J. Aust. Math. Soc. (First published online 2020), page 1 of 19*
doi:10.1017/S1446788719000405
*Provisional—final page numbers to be inserted when paper edition is published

RANK GENERATING FUNCTIONS FOR ODD-BALANCED UNIMODAL SEQUENCES, QUANTUM JACOBI FORMS, AND MOCK JACOBI FORMS

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(Received 30 July 2018; accepted 10 August 2019)

Communicated by M. Coons

Abstract

Let $\mu(m, n)$ (respectively, $\eta(m, n)$) denote the number of odd-balanced unimodal sequences of size $2n$ and rank m with even parts congruent to 2 mod 4 (respectively, 0 mod 4) and odd parts at most half the peak. We prove that two-variable generating functions for $\mu(m, n)$ and $\eta(m, n)$ are simultaneously quantum Jacobi forms and mock Jacobi forms. These odd-balanced unimodal rank generating functions are also duals to partial theta functions originally studied by Ramanujan. Our results also show that there is a single C^∞ function in $\mathbb{R} \times \mathbb{R}$ to which the errors to modularity of these two different functions extend. We also exploit the quantum Jacobi properties of these generating functions to show, when viewed as functions of the two variables w and q, how they can be expressed as the same simple Laurent polynomial when evaluated at pairs of roots of unity. Finally, we make a conjecture which fully characterizes the parity of the number of odd-balanced unimodal sequences of size $2n$ with even parts congruent to 0 mod 4 and odd parts at most half the peak.

2010 *Mathematics subject classification*: primary 05A17, 11f37, 11f50; secondary 11f03, 11f30, 33D70.

Keywords and phrases: partitions, strongly unimodal sequences, generating functions, mock modular forms, quantum modular forms, Jacobi forms.

1. Introduction and statement of results

Many recent papers have explored connections between combinatorial generating functions associated to *unimodal sequences* and modular forms. For example, Andrews *et al.* proved in [2] that such a generating function, there described in terms of *concave compositions*, is a mixed mock modular form and made connections to Mathieu Moonshine. Additionally, first in [9], and later in [12, 15], related unimodal generating functions were shown to be both quantum and mock modular forms. More recently in [5], the notion of a quantum Jacobi form was defined and the first examples of quantum Jacobi forms, given in [4, 5], were presented as certain unimodal rank

The authors are grateful for support from National Science Foundation CAREER Grant DMS-1449679 and from the Simons Foundation Fellows Program in Mathematics (awarded to the second author).

generating functions. Here, we study rank generating functions for so-called *odd-balanced* unimodal sequences and show that these combinatorial q-hypergeometric series are quantum Jacobi and mock Jacobi forms.

To make this precise, we define the notion of a *strongly unimodal sequence of size* n: it is a sequence of integers $\{a_j\}_{j=1}^{s}(s \in \mathbb{N})$ such that for some $k \in \mathbb{N}$,

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0$$

and $a_1 + \cdots + a_s = n$. The *rank* of a strongly unimodal sequence is equal to $2k - s - 1$[1]. *Odd-balanced unimodal sequences*, first defined in [15], allow odd parts to repeat on either side of the peak a_k, but they must be identical on either side and the peak must be even. Here we study a particular unimodal rank statistic defined as follows. Let $\mu(m, n)$ denote the number of odd-balanced unimodal sequences of size $2n$ and rank m with all even parts congruent to 2 modulo 4 and odd parts at most half the peak. The rank generating function for such sequences satisfies

$$\begin{aligned}\frac{1}{(1+w^{-1})}M(z;\tau) &:= \sum_{n=0}^{\infty} \frac{q^{2n+1}(-wq;q^2)_n(-w^{-1}q;q^2)_n}{(q;q^2)_{n+1}} \\ &= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \mu(m,n)w^m q^n,\end{aligned}$$

where $w := e(z), q := e(\tau)$, with $e(\alpha) := e^{2\pi i\alpha}$, and the q-Pochhammer symbol is defined by $(w; q)_n := \prod_{j=0}^{n-1}(1 - wq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$. We will use this notation (in particular, that for w and q just mentioned) throughout the paper. We define a slight normalization of M called $\mathcal{M}^+$ by

$$\mathcal{M}^+(z;\tau) := w^{1/2}q^{-1/16}M\Big(z;\frac{\tau}{2}\Big). \tag{1-1}$$

In Theorem 1.1, we prove that $\mathcal{M}^+$ is a quantum Jacobi form when viewed as a function on an appropriate subset of $\mathbb{Q}\times\mathbb{Q}$ and a mock Jacobi form when viewed as a function on $\mathbb{C}\times\mathbb{H}$, both with respect to the subgroup $\Gamma := \left\langle\left\{\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)\right\}\right\rangle \subseteq \Gamma_0(6) \subseteq \mathrm{SL}_2(\mathbb{Z})$, where we let $\langle S\rangle$ denote the group generated by the set S. We also use the quantum Jacobi properties of $\mathcal{M}^+$ to obtain an expression for $\mathcal{M}^+$ as a Laurent polynomial when evaluated at pairs of rational numbers in Theorem 1.5.

To state Theorem 1.1 precisely, we define the 'errors of modularity'

$$H_1(z;\tau) := -\frac{1}{2}(6\tau+1)^{-1/2}e\Big(\frac{6z^2}{6\tau+1}\Big)h\Big(\frac{z}{6\tau+1};\frac{\tau}{2(6\tau+1)}\Big) - \frac{1}{2}h\Big(z;\frac{\tau}{2}\Big), \tag{1-2}$$

$$\begin{aligned}H_2(z;\tau) &:= \frac{1}{2}w^{1/2}q^{-1/24}(6\tau+1)^{-1/2}e\Big(\frac{9z^2+\tau^2}{6\tau+1}\Big) \\ &\quad\times \frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}\sum_{\pm} e\Big(\pm\Big(\frac{z}{6\tau+1}+\frac{1}{4}\Big)\Big)\end{aligned}$$

[1]Here, we use the definition of rank as in [15]. Other sources such as [6] define rank to be $-(2k - s - 1) = s - 2k + 1$.

$$\times\left[h\left(\frac{3z+(3\mp 1)\tau+\frac{1}{2}}{6\tau+1};\frac{3\tau}{6\tau+1}\right)\right.$$
$$\left.-(6\tau+1)^{1/2}e\left(\frac{-9z^2-\tau^2\pm 6\tau z}{6\tau+1}\right)h\left(3z\mp\tau+\frac{1}{2};3\tau\right)\right], \qquad (1\text{-}3)$$

where, for $z\in\mathbb{C}$, $\tau\in\mathbb{H}$, the Mordell integral h is given by

$$h(z;\tau):=\int_{\mathbb{R}}\frac{e^{\pi i\tau t^2-2\pi zt}}{\cosh(\pi t)}\,dt \qquad (1\text{-}4)$$

and the respective modular and Jacobi forms η and ϑ are defined in (2-1).

Theorem 1.1 below establishes the quantum Jacobi and mock Jacobi transformation properties of the two-variable odd-balanced unimodal rank generating function $\mathcal{M}^+$. The set $Q_{\mathcal{M}^+}\subseteq\mathbb{Q}\times\mathbb{Q}$ is defined in Section 3.

Theorem 1.1. *The following transformation properties hold.*
(i) *For* $(z,\tau)\in(\mathbb{C}\times\mathbb{H})\cup Q_{\mathcal{M}^+}$,

$$\mathcal{M}^+(z;\tau)-e^{\pi i/4}\mathcal{M}^+(z;\tau+2)=0, \qquad (1\text{-}5)$$
$$\mathcal{M}^+(z;\tau)-\mathcal{M}^+(-z;\tau)=0, \qquad (1\text{-}6)$$
$$\mathcal{M}^+(z;\tau)+(6\tau+1)^{-1/2}e\left(\frac{6z^2}{6\tau+1}\right)\mathcal{M}^+\left(\frac{z}{6\tau+1};\frac{\tau}{6\tau+1}\right)$$
$$=H_1(z;\tau)+H_2(z;\tau), \qquad (1\text{-}7)$$
$$\mathcal{M}^+(z;\tau)+\mathcal{M}^+(z+1;\tau)=0, \qquad (1\text{-}8)$$
$$\mathcal{M}^+(z;\tau)-w^{-2}q^{-1}\mathcal{M}^+(z+\tau;\tau)$$
$$=-w^{-1/2}q^{-1/16}+w^{-3/2}q^{-9/16}$$
$$+w^{-2}q^{-11/12}(1-qw^2)\frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}. \qquad (1\text{-}9)$$

(ii) *In particular, for* $(z,\tau)\in Q_{\mathcal{M}^+}$,

$$\mathcal{M}^+(z;\tau)+(6\tau+1)^{-1/2}e\left(\frac{6z^2}{6\tau+1}\right)\mathcal{M}^+\left(\frac{z}{6\tau+1};\frac{\tau}{6\tau+1}\right)=H_1(z;\tau), \qquad (1\text{-}10)$$
$$\mathcal{M}^+(z;\tau)-w^{-2}q^{-1}\mathcal{M}^+(z+\tau;\tau)=-w^{-1/2}q^{-1/16}+w^{-3/2}q^{-9/16}. \qquad (1\text{-}11)$$

The function on the right-hand side of (1-11) extends to a C^∞ *function on* $\mathbb{R}\times\mathbb{R}$ *and the function* H_1 *on the right-hand side of (1-10) extends to a* C^∞ *function on* $(\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{24},\pm\frac{1}{8},\pm\frac{5}{24},\pm\frac{7}{24},\pm\frac{3}{8},\pm\frac{11}{24}\}))\times(\mathbb{R}\setminus\{-\frac{1}{6}\})$.

Remarks 1.2. (1) Theorem 1.1 shows that the function $\mathcal{M}^+$ is a quantum Jacobi form of weight $\frac{1}{2}$ and index -1 with respect to Γ (where Γ is as defined above). Direct calculation shows that $Q_{\mathcal{M}^+}$ is invariant under the action of Γ.

(2) As suggested by the transformations given on $\mathbb{C}\times\mathbb{H}$ in Theorem 1.1, the proof of Theorem 1.1 in Section 4 reveals that $\mathcal{M}^+$ is also a mock Jacobi form. In particular, it is the holomorphic part of a nonholomorphic Jacobi form.

We also study the quantum Jacobi and mock Jacobi properties of a second odd-balanced unimodal rank generating function, namely, the function

$$\frac{1}{(1+w^{-1})}N(z;\tau) := \sum_{n=0}^{\infty} \frac{q^{2n+2}(-wq^2;q^2)_n(-w^{-1}q^2;q^2)_n}{(q;q^2)_{n+1}}$$
$$= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \eta(m,n)w^m q^n,$$

where $\eta(m,n)$ counts the number of odd-balanced unimodal sequences of size $2n$ and rank m with even parts congruent to 0 modulo 4 and odd parts at most half the peak. Our function N was independently studied at around the same time in [7] by Bringmann and Jennings-Shaffer. In [7], the relevant function is called $U2$, studied there for its combinatorial, asymptotic, and mock modular properties. We also point out that the q-hypergeometric series defining N and M are duals to partial theta functions studied by Ramanujan [1, 17]. Parallel to Theorem 1.1, in Theorem 1.3 below, we establish the quantum Jacobi and mock Jacobi properties of the normalized function

$$\mathcal{N}^+(z;\tau) := w^{1/2}q^{-1/16}N\Big(z;\frac{\tau}{2}\Big).$$

The 'error' function H_1 is as in Theorem 1.1 (see (1-2)) and H_3 is defined in (4-16). The set $Q_{\mathcal{N}^+} \subseteq \mathbb{Q}\times\mathbb{Q}$ is defined in Section 3.

Theorem 1.3. *The following transformation properties hold.*
(i) *For* $(z,\tau)\in(\mathbb{C}\times\mathbb{H})\cup Q_{\mathcal{N}^+}$,

$$\mathcal{N}^+(z;\tau) - e^{\pi i/4}\mathcal{N}^+(z;\tau+2) = 0, \tag{1-12}$$
$$\mathcal{N}^+(z;\tau) - \mathcal{N}^+(-z;\tau) = 0, \tag{1-13}$$
$$\mathcal{N}^+(z;\tau) + (6\tau+1)^{-1/2}e\Big(\frac{6z^2}{6\tau+1}\Big)\mathcal{N}^+\Big(\frac{z}{6\tau+1};\frac{\tau}{6\tau+1}\Big) = H_1(z;\tau)+H_3(z;\tau), \Big(\tau\neq-\frac{1}{6}\Big), \tag{1-14}$$
$$\mathcal{N}^+(z;\tau) + \mathcal{N}^+(z+1;\tau) = 0, \tag{1-15}$$
$$\mathcal{N}^+(z;\tau) - w^{-2}q^{-1}\mathcal{N}^+(z+\tau;\tau) = -w^{-1/2}q^{-1/16} + w^{-3/2}q^{-9/16} + w^{-2}q^{-2/3}(1-wq^{1/2})\frac{\vartheta(z+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}. \tag{1-16}$$

(ii) *In particular, for* $(z,\tau)\in Q_{\mathcal{N}^+}$,

$$\mathcal{N}^+(z;\tau) + (6\tau+1)^{-1/2}e\Big(\frac{6z^2}{6\tau+1}\Big)\mathcal{N}^+\Big(\frac{z}{6\tau+1};\frac{\tau}{6\tau+1}\Big) = H_1(z;\tau), \Big(\tau\neq-\frac{1}{6}\Big), \tag{1-17}$$
$$\mathcal{N}^+(z;\tau) - w^{-2}q^{-1}\mathcal{N}^+(z+\tau;\tau) = -w^{-1/2}q^{-1/16} + w^{-3/2}q^{-9/16}. \tag{1-18}$$

The function on the right-hand side of (1-18) extends to a C^∞ function on $\mathbb{R}\times\mathbb{R}$ and the function H_1 on the right-hand side of (1-17) extends to a C^∞ function on $(\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{24},\pm\frac{1}{8},\pm\frac{5}{24},\pm\frac{7}{24},\pm\frac{3}{8},\pm\frac{11}{24}\}))\times(\mathbb{R}\setminus\{-\frac{1}{6}\})$.

REMARKS 1.4.

(1) Theorem 1.3 shows that $\mathcal{N}^+$, like $\mathcal{M}^+$ in Theorem 1.1, is a quantum Jacobi form of weight $\frac{1}{2}$ and index -1 with respect to Γ. Direct calculation shows that $Q_{\mathcal{N}^+}$ is invariant under the action of Γ. As was the case with $\mathcal{M}^+$, the proof of Theorem 1.3 shows that $\mathcal{N}^+$ is also a mock Jacobi form.

(2) In (1-14) and (1-17), we must exclude pairs (z,τ) with $\tau=-\frac{1}{6}$ in order to avoid singularities. For any $\gamma\in\Gamma$, we similarly must exclude pairs (z,τ) with $\tau=\gamma^{-1}(i\infty)$ in the analogues to (1-14) and (1-17).

(3) It is interesting to point out that the so-called 'error functions' for $\mathcal{M}^+$ and $\mathcal{N}^+$ on their respective quantum Jacobi sets are identical and therefore extend to identical functions in an identical subset of $\mathbb{R}\times\mathbb{R}$. That is, the functions on the right-hand sides of (1-5), (1-6), (1-8), (1-12), (1-13), and (1-15) are identically zero and (more interestingly) the functions on the right-hand sides of (1-10) and (1-17) are equal, as are the functions on the right-hand sides of (1-11) and (1-18).

Using quantum Jacobi properties established in Theorems 1.1 and 1.3, we show in Theorem 1.5 that the odd-balanced unimodal rank generating functions $\mathcal{M}^+$ and $\mathcal{N}^+$ can be expressed as simple Laurent polynomials when evaluated at pairs of rationals in $Q_{\mathcal{M}^+}$ and $Q_{\mathcal{N}^+}$, respectively. These evaluations are nontrivial, in that they do not simply follow from the q-hypergeometric definitions of the functions $\mathcal{M}^+$ and $\mathcal{N}^+$. These types of evaluations have been of interest lately specifically as related to radial limits of mock theta functions and can sometimes also be used to give simple polynomial evaluations of Eichler integrals [5, 8, 11–13, 21].

THEOREM 1.5. *The following identities hold.*

(i) *For* $(a/b,h/k)\in Q_{\mathcal{M}^+}$,

$$\mathcal{M}^+\Big(\frac{a}{b};\frac{h}{k}\Big)=-\frac{1}{2}\sum_{j=0}^{k-1}(\zeta_{2b}^{-(4j+1)a}\zeta_{16k}^{-(4j+1)^2h}-\zeta_{2b}^{-(4j+3)a}\zeta_{16k}^{-(4j+3)^2h}).$$

(ii) *For* $(a/b,h/k)\in Q_{\mathcal{N}^+}$,

$$\mathcal{N}^+\Big(\frac{a}{b};\frac{h}{k}\Big)=-\frac{1}{2}\sum_{j=0}^{k-1}(\zeta_{2b}^{-(4j+1)a}\zeta_{16k}^{-(4j+1)^2h}-\zeta_{2b}^{-(4j+3)a}\zeta_{16k}^{-(4j+3)^2h}).$$

REMARK 1.6. It is interesting to note that the same exact Laurent polynomial is used to evaluate the functions $\mathcal{M}^+$ and $\mathcal{N}^+$ in parts (i) and (ii) of Theorem 1.5. See also the remark following Theorem 1.3.

We also give a conjecture (which has since been proved; see the remark below) on the parity of $\eta(n)$, the number of odd-balanced unimodal sequences of size $2n$ with even parts congruent to 0 modulo 4 and odd parts at most half the peak.

 https://doi.org/10.1017/S1446788719000405

Conjecture 1.7. *For all nonnegative integers n, $\eta(n)$ is odd if and only if* $8n-1 = 3p^e\ell^2$ *or* $q^e m^2$, *where* p *and* q *are primes congruent to* 5 *and* 23 *modulo* 24, *respectively,* $e \equiv 1 \mod 4$, $p \nmid \ell$, *and* $q \nmid m$.

This implies Ramanujan-type congruences analogous to those satisfied by $u(n)$, the number of strongly unimodal sequences of size n, and $v(n)$, the number of odd-balanced unimodal sequences of size $2n+2$ given in [9] and [15].

Remark 1.8. Shortly after this paper was written, Lovejoy [16] provided a proof of Conjecture 1.7 using Bailey pairs and the arithmetic of $\mathbb{Z}[\sqrt{6}]$. Around the same time, Bringmann and Jennings-Shaffer [7, Theorem 1.3] also rediscovered this result and gave a proof along the same lines.

The remainder of this paper is organized as follows. In Section 2, we recall the definition of a quantum Jacobi form and give certain properties of some Jacobi forms which we use in our proofs. In Section 3, we introduce some technical lemmas that are instrumental to our work. Our main theorems are proved in Section 4.

2. Preliminaries

2.1. Quantum Jacobi forms. Quantum Jacobi forms were defined in [5] in 2016, naturally combining Zagier's definition of a quantum modular form from 2010 [20] and the definition of a Jacobi form, the theory of which was largely developed in the 1980s by Eichler and Zagier [10].

Definition 2.1. A weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \frac{1}{2}\mathbb{Z}$ quantum Jacobi form is a complex-valued function on $\mathbb{Q}\times\mathbb{Q}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}\times\mathbb{Z}$, the functions $h_\gamma : \mathbb{Q}\times(\mathbb{Q}\setminus\gamma^{-1}(i\infty)) \to \mathbb{C}$ and $g_{(\lambda,\mu)} : \mathbb{Q}\times\mathbb{Q}\to\mathbb{C}$ defined by

$$h_\gamma(z;\tau) := \phi(z;\tau) - \varepsilon_1^{-1}(\gamma)(c\tau+d)^{-k}e^{-2\pi imcz^2/(c\tau+d)}\phi\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big),$$
$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \varepsilon_2^{-1}((\lambda,\mu))e^{2\pi im(\lambda^2\tau+2\lambda z)}\phi(z+\lambda\tau+\mu;\tau)$$

satisfy a 'suitable' property of continuity or analyticity in a subset of $\mathbb{R}\times\mathbb{R}$.

Remarks 2.2.

(1) The complex numbers $\varepsilon_1(\gamma)$ and $\varepsilon_2((\lambda,\mu))$ satisfy $|\varepsilon_1(\gamma)| = |\varepsilon_2((\lambda,\mu))| = 1$; in particular, the $\varepsilon_1(\gamma)$ are such as those appearing in the theory of half-integral weight forms.

(2) We may modify the definition to allow modular transformations on appropriate subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domains of the functions h_γ and $g_{(\lambda,\mu)}$ to be suitable subsets of $\mathbb{Q}\times\mathbb{Q}$.

Prior to the two functions studied in this paper ($\mathcal{M}^+$ and $\mathcal{N}^+$), there were just three known examples of quantum Jacobi forms. As mentioned in Section 1, the first example is provided in [5] and the second and third examples are given in [4]. All of these functions, including $\mathcal{M}^+$ and $\mathcal{N}^+$ studied here, are q-hypergeometric series which are also two-variable combinatorial generating functions.

2.2. Modular forms and related functions. Here we recall some properties of various modular objects and related functions. We begin with the Mordell integral h defined in Section 1, which appeared in Zwegers' thesis on mock theta functions. The following properties were given in [22].

Lemma 2.3. *The following properties hold:*

(i) *h is an even function of z;*
(ii) $h(z;\tau) + h(z+1;\tau) = 2/\sqrt{-i\tau}e^{\pi i(z+1/2)^2/\tau}$;
(iii) $h(z;\tau) + e^{-2\pi iz-\pi i\tau}h(z+\tau;\tau) = 2e^{-\pi i(z+\tau/4)}$;
(iv) $h(z/\tau;-1/\tau) = \sqrt{-i\tau}e^{-\pi iz^2/\tau}h(z;\tau)$.

Under certain conditions, h can be re-written using the weight $3/2$ theta functions $g_{a,b}$, defined for $a, b \in \mathbb{R}$ and $\tau \in \mathbb{H}$ by

$$g_{a,b}(\tau) := \sum_{\nu\in a+\mathbb{Z}} \nu e^{\pi i\nu^2\tau+2\pi i\nu b}.$$

The functions $g_{a,b}$ transform as follows [19, 22].

Lemma 2.4. *With hypotheses as above, the functions $g_{a,b}$ satisfy:*

(i) $g_{a+1,b}(\tau) = g_{a,b}(\tau)$;
(ii) $g_{a,b+1}(\tau) = e^{2\pi ia}g_{a,b}(\tau)$;
(iii) $g_{a,b}(\tau+1) = e^{-\pi ia(a+1)}g_{a,a+b+1/2}(\tau)$;
(iv) $g_{a,b}(-1/\tau) = ie^{2\pi iab}(-i\tau)^{3/2}g_{b,-a}(\tau)$;
(v) $g_{-a,-b}(\tau) = -g_{a,b}(\tau)$.

The following result relates the functions h and $g_{a,b}$ [22].

Lemma 2.5. *For $a, b \in (-\frac{1}{2}, \frac{1}{2})$,*

$$\int_0^{i\infty} \frac{g_{a+1/2,b+1/2}(z)}{\sqrt{-i(z+\tau)}}\,dz = -e^{-\pi ia^2\tau+2\pi ia(b+1/2)}h(a\tau-b;\tau).$$

We also make use of the weight $1/2$ modular form η and the weight $1/2$ Jacobi form ϑ, defined for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ by

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \vartheta(z;\tau) := \sum_{n\in\mathbb{Z}+1/2} e^{\pi in^2\tau+2\pi in(z+1/2)}. \tag{2-1}$$

These functions satisfy the transformation properties given in Lemmas 2.6 and 2.7 below [18].

Lemma 2.6. *For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,*

$$\eta(\gamma\tau) = \chi_\gamma(c\tau+d)^{1/2}\eta(\tau),$$

where, for $c > 0$,

$$\chi_\gamma = \begin{cases} \dfrac{1}{\sqrt{i}}\Big(\dfrac{d}{c}\Big) i^{(1-c)/2} e^{\pi i(bd(1-c^2)+c(a+d))/12} & \textit{if } c \textit{ is odd,} \\ \dfrac{1}{\sqrt{i}}\Big(\dfrac{c}{d}\Big) e^{\pi i d/4} e^{\pi i(ac(1-d^2)+d(b-c))/12} & \textit{if } d \textit{ is odd.} \end{cases} \tag{2-2}$$

LEMMA 2.7. *For* $\lambda, \mu \in \mathbb{Z}, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *and* $(z, \tau) \in \mathbb{C} \times \mathbb{H}$:

(i) $\vartheta(z + \lambda\tau + \mu; \tau) = (-1)^{\lambda+\mu} q^{-\lambda^2/2} e^{-2\pi i \lambda z} \vartheta(z; \tau)$;

(ii) $\vartheta(z/(c\tau + d); \gamma\tau) = \chi_\gamma^3 (c\tau + d)^{1/2} e^{\pi i c z^2/(c\tau+d)} \vartheta(z; \tau)$;

(iii) $\vartheta(z; \tau) = -iq^{1/8} w^{-1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 - wq^{n-1})(1 - w^{-1}q^n)$.

Next we define Zwegers' mock Jacobi form μ, defined for $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C}$ ($u, v \notin \mathbb{Z}\tau + \mathbb{Z}$) by [22]

$$\mu(u, v; \tau) := \frac{e^{\pi i u}}{\vartheta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n\tau + 2\pi i u}}. \tag{2-3}$$

This form may be completed into a nonholomorphic Jacobi form by adding to it a certain nonholomorphic function as follows [22]:

$$\widehat{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau), \tag{2-4}$$

where the nonholomorphic function R is defined by

$$R(u; \tau) := \sum_{\nu \in 1/2+\mathbb{Z}} \{\mathrm{sgn}(\nu) - E((\nu + \alpha)\sqrt{2y})\}(-1)^{\nu-1/2} e^{-\pi i \nu^2 \tau - 2\pi i \nu u}, \tag{2-5}$$

with $y := \mathrm{Im}(\tau)$, $\alpha := \mathrm{Im}(u)/\mathrm{Im}(\tau)$ and

$$E(z) := 2 \int_0^z e^{-\pi u^2}\, du.$$

From [22, Propositions 1.9 and 1.10 and Theorem 1.11], we have the following transformation properties of R and $\widehat{\mu}$.

LEMMA 2.8. *With hypotheses as above, R satisfies the following transformation properties:*

(i) $R(u; \tau + 1) = e^{-\pi i/4} R(u; \tau)$;

(ii) $1/\sqrt{-i\tau}\, e^{\pi i u^2/\tau} R(u/\tau; -1/\tau) + R(u; \tau) = h(u; \tau)$;

(iii) $R(u; \tau) = R(-u; \tau)$;

(iv) $R(u; \tau) + e^{-2\pi i u - \pi i \tau} R(u + \tau; \tau) = 2e^{-\pi i u - \pi i \tau/4}$.

LEMMA 2.9. *With hypotheses as above, for* $k, l, m, n \in \mathbb{Z}$ *and* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *the function* $\widehat{\mu}$ *satisfies the following (nonholomorphic) Jacobi transformation properties:*

(i) $\widehat{\mu}(u+k\tau+l, v+m\tau+n;\tau) = (-1)^{k+l+m+n} e^{\pi i(k-m)^2\tau+2\pi i(k-m)(u-v)}\widehat{\mu}(u,v;\tau)$;

(ii) $\widehat{\mu}(u/(c\tau+d),\ v/(c\tau+d);\ (a\tau+b)/(c\tau+d)) = \chi^{-3}_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}(c\tau+d)^{1/2}e^{-\pi i c(u-v)^2/c\tau+d}$ $\widehat{\mu}(u,v;\tau)$.

Next, after Zwegers [23], we define the level-ℓ Appell functions A_ℓ for $\ell \in \mathbb{N}$, $z_1, z_2 \in \mathbb{C}$, and $\tau \in \mathbb{H}$ by

$$A_\ell(z_1,z_2;\tau) := \rho_1^{\ell/2}\sum_{n\in\mathbb{Z}} \frac{(-1)^{\ell n}\rho_2^n q^{\ell n(n+1)/2}}{1-\rho_1 q^n},$$

where $\rho_j = e^{2\pi i z_j}$, $j \in \{1,2\}$. As with Zwegers' μ function, the functions A_ℓ are completed with the function R. The completed level-ℓ Appell functions $\widehat{A}_\ell$ are defined by

$$\widehat{A}_\ell(z_1,z_2;\tau) := A_\ell(z_1,z_2;\tau) + \frac{i}{2}\sum_{j=0}^{\ell-1} e^{2\pi i j z_1}\vartheta\Big(z_2+j\tau+\frac{\ell-1}{2};\ell\tau\Big)R\Big(\ell z_1 - z_2 - j\tau - \frac{\ell-1}{2};\ell\tau\Big). \tag{2-6}$$

We have the following transformation properties of $\widehat{A}_\ell$ [23].

Lemma 2.10. *With hypotheses as above, for $n_1,n_2,m_1,m_2 \in \mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the functions $\widehat{A}_\ell$ satisfy the following transformation properties:*

(i) $\widehat{A}_\ell(-z_1,-z_2;\tau) = -\widehat{A}_\ell(z_1,z_2;\tau)$;

(ii) $\widehat{A}_\ell(z_1+n_1\tau+m_1, z_2+n_2\tau+m_2;\tau) = (-1)^{\ell(n_1+m_1)}\rho_1^{\ell n_1 - n_2}\rho_2^{-n_1}q^{\ell n_1^2/2-n_1n_2}\widehat{A}_\ell(z_1,z_2;\tau)$;

(iii) $\widehat{A}_\ell(z_1/(c\tau+d), z_2/(c\tau+d);\gamma\tau) = (c\tau+d)e^{\pi i c/(c\tau+d)(-\ell z_1^2+2z_1z_2)}\widehat{A}_\ell(z_1,z_2;\tau)$.

3. Lemmas

Here we establish various auxiliary lemmas which are used in Section 4 in the proofs of our main results. First we define the infinite subsets $Q_{\mathcal{M}^+}$ and $Q_{\mathcal{N}^+}$ of $\mathbb{Q}\times\mathbb{Q}$ on which $\mathcal{M}^+$ and $\mathcal{N}^+$ converge, respectively:

$$Q_{\mathcal{M}^+} := \left\{\left(\frac{a}{b},\frac{h}{k}\right)\in\mathbb{Q}\times\mathbb{Q}\ \middle|\ \begin{array}{l} a,h\in\mathbb{Z}, b,k\in\mathbb{N}, h,k \text{ odd},\\ \gcd(a,b)=\gcd(h,k)=1, \text{and } b\mid k\end{array}\right\},$$

$$Q_{\mathcal{N}^+} := \left\{\left(\frac{a}{b},\frac{h}{k}\right)\in\mathbb{Q}\times\mathbb{Q}\ \middle|\ \begin{array}{l} a\in\mathbb{Z}, b,k\in\mathbb{N}, h \text{ odd}, \gcd(a,b)=1,\\ \gcd(h,k)=1, \text{and either } k \text{ even and } b\mid k,\\ \text{or } k \text{ odd}, b \text{ even, and } b/2\mid k\end{array}\right\}.$$

Lemma 3.1. *The following identities hold.*

(i) *If $(z,\tau) = (a/b, h/k) \in Q^+_{\mathcal{M}}$, then there is a constant $C_{a,b,h,k}$ such that*

$$\mathcal{M}^+(z;\tau) = 2\cos\left(\frac{a\pi}{b}\right)\zeta_{16k}^{-h}\sum_{n=0}^{C_{a,b,h,k}}\frac{\zeta_{2k}^{h(2n+1)}(-\zeta_b^a\zeta_{2k}^h;\zeta_k^h)_n(-\zeta_b^{-a}\zeta_{2k}^h;\zeta_k^h)_n}{(\zeta_{2k}^h;\zeta_k^h)_{n+1}}. \tag{3-1}$$

(ii) *If* $(z,\tau) = (a/b, h/k) \in \mathcal{Q}_{\mathcal{N}^+}$, *then there is a constant* $D_{a,b,h,k}$ *such that*

$$\mathcal{N}^+(z;\tau) = 2\cos\Big(\frac{a\pi}{b}\Big)\zeta_{16k}^{-h}\sum_{n=0}^{D_{a,b,h,k}} \frac{\zeta_k^{h(n+1)}(-\zeta_b^a\zeta_k^h;\zeta_k^h)_n(-\zeta_b^{-a}\zeta_k^h;\zeta_k^h)_n}{(\zeta_{2k}^h;\zeta_k^h)_{n+1}}.$$

Proof. The proof of (ii) is similar to the proof of (i), so for brevity we omit it and prove (i) here. If h is odd, then the denominators of the summands of (3-1) are nonzero for every term in the series. We will show that the numerators of the summands eventually vanish by showing that there exists a $j \in \mathbb{N}$ such that

$$\pm\frac{a}{b} + \frac{h}{k}\Big(j + \frac{1}{2}\Big) \in \mathbb{Z} + \frac{1}{2}. \tag{3-2}$$

Since $b \mid k$, there exists $b' \in \mathbb{Z}$ such that $bb' = k$. Thus, (3-2) is equivalent to the congruence $\pm 2ab' + 2hj + h - k \equiv 0 \mod 2k$. Since h and k are odd, $(h-k)/2 \in \mathbb{Z}$. Moreover, since $(a/b, h/k) \in \mathcal{Q}_{\mathcal{M}^+}$, there exists h' such that $hh' \equiv -1 \mod 2k$. Then we may let j be a positive integer such that $j \equiv h'(\pm ab' + (h-k)/2) \mod 2k$. □

We will also use slightly different normalizations of the functions ϑ, μ, and A_3 introduced in the previous section. As in [17], we let

$$\begin{aligned} j(x;q) &:= (x;q)_\infty(x^{-1}q;q)_\infty(q;q)_\infty, \\ m(\rho,q,x) &:= \frac{1}{j(x;q)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(n-1)/2}x^n}{1-\rho x q^{n-1}}, \\ g(x,q) &:= x^{-1}\Big(-1 + \sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(x^{-1}q;q)_n}\Big). \end{aligned}$$

The functions j and m are nearly identical to the functions ϑ and μ defined in (2-1) and (2-3), respectively, after suitable changes of variables, and the function g is directly related to the function A_3. Note that g is a universal mock theta function and is also often referred to as g_3 in the literature, as in [14].

Lemma 3.2. *The following identities hold:*

(i) $j(w;q) = iw^{1/2}q^{-1/8}\vartheta(z;\tau)$;
(ii) $m(\rho_1\rho_2^{-1}, q, \rho_2) = i\rho_2^{1/2}\rho_1^{-1/2}q^{1/8}\mu(z_1,z_2;\tau)$, *where* $\rho_j = e(z_j)$, $j \in \{1,2\}$;
(iii) $1 - wg(-w,q^2) = iw^{-3/2}q^{1/12}\eta^{-1}(2\tau)A_3(z+\frac{1}{2}, -2\tau; 2\tau)$.

Proof. Identities (i) and (ii) follow immediately from the definitions of j, m, and μ and from Lemma 2.7. (In verifying (ii), the reader may find it convenient to perform a simple shift in the index of summation in the series defining m, as in [17, (2.6)].) Identity (iii) follows from the equation above equation (3-2) in [6], originally due to Atkin and Swinnerton-Dyer [3][1]. □

[1] We correct a minor typographical error in [6]: the left-hand side of the referenced equation should be multiplied by w.

Using Lemma 3.2 and equations (4.20) and (4.29) in [17], we establish the following expressions for $\mathcal{M}^+$ and $\mathcal{N}^+$ in terms of the Jacobi ϑ-function, Zwegers' mock Jacobi form μ, and the Appell function A_3.

Lemma 3.3. *We have that*

$$\mathcal{M}^+(z;\tau) = \mathcal{M}_1^+(z;\tau) + \mathcal{M}_2^+(z;\tau) + \mathcal{M}_3^+(z;\tau)$$

and

$$\mathcal{N}^+(z;\tau) = \mathcal{N}_1^+(z;\tau) + \mathcal{N}_2^+(z;\tau) + \mathcal{N}_3^+(z;\tau),$$

where

$$\mathcal{M}_1^+(z;\tau) := -i\mu\Big(z+\frac{1}{2}, \frac{1}{2}; \frac{\tau}{2}\Big),$$
$$\mathcal{M}_2^+(z;\tau) := -iw^{-1/2}q^{1/8}\frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\tau)\eta(\frac{\tau}{2})}A_3\Big(z+\frac{1}{2}, -\tau;\tau\Big),$$
$$\mathcal{M}_3^+(z;\tau) := \frac{\eta^4(\frac{\tau}{2})}{2\eta^2(\tau)\vartheta(z+\frac{1}{2};\frac{\tau}{2})}$$

and

$$\mathcal{N}_1^+(z;\tau) := -i\mu\Big(z+\frac{1}{2}, \frac{1}{2}; \frac{\tau}{2}\Big),$$
$$\mathcal{N}_2^+(z;\tau) := -q^{-1/24}\frac{\vartheta(z+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}\sum_{\pm} \pm w^{\pm 1/2}\mu\Big(\Big(1\pm\frac{1}{2}\Big)\tau - z + \frac{1}{2}, 2z+\tau; 3\tau\Big),$$
$$\mathcal{N}_3^+(z;\tau) := \frac{\vartheta(z;\frac{\tau}{2})\eta^2(\frac{\tau}{2})}{2\vartheta(2z;\tau)\eta(\tau)}.$$

We will also use Lemma 3.4 below in Section 4.

Lemma 3.4. *The following are true.*

(i) *If* $(z,\tau)\in Q_{\mathcal{M}^+}$, *then* $\vartheta(z+\tau/2+1/2;\tau)/\eta(\tau/2) = 0$.
(ii) *If* $(z,\tau)\in Q_{\mathcal{N}^+}$, *then* $\vartheta(z+1/2;\tau)/\eta(\tau/2) = 0$.

Proof. Using Lemma 2.7 and simplifying,

$$\frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})} = -q^{-7/48}w^{-1/2}\prod_{n=1}^{\infty}(1+wq^{n-1/2})(1+w^{-1}q^{n-1/2})(1+q^{n/2}).$$

By an argument similar to the proof of Lemma 3.1, there is a positive n such that $(1+wq^{n-\frac{1}{2}})$ is zero, so the infinite product vanishes. The proof of (ii) follows similarly using Lemma 2.7 and the definition of $Q_{\mathcal{N}^+}$. □

4. Proofs

4.1. Proof of Theorem 1.1. Equations (1-5), (1-6), and (1-8) follow directly from the definition of $\mathcal{M}^+$ in (1-1). Equations (1-10) and (1-11) follow directly from (1-7) and (1-9), respectively, using Lemma 3.4. Thus, we are left to prove (1-7), (1-9), and the analytic properties claimed in Theorem 1.1.

We begin with (1-9). Using the functional equation (4.21) in [17], as well as Lemma 3.2 and the definition of $\mathcal{M}^+$ in (1-1),

$$\mathcal{M}^+(z+\tau) - qw^2\mathcal{M}^+(z) = w^{1/2}q^{7/16}\Big(q^{1/2}w - 1 - w^{-1/2}q^{-17/48}(1-w^2q)\frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}\Big).$$

Equation (1-9) follows after simplification and re-arrangement of terms.

To prove (1-7), we begin with the expressions given for $\mathcal{M}^+$ in Lemma 3.3. We define the following completed functions:

$$\widehat{\mathcal{M}}_1(z;\tau) := -i\widehat{\mu}\Big(z+\frac{1}{2},\frac{1}{2};\frac{\tau}{2}\Big),$$

$$\widehat{\mathcal{M}}_2(z;\tau) := -iw^{-1/2}q^{1/8}\frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\tau)\eta(\frac{\tau}{2})}\widehat{A}_3\Big(z+\frac{1}{2},-\tau;\tau\Big),$$

$$\widehat{\mathcal{M}}_3(z;\tau) := \frac{\eta^4(\frac{\tau}{2})}{2\eta^2(\tau)\vartheta(z+\frac{1}{2};\frac{\tau}{2})}.$$

Note that $\widehat{\mathcal{M}}_3 = \mathcal{M}_3^+$ because, as we shall show, $\mathcal{M}_3^+$ is a Jacobi form (see (4-6)). Using Lemmas 2.6, 2.7, 2.9, and 2.10, after some calculation and simplification, we obtain the following Jacobi transformation properties, which hold for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma^0(2)\cap\Gamma_0(6)$ (noting that these congruence conditions are imposed in order to apply portions of the aforementioned lemmas) and $j\in\{1,2,3\}$:

$$\widehat{\mathcal{M}}_j\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big) = \psi_j(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}\widehat{\mathcal{M}}_j(z;\tau), \tag{4-1}$$

where

$$\psi_1 = \psi_1\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \chi^{-3}_{\left(\begin{smallmatrix} a & b/2 \\ 2c & d\end{smallmatrix}\right)}(-1)^{c+d-1},$$

$$\psi_2 = \psi_2\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \chi^{2}_{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)}\chi^{-1}_{\left(\begin{smallmatrix} a & b/2 \\ 2c & d\end{smallmatrix}\right)}(-1)^{(a+b-1)/2}e\big(\tfrac{1}{8}(-2+2a-2cd-2bc+ab)\big),$$

$$\psi_3 = \psi_3\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \chi_{\left(\begin{smallmatrix} a & b/2 \\ 2c & d\end{smallmatrix}\right)}\chi^{-2}_{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)}(-1)^{c+(1-d)/2}e\Big(\frac{c}{2}-\frac{cd}{4}\Big).$$

A lengthy but straightforward calculation using (2-2) reveals that ψ_1,ψ_2, and ψ_3 are equal for any $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma^0(2)\cap\Gamma_0(6)$, so that for $j\in\{1,2,3\}$,

$$\widehat{\mathcal{M}}_j\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big) = \psi(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}\widehat{\mathcal{M}}_j(z;\tau),$$

where $\psi = \psi\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \psi_1 = \psi_2 = \psi_3$.

By definition and (2-4),

$$\widehat{\mathcal{M}_1}(z;\tau) = \mathcal{M}_1^+(z;\tau) + \frac{1}{2}R\Big(z;\frac{\tau}{2}\Big). \tag{4-2}$$

We also apply the definition of $\widehat{\mathcal{M}_2}(z;\tau)$ and (2-6), as well as the facts that $\vartheta(1;\tau)=0$, that $\vartheta(-z;\tau) = -\vartheta(z;\tau)$, and that $\vartheta(\tau;3\tau) = -iq^{-1/6}\eta(\tau)$, together with Lemma 2.7(i), to conclude that

$$\widehat{\mathcal{M}_2}(z;\tau) = \mathcal{M}_2^+(z;\tau) + \frac{1}{2}w^{1/2}q^{-1/24}\frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}\sum_{\pm}(iw)^{\pm 1}R\Big(3z\mp\tau+\frac{1}{2};3\tau\Big). \tag{4-3}$$

Thus, we obtain from (4-1) and the discussion following, including (4-2) and (4-3), that for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma^0(2)\cap\Gamma_0(6)$,

$$\mathcal{M}_1^+\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big) - \psi(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}\mathcal{M}_1^+(z;\tau) = -\frac{1}{2}R\Big(\frac{z}{c\tau+d};\frac{a\tau+\frac{b}{2}}{2c\tau+d}\Big) + \frac{1}{2}\psi(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}R(z;\tfrac{\tau}{2}), \tag{4-4}$$

$$\begin{aligned}\mathcal{M}_2^+\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big) &- \psi(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}\mathcal{M}_2^+(z;\tau)\\ &= -\frac{1}{2}e\Big(\frac{z/2-(a\tau+b)/24}{c\tau+d}\Big)\frac{\vartheta(\frac{2z+a\tau+b+c\tau+d}{2(c\tau+d)};\frac{a\tau+b}{c\tau+d})}{\eta(\frac{a\tau+b/2}{2c\tau+d})}\\ &\quad\times\sum_{\pm}e\Big(\pm\Big(\frac{z}{c\tau+d}+\frac{1}{4}\Big)\Big)R\Big(\frac{3z\mp(a\tau+b)}{c\tau+d}+\frac{1}{2};\frac{3(a\tau+b)}{c\tau+d}\Big)\\ &\quad+\frac{1}{2}\psi(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}w^{1/2}q^{-1/24}\frac{\vartheta(z+\frac{\tau}{2}+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}\\ &\quad\times\sum_{\pm}(iw)^{\pm1}R(3z\mp\tau+\tfrac{1}{2};3\tau),\end{aligned} \tag{4-5}$$

$$\mathcal{M}_3^+\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big) - \psi(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}\mathcal{M}_3^+(z;\tau) = 0. \tag{4-6}$$

If we denote the right-hand sides of (4-4) and (4-5) by $F_1(z;\tau)$ and $F_2(z;\tau)$, respectively, then we have shown (for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$) that

$$\mathcal{M}^+\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big) - \psi(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}\mathcal{M}^+(z;\tau) = F_1(z;\tau)+F_2(z;\tau). \tag{4-7}$$

Next we write $\left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)$ as $S^{-1}T^{-6}S$ (where $S := \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), T := \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$) and apply Lemma 2.8 repeatedly. After some lengthy but straightforward calculations,

$$R\Big(\frac{z}{6\tau+1};\frac{\tau}{2(6\tau+1)}\Big) = h\Big(\frac{z}{6\tau+1};\frac{\tau}{2(6\tau+1)}\Big) + (6\tau+1)^{1/2}e\Big(\frac{-6z^2}{6\tau+1}\Big)\Big(h\Big(z;\frac{\tau}{2}\Big) - R\Big(z;\frac{\tau}{2}\Big)\Big), \tag{4-8}$$

$$R\Big(\frac{3z+(3\mp 1)\tau+\frac{1}{2}}{6\tau+1};\frac{3\tau}{6\tau+1}\Big) = h\Big(\frac{3z+(3\mp 1)\tau+\frac{1}{2}}{6\tau+1};\frac{3\tau}{6\tau+1}\Big) - (6\tau+1)^{1/2}e\Big(\frac{-\tau^2\pm 6\tau z-9z^2}{6\tau+1}\Big)\Big[h\Big(3z\mp\tau+\frac{1}{2};3\tau\Big) - R\Big(3z\mp\tau+\frac{1}{2};3\tau\Big)\Big]. \tag{4-9}$$

Substituting (4-8) and (4-9) into (4-7) with $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\6&1\end{smallmatrix}\right)$, applying Lemmas 2.6 and 2.7, and re-arranging, we obtain (1-7).

It remains to show that the 'errors' on the right-hand sides of (1-10) and (1-11) extend to C^∞ functions on $(\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{24},\pm\frac{1}{8},\pm\frac{5}{24},\pm\frac{7}{24},\pm\frac{3}{8},\pm\frac{11}{24}\}))\times(\mathbb{R}\setminus\{-\frac{1}{6}\})$, and $\mathbb{R}\times\mathbb{R}$, respectively. The claim with respect to (1-11) is clear. Regarding (1-10), we first prove that H_1 is C^∞ on $(-\frac{1}{24},\frac{1}{24})\times(\mathbb{R}\setminus\{-\frac{1}{6}\})$ and then explain why it suffices to prove the result on this restricted interval in z. To begin, we apply Lemmas 2.4 and 2.5, with $a=0$ and $b=-z$, with $z\in(-\frac{1}{2},\frac{1}{2})$, to write the Mordell integral $h(z;\frac{\tau}{2})$ as the period integral

$$h\Big(z;\frac{\tau}{2}\Big) = -\int_0^{i\infty}\frac{g_{1/2,-z+1/2}(\rho)}{\sqrt{-i(\frac{\tau}{2}+\rho)}}\,d\rho. \tag{4-10}$$

Applying Lemma 2.3 and using the substitutions $a=-z, b=12z$, where $z\in(-\frac{1}{24},\frac{1}{24})$,

$$\begin{aligned} h\Big(\frac{z}{6\tau+1};\frac{\tau}{2(6\tau+1)}\Big) &= \frac{1}{\sqrt{-i(\frac{\tau}{2(6\tau+1)})}}e\Big(\frac{z^2}{\tau(6\tau+1)}\Big)h\Big(\frac{2z}{\tau};-12-\frac{2}{\tau}\Big) \\ &= \frac{-1}{\sqrt{-i(\frac{\tau}{2(6\tau+1)})}}e\Big(\frac{z^2}{\tau(6\tau+1)}-\frac{z^2}{\tau}+6z^2+\frac{z}{2}\Big)\int_0^{i\infty}\frac{g_{-z+1/2,12z+1/2}(\rho)}{\sqrt{-i(\rho-12-\frac{2}{\tau})}}\,d\rho. \end{aligned} \tag{4-11}$$

Making the change of variable $\rho\mapsto 12-1/\rho$ and applying Lemmas 2.4 and 2.5, we see that the integral on the right-hand side of (4-11) equals

$$\int_{1/12}^{0} \frac{g_{-z+1/2,12z+1/2}(12-\frac{1}{\rho})}{\rho^2\sqrt{-i(-\frac{1}{\rho}-\frac{2}{\tau})}}\,d\rho$$
$$= -(-i)^{5/2}\sqrt{\frac{\tau}{2}}e\Big(-\frac{z}{2}-6z^2\Big)\int_{1/12}^{0}\frac{g_{1/2,-z+1/2}(\rho)}{\sqrt{-i(\rho+\frac{\tau}{2})}}\,d\rho. \tag{4-12}$$

Combining (4-11) and (4-12) gives

$$h\Big(\frac{z}{6\tau+1};\frac{\tau}{2(6\tau+1)}\Big) = -(6\tau+1)^{1/2}e\Big(\frac{-6z^2}{6\tau+1}\Big)\int_{1/12}^{0}\frac{g_{1/2,-z+1/2}(\rho)}{\sqrt{-i(\rho+\frac{\tau}{2})}}\,d\rho. \tag{4-13}$$

Now substituting (4-10) and (4-13) into (1-2) gives

$$H_1(z;\tau) = \frac{1}{2}\int_{1/12}^{i\infty}\frac{g_{1/2,-z+1/2}(\rho)}{\sqrt{-i(\rho+\frac{\tau}{2})}}\,d\rho. \tag{4-14}$$

The proof that the function in (4-14) is C^∞ follows by an almost identical argument as given by Bringmann and the second author in [5] and in the follow-up paper [4]; note that the same integrand, namely the function $g_{1/2,-z+1/2}(\rho)/\sqrt{-i(\tau+\rho)}$, appears in [5, Proof of Theorem 1.1, page 375] (and also in [4]). By Lemma 3.4, $H_2(z;\tau)$ vanishes on $\mathcal{Q}_{\mathcal{M}^+}$. The proof that $H_1(z;\tau)$ is C^∞ is similar to the argument given in [4]: we integrate along the path from $\frac{1}{12}$ to $i\infty$ and the path from $\frac{1}{12}+i\infty$ to $i\infty$. Using the following bound given in [5],

$$\frac{\partial^\ell}{\partial z^\ell}g_{1/2,-z+1/2}\Big(\frac{1}{12}+it\Big) \ll e^{-\pi/4t},$$

we have that $H_1(z;\tau)$ is C^∞ on $(-\frac{1}{24},\frac{1}{24})\times(\mathbb{R}\setminus\{-\frac{1}{6}\})$ by the Leibniz rule.

To establish the claim for more general z which lie in the set $(\mathbb{R}\setminus(\mathbb{Z}+\{\frac{1}{2},\pm\frac{1}{24},\pm\frac{1}{8},\pm\frac{5}{24},\pm\frac{7}{24},\pm\frac{3}{8},\pm\frac{11}{24}\}))$, we argue as in [4, 5]. Briefly speaking, this is made possible by Lemma 2.3: we use Lemma 2.3 to translate the Mordell integral h in the elliptic variable by integers or integer multiples of τ up to addition of analytic functions ($\tau \neq 0$). After doing so, we may proceed as with (4-10) and re-write the h-function which appears as a period integral. The rest of the argument follows as above. In particular, we point out that the rationals excluded from the domain of z arise from the hypotheses given in Lemma 2.5 (as is made explicit above in the case $z\in(-\frac{1}{24},\frac{1}{24})$). We refer the interested reader to [5, Proof of Theorem 1.1, page 375] or [4, Proof of Theorem 1.1, page 31] for more details carried out there.

4.2. Proof of Theorem 1.3. The proof of Theorem 1.3 is very similar to the proof of Theorem 1.1, so we provide a detailed sketch. Equations (1-12), (1-13), (1-15), and (1-16) follow in a straightforward manner, analogous to the proofs of their counterparts in Theorem 1.1. Equation (1-17) follows from (1-14) and Lemma 3.4, and equation (1-18) follows from (1-16) and Lemma 3.4. Thus, we are left to prove (1-14) and the analytic properties claimed in Theorem 1.3.

Define

$$\widehat{\mathcal{N}}_1(z;\tau) := -i\widehat{\mu}\Big(z+\frac{1}{2},\frac{1}{2};\frac{\tau}{2}\Big),$$

$$\widehat{\mathcal{N}}_2(z;\tau) := -q^{-1/24}\frac{\vartheta(z+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})}\sum_{\pm}\pm w^{\pm 1/2}\widehat{\mu}\Big(\Big(1\pm\frac{1}{2}\Big)\tau - z + \frac{1}{2}, 2z+\tau; 3\tau\Big),$$

$$\widehat{\mathcal{N}}_3(z;\tau) := \frac{\vartheta(z;\frac{\tau}{2})\eta^2(\frac{\tau}{2})}{2\vartheta(2z;\tau)\eta(\tau)}.$$

Using Lemmas 2.6, 2.7, and 2.9, we find, after some simplification, that for any $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $a \equiv 1 \mod 6$, $b \equiv 0 \mod 2$, and $c \equiv 0 \mod 6$, and $j \in \{1,2,3\}$, that

$$\widehat{\mathcal{N}}_j\Big(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\Big) = \sigma_j(c\tau+d)^{1/2}e^{-2\pi icz^2/(c\tau+d)}\widehat{\mathcal{N}}_j(z;\tau), \tag{4-15}$$

where

$$\sigma_1 = \sigma_1(\gamma) := \chi^{-3}_{\left(\begin{smallmatrix} a & b/2\\ 2c & d\end{smallmatrix}\right)},$$

$$\sigma_2 = \sigma_2(\gamma) := \chi^{3}_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}\chi^{-1}_{\left(\begin{smallmatrix} a & b/2\\ 2c & d\end{smallmatrix}\right)}\chi^{-3}_{\left(\begin{smallmatrix} a & 3b\\ c/3 & d\end{smallmatrix}\right)}(-1)^{c/6+b/2}\zeta_{24}^{-ab},$$

$$\sigma_3 = \sigma_3(\gamma) := \chi^{5}_{\left(\begin{smallmatrix} a & b/2\\ 2c & d\end{smallmatrix}\right)}\chi^{-4}_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}.$$

A calculation shows that under the given conditions on γ, $\sigma_1 = \sigma_2 = \sigma_3$. Using this fact, as well as (4-15) and Lemma 3.3, we obtain (with $\gamma = \left(\begin{smallmatrix} 1 & 0\\ 6 & 1\end{smallmatrix}\right)$) that

$$\mathcal{N}^+(z;\tau) + (6\tau+1)^{-1/2}e^{12\pi iz^2/(6\tau+1)}\mathcal{N}^+\Big(\frac{z}{6\tau+1};\frac{\tau}{6\tau+1}\Big) = F_1(z;\tau) + F_3(z;\tau),$$

where the 'error function' F_1 is exactly as given in the proof of Theorem 1.1 (see (4-7)) and F_3 is defined similarly using the Mordell integral h in (1-4) and the nonholomorphic function R in (2-5). As was the case in the proof of Theorem 1.1, we have that $F_1 = H_1$, where H_1 is as in (1-3). Similarly, we apply Lemma 2.8 repeatedly and after some calculation obtain that $F_3 = H_3$, where

$$\begin{aligned} H_3(z;\tau) := \frac{i}{2}q^{-1/24}\frac{\vartheta(z+\frac{1}{2};\tau)}{\eta(\frac{\tau}{2})} &\\ \times\Big(\sum_{\pm}\pm w^{\pm 1/2}h\Big(\mp\frac{\tau}{2}+3z-\frac{1}{2};3\tau\Big) &- (6\tau+1)^{-1/2}e\Big(\frac{1}{2}\Big(\frac{18z^2}{6\tau+1}+\frac{\tau^2/2}{6\tau+1}\Big)\Big)\\ &\times\sum_{\pm}\pm\alpha_{\pm}\Big(z;\frac{\tau}{2}\Big)h\Big(\mp\frac{1}{6}+\frac{z}{\tau}-1-\frac{1}{6\tau};-2-\frac{1}{3\tau}\Big)\Big), \end{aligned}\tag{4-16}$$

where

$$\alpha_{\pm}(z;\tau) := \sqrt{-i\Big(-2-\frac{1}{6\tau}\Big)}e\Big(\frac{\pm z}{2(12\tau+1)}\Big)e\Big(\frac{-(\mp\frac{1}{6}+\frac{z}{2\tau}-1-\frac{1}{12\tau})^2}{2(-2-\frac{1}{6\tau})}\Big).$$

This proves (1-14).

The functions on the right-hand sides of (1-17) and (1-18) are exactly the same as on the right-hand sides of (1-10) and (1-11), respectively, so by the argument given regarding the C^∞ properties of these functions given in the proof of Theorem 1.1, the proof of Theorem 1.3 is complete.

4.3. Proof of Theorem 1.5. We proceed as in [4, 5]. Suppose that a function $f(z;\tau)$ transforms in z as

$$f(z+\tau;\tau) = w^2 q(f(z;\tau) + r(z;\tau)) \tag{4-17}$$

for some function $r(z;\tau)$. By induction on $m \in \mathbb{N}_0$, it can be shown that

$$f(z+m\tau;\tau) = w^{2m} q^{m^2} f(z;\tau) + \sum_{j=0}^{m-1} r(z+j\tau;\tau) w^{2(m-j)} q^{m^2-j^2}.$$

We can see from (1-11) and (1-18) that $\mathcal{M}^+$ and $\mathcal{N}^+$ satisfy (4-17) with $r(z;\tau) = w^{-1/2}q^{-1/16} - w^{-3/2}q^{-9/16}$, so

$$\begin{aligned}
&\mathcal{M}^+(z+h;\tau)|_{(z,\tau)=(a/b,h/k)\in Q_{\mathcal{M}^+}} \\
&\qquad = w^{2k} q^{k^2} \mathcal{M}^+(z;\tau) + \sum_{j=0}^{k-1} r(z+j\tau;\tau) w^{2(k-j)} q^{k^2-j^2}, \\
&\mathcal{N}^+(z+h;\tau)|_{(z,\tau)=(a/b,h/k)\in Q_{\mathcal{N}^+}} \\
&\qquad = w^{2k} q^{k^2} \mathcal{N}^+(z;\tau) + \sum_{j=0}^{k-1} r(z+j\tau;\tau) w^{2(k-j)} q^{k^2-j^2}.
\end{aligned}$$

From (1-8) and (1-15), we have that for any $h \in \mathbb{Z}$,

$$\mathcal{M}^+(z+h,\tau) = (-1)^h \mathcal{M}^+(z;\tau) \quad \text{and} \quad \mathcal{N}^+(z+h,\tau) = (-1)^h \mathcal{N}^+(z;\tau).$$

It follows that

$$\begin{aligned}
[(-1)^h - w^{2k} q^{k^2}]\mathcal{M}^+(z;\tau) &= \sum_{j=0}^{k-1} r(z+j\tau;\tau) w^{2(k-j)} q^{k^2-j^2}, \\
[(-1)^h - w^{2k} q^{k^2}]\mathcal{N}^+(z;\tau) &= \sum_{j=0}^{k-1} r(z+j\tau;\tau) w^{2(k-j)} q^{k^2-j^2}.
\end{aligned}$$

For the function $\mathcal{M}^+$, h is odd and $b \mid k$, so the factor in front of $\mathcal{M}^+$ in the above equation is -2. For the function $\mathcal{N}^+$, there are two cases to consider (arising from the definition of $Q_{\mathcal{N}^+}$). If h is odd, k is even, and $b \mid k$, then the factor in front of $\mathcal{N}^+$ in the above equation is -2. If h is odd, k is odd, b is even, and $b/2 \mid k$, then again the factor in front of $\mathcal{N}^+$ in the above equation is -2. The result of Theorem 1.5 now follows after simplification.

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