}{\sqrt{2p}}; \frac{\tau}{4p} \right) - F_{2p+s-pr,2p} \left(\frac{-i\varepsilon}{\sqrt{2p}}; \frac{\tau}{4p} \right).$$

Our next results further relate the regularized atypical characters to some of the functions studied earlier.

Lemma 6.1. For $\varepsilon \in \mathbb{C}$, we have that

$$C_{r,s}(\varepsilon; \tau) = -G_{s+p(r-2),2p} \left(-\frac{i\varepsilon}{\sqrt{2p}}; \frac{\tau}{4p} \right) + \sum_{a \geq 0} \left[\mathcal{D}_z^{2a} \left(\frac{\zeta^{s+(r-2)p} (1 - \zeta^{-2p(r-2)})}{1 - \zeta^{2p}} \right) \right]_{z=-\frac{i\varepsilon}{\sqrt{2p}}} \frac{\left(\frac{\pi i \tau}{2p}\right)^a}{a!}.$$

Proof. The claim follows by applying Lemma 2.3. □

For the remainder of this section, as introduced in the Introduction, we write $\varepsilon = (\varepsilon_0 + ik)/\sqrt{2p}$, with $k \in \mathbb{Z}$, $\varepsilon_0 = u_0 + iv_0$ with $u_0, v_0 \in \mathbb{R}$, and $-1/2 < v_0 \leq 1/2$. Using Lemma 6.1 and results from Section V, we establish the asymptotic expansions of the functions $C_{r,s}$ in Theorem 6.2.

Theorem 6.2. We have the following behavior, as $t \rightarrow 0^+$, for any $N \in \mathbb{N}_0$.

(i) If $\text{Re}(\varepsilon) < 0$ or $(\text{Re}(\varepsilon) > 0$ and $|v_0| > |u_0|)$ or $(u_0 = 0$ and $v_0 \neq 0)$, then

$$C_{r,s}(\varepsilon; it) = \sum_{a=0}^N \mathcal{D}_\varepsilon^{2a} \left(\frac{e^{\pi\sqrt{2p}(1-r)\varepsilon} \sinh\left(\sqrt{\frac{2}{p}}\pi s\varepsilon\right)}{\sinh(\sqrt{2p}\pi\varepsilon)} \right) \frac{(\pi t)^a}{a!} + O(t^{N+1}).$$

(ii) If $\text{Re}(\varepsilon) > 0$, and $|v_0| \leq |u_0|$, then

$$C_{r,s}(\varepsilon; it) = -2i(2pt)^{-\frac{1}{2}} e^{\frac{\pi\varepsilon_0^2}{2pt}} \sum_{\substack{n \in \mathbb{Z} \\ |n+v_0| \leq |u_0|}} (-1)^{r(k+n)} \sin\left(\frac{\pi s(k-n)}{p}\right) e^{-\frac{\pi n^2}{2pt} + \frac{2\pi i\varepsilon_0 n}{t}} + \sum_{a=0}^N \mathcal{D}_\varepsilon^{2a} \left(\frac{e^{\pi\sqrt{2p}(1-r)\varepsilon} \sinh\left(\sqrt{\frac{2}{p}}\pi s\varepsilon\right)}{\sinh(\sqrt{2p}\pi\varepsilon)} \right) \frac{(\pi t)^a}{a!} + O(t^{N+1}).$$

(iii) If $\varepsilon_0 = 0$, then

$$C_{r,s}(\varepsilon; it) = -i(-1)^{rk} \sin\left(\frac{\pi ks}{p}\right) (2pt)^{-\frac{1}{2}} + (-1)^{kr} \sum_{a=0}^N \left(e^{\frac{\pi i s k}{p}} B_{2a+1}\left(\frac{s+p(r-2)}{2p}\right) - e^{-\frac{\pi i s k}{p}} B_{2a+1}\left(-\frac{s+p(r-2)}{2p}\right) \right) \frac{(-2\pi t)^a}{(2a+1)a!} - \sum_{a=0}^N \mathcal{D}_\varepsilon^{2a} \left(e^{2\pi\varepsilon\frac{(s-p)}{\sqrt{2p}}} \frac{\sinh(\sqrt{2p}(r-2)\pi\varepsilon)}{\sinh(\sqrt{2p}\pi\varepsilon)} \right) \frac{(\pi t)^a}{a!} + O(t^{N+1}).$$

Proof. We use Lemma 6.1 and Theorem 5.1 with $z = -i\varepsilon/\sqrt{2p}$ and $\tau \mapsto \tau/(4p)$. Recalling that $z = (z_0 + j)/\ell$ and $\varepsilon = (\varepsilon_0 + ik)/\sqrt{2p}$, we thus take $\ell = 2p$, $j = k$, $z_0 = -i\varepsilon_0$, $x_0 = v_0$, and $y_0 = -u_0$. The conditions from Theorem 5.1 then translate as follows:

$$\text{Im}(z) > 0 \Leftrightarrow \text{Re}(\varepsilon) < 0, \quad |x_0| > |y_0| \Leftrightarrow |v_0| > |u_0|, \quad z \in \mathbb{R} \setminus \frac{1}{\ell}\mathbb{Z} \Leftrightarrow u_0 = 0 \text{ and } v_0 \neq 0.$$

Parts (i) and (ii) follow by combining Lemma 6.1 and Theorem 5.1 (i) and (ii), respectively. To prove (iii), we proceed similarly and combine Lemma 6.1 and Theorem 5.1 (iii). The first term

arising from (iii) of Theorem 5.1 simplifies to be the first term of the statement. For the second term arising from (iii) of Lemma 6.1 we use a direct substitution. Finally the remaining term from Lemma 6.1 yields the third term after a change of variables. \square

Remark. Using the fact that $C_{r,s} = F_{2p-s-pr,2p} - F_{2p+s-pr,2p}$, one may alternatively establish asymptotic results for the functions $C_{r,s}$ using the asymptotic results for the functions $F_{d,\ell}$ obtained in Secs. I–V.

Remark. Define $C_{r,s}^{(m)}(\varepsilon; \tau) := \mathcal{D}_\varepsilon^m(C_{r,s}(\varepsilon; \tau))$. For $m = 1$, these and related “weight 3/2” false theta functions were studied in Refs. 8 and 16 in connection to atypical characters of the (p, p') -singlet vertex algebras. Their asymptotic expansion can be computed from the previous theorem by differentiating the asymptotic expansion in (i)–(iii) term by term.

Using Theorem 6.2 as well as some earlier results, we establish the asymptotic behavior of the normalized characters $C_{r,s}$ in Corollary 6.3. Recall that $1 \leq s \leq p - 1$; for this reason, if we encounter the hypothesis $p \nmid s$ in the proof of Corollary 6.3, we refrain from writing it down. Similarly, $p \mid s$ cannot occur, so if tabulating previous results to formulate Corollary 6.3 we (must) omit such cases.

Corollary 6.3. We have the following behavior, as $t \rightarrow 0^+$.

- (i) If $\text{Re}(\varepsilon) < 0$ or $(\text{Re}(\varepsilon) > 0 \text{ and } |v_0| > |u_0|)$ or $(\text{Re}(\varepsilon) > 0, p \mid ks, |u_0| < 1 - |v_0|, \text{ and } v_0 \neq 1/2)$ or $(u_0 = 0, v_0 \neq 0, \text{ and } s(v_0 + k)/p \notin \mathbb{Z})$, then

$$C_{r,s}(\varepsilon; it) \sim e^{\pi\varepsilon\sqrt{2p(1-r)}} \frac{\sinh\left(\frac{\sqrt{2}\pi s\varepsilon}{\sqrt{p}}\right)}{\sinh\left(\sqrt{2p}\pi\varepsilon\right)}.$$

- (ii) If $\text{Re}(\varepsilon) > 0, |v_0| \leq |u_0|, v_0 \neq 1/2$, and $p \nmid ks$, then

$$C_{r,s}(\varepsilon; it) \sim -2i(-1)^{kr}(2pt)^{-\frac{1}{2}} \sin\left(\frac{\pi ks}{p}\right) e^{\frac{\pi\varepsilon_0^2}{2pt}}.$$

- (iii) If $\text{Re}(\varepsilon) > 0, p \mid ks, 1 - |u_0| \leq |v_0| \leq |u_0|$, and $v_0 \neq 1/2$, then

$$C_{r,s}(\varepsilon; it) \sim -2i(-1)^{r(k+1)+\frac{sk}{p}}(2pt)^{-\frac{1}{2}} e^{\frac{\pi\varepsilon_0^2}{2pt} - \frac{\pi}{2pt} - \frac{\pi i \text{sgn}(v_0)\varepsilon_0}{pt}} \text{sgn}(v_0) \sin\left(\frac{\pi s}{p}\right).$$

- (iv) If $\text{Re}(\varepsilon) > 0, v_0 = 1/2$, and $|u_0| \geq 1/2$, then

$$C_{r,s}(\varepsilon; it) \sim -2(2pt)^{-\frac{1}{2}} e^{-\frac{\pi}{8pt} + \frac{\pi u_0^2}{2pt}} \sum_{\pm} \pm \cos\left(\frac{\pi}{2p}\left(s + pr \mp \frac{u_0}{t}\right)\right) e^{\pm \frac{\pi i(2k+1)(s+pr)}{2p}}.$$

- (v) If $\varepsilon_0 = 0$ and $p \nmid ks$, then

$$C_{r,s}(\varepsilon; it) \sim -i(-1)^{kr}(2pt)^{-\frac{1}{2}} \sin\left(\frac{\pi ks}{p}\right).$$

- (vi) If $\varepsilon_0 = 0$ and $p \mid ks$, then

$$C_{r,s}(\varepsilon; it) \sim (-1)^{kr+\frac{ks}{p}} \frac{s}{p}.$$

- (vii) If $u_0 = 0, v_0 \neq 0$, and $s(v_0 + k)/p \in \mathbb{Z}$, then

$$C_{r,s}(\varepsilon; it) \sim 2\pi s(-1)^{\frac{s(v_0+k)}{p}+kr} e^{-\pi i r v_0} \frac{(r - 1 - i \cot(\pi v_0))}{1 - e^{-2\pi i v_0}} t.$$

Proof. (i) Under the first, second, and fourth sets of hypotheses given, we use Theorem 6.2. Under the third set of hypotheses given, we apply Lemma 6.1 and appeal to the fourth set of hypotheses in Corollary 5.2 (i). The main term in the asymptotic expansion then follows from Theorem 6.2, taking $a = 0$.

- (ii) Corollary 1.2 directly yields the claim.
- (iii) The claim follows from Corollary 1.2, after simplifying.
- (iv) and (v) follow from Corollary 5.2, Lemma 6.1, and Theorem 6.2.
- (vi) The claim follows directly from Corollary 5.2 and Lemma 6.1 since

$$\lim_{t \rightarrow 0} \frac{1 - e^{-2pt(r-2)}}{1 - e^{2pt}} = (2 - r).$$

- (vii) The claim follows from Theorem 6.2 (i), using the $a = 1$ term of the sum given. □

We finally prove Theorem 1.3, which establishes the (ε -regularized) quantum dimensions of the singlet algebra modules. Recall from (1.3), that

$$\text{qdim}[M_{r,s}^\varepsilon] = \lim_{t \rightarrow 0^+} \frac{\text{ch}[M_{r,s}^\varepsilon](it)}{\text{ch}[M_{1,1}^\varepsilon](it)} \quad \text{and} \quad \text{qdim}[F_\lambda^\varepsilon] = \lim_{t \rightarrow 0^+} \frac{\text{ch}[F_\lambda^\varepsilon](it)}{\text{ch}[M_{1,1}^\varepsilon](it)}.$$

Proof of Theorem 1.3. For $\text{qdim}[M_{r,s}^\varepsilon]$, all of the statements follow from Corollary 6.3 in a straightforward manner, except for certain parts of (i), (ii), and (iv), which we now elaborate upon.

For part (i), we first note that in establishing (1.4) for $\text{qdim}[M_{r,s}^\varepsilon]$ in the case ($\text{Re}(\varepsilon) > 0, p|k, |u_0| < 1 - |v_0|$, and $v_0 \neq 1/2$), we apply Corollary 6.3 (i) twice, using that $p | k$ implies $p | ks$, to establish the asymptotic behaviors of $C_{r,s}$ and $C_{1,1}$, from which the result follows in this case. Next, we consider the two cases ($\text{Re}(\varepsilon) > 0$, and $|v_0| > |u_0|$), and $\text{Re}(\varepsilon) < 0$. The claimed results for $\text{qdim}[M_{r,s}^\varepsilon]$ in these cases again follow by applying Corollary 6.3 (i) twice, once to $C_{r,s}$ and once to $C_{1,1}$. Turning to the last set of hypotheses ($u_0 = 0$ and $v_0 \neq 0$), the proof splits into two cases: $s(k + v_0)/p \notin \mathbb{Z}$ and $s(k + v_0)/p \in \mathbb{Z}$. In the former case, we again apply Corollary 6.3 (i) to establish the asymptotic behaviors of $C_{r,s}$ and $C_{1,1}$. For this we additionally require that $(v_0 + k)/p \notin \mathbb{Z}$, but since $0 < |v_0| \leq 1/2$, this condition always holds. In the latter case ($s(k + v_0)/p \in \mathbb{Z}$) we apply Corollary 6.3 (vii) for s , and Corollary 6.3 (i) for $s = 1$, to establish the asymptotic behaviors of $C_{r,s}$ and $C_{1,1}$, respectively. We find that $\text{qdim}[M_{r,s}^\varepsilon] = 0$ in this case, which agrees with (1.4) under the hypotheses given.

To establish part (ii) for $\text{qdim}[M_{r,s}^\varepsilon]$, we distinguish three subcases of the hypotheses ($\text{Re}(\varepsilon) > 0, p \nmid k, |v_0| \leq |u_0|$, and $v_0 \neq 1/2$). First, if we additionally have that $p \nmid ks$, then we may apply Corollary 6.3 (ii) twice to obtain the claim. Next, if in addition to the original hypotheses we have that $p | ks$ and $|u_0| < 1 - |v_0|$, then we apply Corollary 6.3 (i) for s and part (ii) for $s = 1$ yielding $\text{qdim}[M_{r,s}^\varepsilon] = 0$ which is compatible with the claimed formula in (ii) in this case. Finally, if we additionally have that $|u_0| > 1 - |v_0|$, we apply Corollary 6.3 (iii), and Corollary 6.3 (ii) for $s = 1$, to establish the asymptotic behaviors of $C_{r,s}$ and $C_{1,1}$, respectively, and again find that $\text{qdim}[M_{r,s}^\varepsilon] = 0$ under the hypotheses given, as claimed.

To prove (iv), we use Corollary 6.3 (iv) to obtain

$$\frac{C_{r,s}(\varepsilon; it)}{C_{1,1}(\varepsilon; it)} \sim \frac{\sum_{\pm} \pm \cos\left(\frac{\pi}{2p}\left(s + pr \mp \frac{u_0}{2pt}\right)\right) e^{\pm \frac{\pi i(2k+1)(s+pr)}{2p}}}{\sum_{\pm} \pm \cos\left(\frac{\pi}{2p}\left(1 + p \mp \frac{u_0}{2pt}\right)\right) e^{\pm \frac{\pi i(2k+1)(1+p)}{2p}}}. \tag{6.3}$$

With $\alpha := e^{\frac{\pi i(2k+1)(s+pr)}{2p}}$, $\beta := e^{\frac{\pi i(2k+1)(1+p)}{2p}}$, $a := \pi \frac{s+pr}{2p}$, $b := \pi \frac{1+p}{2p}$, and $T := \frac{-u_0 \pi}{4p^2 t}$ (so that $T \rightarrow \infty$), (6.3) equals

$$\frac{\sum_{\pm} \pm \cos(a \pm T) \alpha^{\pm 1}}{\sum_{\pm} \pm \cos(b \pm T) \beta^{\pm 1}} = \frac{i \cos(a) \sin(\text{Arg}(\alpha)) - \sin(a) \cos(\text{Arg}(\alpha)) \tan(T)}{i \cos(b) \sin(\text{Arg}(\beta)) - \sin(b) \cos(\text{Arg}(\beta)) \tan(T)}.$$

Thus, we need to investigate for which $A, B, C, D \in \mathbb{C}$

$$\lim_{T \rightarrow \infty} \frac{A + B \tan(T)}{C + D \tan(T)}$$

exists. If we consider the special sequence $T = \pi(j + 1/2)$ with $j \in \mathbb{Z}$, we see that the above limit must be B/D . At the same time, with $T = \pi j$ where $j \in \mathbb{Z}$, this limit equals A/C if $C \neq 0$. Therefore

we must have $AD = BC$. In this case, the above quotient (and the limit) exists and equals B/D . Note that if $C = 0$ and the limit exists, then $A = 0$ and we again obtain B/D . In our situation, this gives the condition

$$\tan(\text{Arg}(a)) \tan(b) = \tan(a) \tan(\text{Arg}(\beta)).$$

Substituting into this expression the definitions of α, β, a , and b gives the condition stated in Theorem 1.3. Substituting the appropriate values for B/D and simplifying give the claimed limit.

For $\text{qdim}[F_\lambda^\varepsilon]$, we first recall (6.1) and hence

$$\eta(\tau) \text{ch}[F_\lambda^\varepsilon](\tau) = e^{2\pi\varepsilon(\lambda - \frac{\alpha_0}{2})} q^{\frac{1}{2}(\lambda - \frac{\alpha_0}{2})^2}.$$

Therefore, $\eta(it) \text{ch}[F_\lambda^\varepsilon](it) \sim e^{2\pi\varepsilon(\lambda - \frac{\alpha_0}{2})} + O(t)$, for all ε . The results claimed in Theorem 1.3 pertaining to $\text{qdim}[F_\lambda^\varepsilon]$ follow again from Corollary 6.3. \square

VII. FUTURE WORK

This work has several possible extensions. Here we briefly propose its “higher rank” generalization in connection with representation theory.

As explained in Ref. 13, the atypical singlet characters $\text{ch}[M_{r,s}](\tau)$ are in fact parametrized by the elements of the dual lattice of $L = \sqrt{2}p\mathbb{Z}$, which can be viewed as a dilation of the \mathfrak{sl}_2 root lattice. This construction generalizes to higher rank simple Lie algebras. More precisely, for every root lattice Q of ADE type and $p \in \mathbb{N}_{\geq 2}$, there exist vertex operator algebras whose (atypical) irreducible characters are in one-to-one correspondence with the elements of the dual lattice of $\sqrt{2}pQ$ (for details see Ref. 23). These characters are further studied in Ref. 9 where we denoted them by $\text{ch}[W^0(p, \lambda)_Q]$. In parallel with Ref. 13, it is straightforward to regularize them by inclusion of an additional (Jacobi) variable $\varepsilon \in \mathbb{C}^n$.¹⁴ The resulting expression $\text{ch}[W^0(p, \lambda)_Q](\varepsilon; \tau)$, modulo a power of the Dedekind η -function, can be expressed in terms of certain higher rank (Jacobi) partial theta functions and their derivatives.

We propose to study asymptotic properties (as $t \rightarrow 0^+$) of $\text{ch}[W^0(p, \lambda)_Q](\varepsilon; \tau)$. This requires a suitable extension of Theorem 1.1 to higher ranks, which we (jointly with T. Creutzig) intend to address and solve in our future work (see also Ref. 14).

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