Carroll et al. *Res Math Sci(2019)6:6*
https://doi.org/10.1007/s40687-018-0169-6

Research in
the Mathematical Sciences

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Universal mock theta functions as quantum Jacobi forms

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Abstract

Quantum Jacobi forms were defined in 2016, naturally combining Zagier's definition of a quantum modular form with that of a Jacobi form. To date, just three examples of such functions exist in the literature. Here, we prove that the universal mock theta function g_2, as well as the universal mock theta functions K, K_1, K_2, and κ, gives rise to an infinite family of quantum Jacobi forms $G_{a,b}(z;\tau)$ of weight 1/2 in dense subsets $\mathcal{Q}_{a,b} \subseteq \mathbb{Q} \times \mathbb{Q}$. We then use these quantum Jacobi transformation properties to establish polynomial expressions for $G_{a,b}$ at pairs of rational numbers, as well as simple closed-form expressions for sums of Mordell integrals.

Mathematics Subject Classification: 11F03, 11F37, 11F50, 11F99, 33D70

1 Introduction and statement of results

Understanding the relationship between mock modular forms and quantum modular forms is a current problem of interest (see, for example, [4–8,12,14,16,17,22], and references therein). Both transform with respect to $\mathrm{SL}_2(\mathbb{Z})$ (or a subgroup) up to certain error functions, but their domains are notably different: by definition, mock modular forms are defined on the upper half-plane $\mathbb{H} := \{x + iy : x, y \in \mathbb{R}, y > 0\}$, while quantum modular forms are defined on $\mathbb{Q}$. Quantum Jacobi forms, two variable analogues to quantum modular forms, were defined in 2016 [2], naturally combining Zagier's definition of a quantum modular form with that of a Jacobi form [9]. These forms transform on $\mathbb{Q} \times \mathbb{Q}$ up to suitably analytic or continuous error functions in $\mathbb{R} \times \mathbb{R}$ (see Sect. 2 for a more precise definition). To date, three examples of quantum Jacobi forms exist in the literature [1,2], each of which is a combinatorial generating function.

Here, we establish an infinite family of quantum Jacobi forms, arising from the universal mock theta function g_2, defined by

$$g_2(w;q) := \sum_{n\geq 0} \frac{(-q;q)_n q^{n(n+1)/2}}{(w;q)_{n+1}(w^{-1}q;q)_{n+1}},$$

where the q-Pochhammer symbol is defined for non-negative integers n by $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. The function g_2 is aptly named "universal", as specializations of this function in the variables w and q yield all of Ramanujan's mock theta functions (up to the addition of modular forms) [13]. Other q-hypergeometric series may also be viewed as universal mock theta functions in the same sense as g_2. Namely (adopting notation from [5,13,19]), we define the universal mock theta functions

$$K(w;q):=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(wq^2;q^2)_n(w^{-1}q^2;q^2)_n},\quad K_1(w;q):=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(wq;q^2)_{n+1}(w^{-1}q;q^2)_{n+1}},$$

$$K_2(w;q):=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}(-1;q)_n}{(wq;q)_n(w^{-1}q;q)_n},\quad \kappa(w;q):=\sum_{n=0}^{\infty}\frac{q^{n+1}w^{-n}(wq^2;q^2)_n}{(wq;q^2)_{n+1}}.$$

These q-hypergeometric series also carry combinatorial meaning. For example, it was proved in [18] that K_2 is a two-variable generating function for overpartition ranks:

$$K_2(w;q)=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{p}(m,n)w^m q^n,$$

where $\overline{p}(m,n):=\#\{\text{overpartitions of } n \text{ with D-rank } m\}$. We deduce from results in [5,19] that the universal mock theta functions K, K_1, K_2, and κ are related to g_2 by the following identities:

$$\begin{aligned}
g_2(w;q)&=\frac{-w}{q+w^2}K(-w^2q^{-1};q^2)+\frac{iw^{-\frac{1}{2}}q^{-\frac{1}{8}}\eta^3(2\tau)}{\vartheta(z;\tau)\vartheta(\frac{1}{2}+2z-\tau;4\tau)}\\
&=-(wq^{-1}+w^{-1})K_1(-w^2q^{-1};q^2)+\frac{iw^{-\frac{1}{2}}q^{-\frac{1}{8}}\eta^3(2\tau)}{\vartheta(z;\tau)\vartheta(\frac{1}{2}+2z-\tau;4\tau)}\\
&\quad+\frac{q^{-\frac{1}{4}}\eta^4(2\tau)}{\eta^2(4\tau)\vartheta(\frac{1}{2}+2z-\tau;2\tau)}\\
&=\frac{1}{2w}\left(-1+\frac{1+w}{1-w}K_2(w;q)\right)\\
&=-(wq^{-1}+w^{-1})\kappa(-w^2q^{-1};q^2)+\frac{iw^{-\frac{1}{2}}q^{-\frac{1}{8}}\eta^3(2\tau)}{\vartheta(z;\tau)\vartheta(\frac{1}{2}+2z-\tau;4\tau)}\\
&\quad+\frac{q^{-\frac{1}{4}}\eta^4(2\tau)}{\eta^2(4\tau)\vartheta(\frac{1}{2}+2z-\tau;2\tau)},
\end{aligned}$$

where the weight $1/2$ modular form η is defined in (2.1), and the weight $1/2$ Jacobi form ϑ is defined in (2.2), and we have let $w:=e^{2\pi iz}$, $q:=e^{2\pi i\tau}$.

Our infinite family of functions $G_{a,b}$ is defined in terms of g_2 for positive integers a and b, with $a<b$ and even $b\geq 4$, as follows:

$$G_{a,b}(z;\tau):=iq^{-a^2/b^2+a/b}w^{a/b-1/2}g_2(-iq^{\frac{a}{b}}w^{-\frac{1}{2}};q),\tag{1.1}$$

where as above, $w:=e^{2\pi iz}$, $q:=e^{2\pi i\tau}$. In Sect. 5, we explain that the functions $G_{a,b}$ are defined away from particular subsets $\Lambda_{a,b}$ (see (4.3)) of $\mathbb{C}\times\mathbb{H}$, and are also defined on subsets $\mathcal{Q}_{a,b}\subseteq\mathbb{Q}\times\mathbb{Q}$ (see (5.1)), which we prove are dense in Proposition 5.1.

Our first result shows that the functions $G_{a,b}$ form an infinite family of quantum Jacobi forms with respect to the subgroups $\Gamma'_{b,1}\subseteq \mathrm{SL}_2(\mathbb{Z})$ (see (3.5)).

Theorem 1.1 *For all* $(z,\tau)\in((\mathbb{C}\times\mathbb{H})\backslash\Lambda_{a,b})\cup\mathcal{Q}_{a,b}$, $A=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\in\Gamma'_{b,1}$, *and* $m\in\mathbb{Z}$,

(i) $G_{a,b}(z;\tau)-\Psi_{a,b}^{-1}\left(\begin{smallmatrix}\alpha&2\beta\\ \frac{\gamma}{2}&\delta\end{smallmatrix}\right)(\gamma\tau+\delta)^{-1/2}e^{\pi i\gamma z^2/(2(\gamma\tau+\delta))}G_{a,b}\left(\frac{z}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$
$=H_A(z;\tau)$,

(ii) $G_{a,b}(z;\tau) - w^{-m}q^{-m^2}G_{a,b}(z+2m\tau;\tau) = -\mathrm{sgn}(m)q^{\frac{-a^2}{b^2}}w^{\frac{a}{b}}\sum_{j=\frac{m}{2}(1-\mathrm{sgn}(m))+1}^{\frac{m}{2}(1+\mathrm{sgn}(m))} q^{-j^2}w^{-j}q^{\frac{2aj}{b}}$,

(iii) $G_{a,b}(z;\tau) - e^{\frac{-4\pi iam}{b}}G_{a,b}(z+2m;\tau) = 0$,

where the function H_A is defined in (4.2). Moreover, H_A is C^∞ on $\mathbb{R}\times(\mathbb{R}\backslash Z_A)$. In particular, $G_{a,b}$ is a quantum Jacobi form of weight 1/2 and index $-1/4$ on $\mathcal{Q}_{a,b}$ with Jacobi group $\Gamma'_{b,1} \ltimes (2\mathbb{Z}\times 2\mathbb{Z})$.

Remarks (1) The Dirichlet character $\Psi_{a,b}$ in Theorem 1.1 is defined explicitly in [17, equation (1.5.8)], and the subset $Z_A \subseteq \mathbb{Q}$ is defined in (3.8).

(2) As suggested by the transformations in $\mathbb{C}\times\mathbb{H}$ given in Theorem 1.1, we also have that the functions $G_{a,b}$ are mock Jacobi forms. In particular, these functions can be realized as holomorphic parts of non-holomorphic Jacobi forms (see [17, Theorem 2.1.4]).

(3) Note that the definition of $G_{a,b}$ in (1.1) depends only on a/b (as opposed to a and b). Here and throughout, however, we do not require a and b to be coprime, as $G_{a,b}$ is shown in [17, equation (2.1.14)] to be directly related to $\Phi_{a,b}$ from [3, equation (3.3)], in which a and b are not required to be coprime.

We exploit the quantum Jacobi properties of the functions $G_{a,b}$ given in Theorem 1.1 to provide polynomial expressions for these functions at pairs of rational numbers in $\mathcal{Q}_{a,b}$. Special values of quantum modular (and quantum Jacobi) forms have been of recent interest, as they have been shown to equal radial limits of mock theta functions [6,12,14]. Moreover, they may sometimes be used to provide simple exact expressions for Eichler integrals [10,11]. In Theorem 1.2 and throughout, we let $\zeta_N := e^{\frac{2\pi i}{N}}$.

Theorem 1.2 *Let $(\frac{r}{s},\frac{h}{k}) \in \mathcal{Q}_{a,b}$ with $\frac{2ah}{b} - \frac{rk}{s} \notin \mathbb{Z}$. Then we have that*

$$G_{a,b}\left(\frac{r}{s};\frac{h}{k}\right) = \frac{\zeta_{kb^2}^{-ha^2}\zeta_{sb}^{ra}}{\zeta_b^{2ah}-\zeta_s^{rk}}\sum_{j=1}^{k}\zeta_k^{-j^2h}\zeta_s^{r(k-j)}\zeta_{kb}^{2ahj}.$$

Remark We point out that the subset of $\mathbb{Q}\times\mathbb{Q}$ on which Theorem 1.2 holds is also dense in $\mathbb{R}\times\mathbb{R}$. (See also Proposition 5.1.)

Combining Theorems 1.1 and 1.2, we are able to evaluate the function H_A, initially given in (4.2) in terms of Mordell integrals

$$h(z;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau x^2 - 2\pi zx}}{\cosh(\pi x)}\mathrm{d}x, \tag{1.2}$$

as a simple sum of polynomials in roots of unity.

Corollary 1.3 *Let $(\frac{r}{s},\frac{h}{k}) \in \mathcal{Q}_{a,b}$ with $\frac{2ah}{b} - \frac{rk}{s} \notin \mathbb{Z}$, and let $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma'_{b,1}$. Moreover, let $h', k', r', s' \in \mathbb{Z}$ with $k', s' > 0$ and $\gcd(h',k') = \gcd(r',s') = 1$ such that*

$$\frac{h'}{k'} = \frac{\alpha h + \beta k}{\gamma h + \delta k}, \quad \textit{and} \quad \frac{r'}{s'} = \frac{rk}{\gamma hs + \delta ks}.$$

Then, the sum of Mordell integrals $H_A\left(\frac{r}{s},\frac{h}{k}\right)$ *[defined in (4.2)] may be explicitly evaluated as*

$$H_A\left(\frac{r}{s},\frac{h}{k}\right)=\frac{\zeta_{kb^2}^{-ha^2}\zeta_{sb}^{ra}}{\zeta_b^{2ah}-\zeta_s^{rk}}\sum_{j=1}^{k}\zeta_k^{-j^2h}\zeta_s^{r(k-j)}\zeta_{kb}^{2ahj}$$

$$-\Psi_{a,b}^{-1}\begin{pmatrix}\alpha & 2\beta\\ \frac{\gamma}{2} & \delta\end{pmatrix}\left(\gamma\frac{h}{k}+\delta\right)^{-1/2}e^{\pi i\gamma\frac{r^2}{s^2}/(2(\gamma\frac{h}{k}+\delta))}\frac{\zeta_{k'b^2}^{-h'a^2}\zeta_{s'b}^{r'a}}{\zeta_b^{2ah'}-\zeta_{s'}^{r'k}}\sum_{j=1}^{k'}\zeta_{k'}^{-j^2h'}\zeta_{s'}^{r'(k'-j)}\zeta_{k'b}^{2ah'j}. \quad (1.3)$$

Example We illustrate the corollary in a particular case. With notation as in Sects. 3 and 4, we let $N=2, m_0=m_2=0$. In this case, $B=\left(\begin{smallmatrix}1 & 0\\ -m_1 & 1\end{smallmatrix}\right)$, and $A=\left(\begin{smallmatrix}1 & 0\\ -2m_1 & 1\end{smallmatrix}\right)$. We let $(\frac{r}{s},\frac{h}{k})\in\mathcal{Q}_{a,b}$ with $2ah/b-rk/s\notin\mathbb{Z}$. Under these choices of parameters, the sum of Mordell integrals H_A appearing on the left-hand side of (1.3) becomes particularly simple: we have that

$$H_A\left(\frac{r}{s};\frac{h}{k}\right)=f_A\left(\frac{r}{s};\frac{h}{k}\right)\int_{\mathbb{R}}\frac{\Lambda_1e^{\pi ih_1x^2/k_1-2\pi r_1x/s_1}-e^{\pi ih_2x^2/k_2-2\pi r_2x/s_2}}{\cosh(\pi x)}\mathrm{d}x. \quad (1.4)$$

The $f_A(\frac{r}{s},\frac{h}{k})$ and Λ_1 (for which we suppress various additional parameters on which they depend) are constants, and the h_j/k_j and r_j/s_j $(j\in\{1,2\})$ are rational numbers—all of these are easily explicitly determined using the definition of H_A in (4.2). (In particular, Λ_1 is as defined in (3.7).) It appears not to be a trivial result, but rather a direct consequence of the quantum Jacobi properties of the functions $G_{a,b}$, that the Mordell integral on the right-hand side of (1.4) (and the more general function H_A in (4.2)) may be evaluated as on the right-hand side of (1.3) in terms of simple sums of roots of unity.

The remainder of the paper is structured as follows: In Sect. 2, we review definitions and properties of some modular, Jacobi, and mock modular forms used in the proofs of our results. In Sect. 3, we define the group $\Gamma'_{b,1}$ and related matrices and functions. In Sect. 4, we provide some definitions and preliminary results. In Sect. 5, we define the set $\mathcal{Q}_{a,b}$, prove that it is dense in $\mathbb{R}\times\mathbb{R}$, and prove that the usual Jacobi action of $\Gamma'_{b,1}\ltimes(2\mathbb{Z}\times2\mathbb{Z})$ on $\mathcal{Q}_{a,b}$ is well defined. Finally, in Sect. 6, we prove Theorems 1.1, 1.2, and Corollary 1.3.

2 The functions η, ϑ, and μ

In our proof of part (ii) of Theorem 1.1, we rewrite our functions $G_{a,b}$ in terms of the η, ϑ, and μ functions and then make use of some already-established transformation properties of those functions. We define those functions and describe some of their transformations below.

To begin, the Dedekind η-function is defined for $\tau\in\mathbb{H}$ by

$$\eta(\tau):=q^{1/24}\prod_{j=1}^{\infty}\left(1-q^j\right), \quad (2.1)$$

and the Jacobi ϑ-function is defined for $z\in\mathbb{C}$ and $\tau\in\mathbb{H}$ by

$$\vartheta(z;\tau):=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}e^{\pi i\nu^2\tau+2\pi i\nu(z+\frac{1}{2})}. \quad (2.2)$$

We also need the function μ defined by Zwegers in [23, Proposition 1.4]. For $u, v \in \mathbb{C}\backslash(\mathbb{Z}\tau + \mathbb{Z})$ and $\tau \in \mathbb{H}$, define

$$\mu(u, v; \tau) := \frac{e^{\pi iu}}{\vartheta(v; \tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi inv}}{1 - e^{2\pi in\tau+2\pi iu}}.$$

The functions ϑ and μ satisfy the following elliptic transformation properties:

$$\vartheta(z + \tau; \tau) = -e^{-\pi i\tau-2\pi iz}\vartheta(z; \tau), \tag{2.3}$$

$$\vartheta(z + 1; \tau) = -\vartheta(z; \tau), \tag{2.4}$$

and

$$\mu(u, v; \tau) + e^{-2\pi i(u-v)-\pi i\tau}\mu(u + \tau, v; \tau) = -ie^{-\pi i(u-v)-\pi i\tau/4} \tag{2.5}$$

(see [21] and [23, Proposition 1.4] for more). We will now make precise the notion of a quantum Jacobi form, first defined in [2].

Definition 2.1 A weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \frac{1}{2}\mathbb{Z}$ quantum Jacobi form is a complex-valued function ϕ on $\mathbb{Q} \times \mathbb{Q}$ such that for all $M = \left(\begin{smallmatrix} c_1 & c_1 \\ c_3 & c_4 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$ the functions

$$h_M(z; \tau) := \phi(z; \tau) - \epsilon_1(M)(c_3\tau + c_4)^{-k}e^{-\frac{2\pi imc_3z^2}{c_3\tau+c_4}}\phi\left(\frac{z}{c_3\tau + c_4}; \frac{c_1\tau + c_2}{c_3\tau + c_4}\right), \tag{2.6}$$

$$g_{\lambda,\mu}(z; \tau) := \phi(z, \tau) - \epsilon_2(\lambda, \mu)e^{2\pi im(\lambda^2\tau+2\lambda z)}\phi(z + \lambda\tau + \mu; \tau) \tag{2.7}$$

satisfy a "suitable" property of continuity or analyticity in (a subset of) $\mathbb{R} \times \mathbb{R}$.

Remarks (1) The property of continuity or analyticity that the functions h_M and $g_{\lambda,\mu}$ appearing in Definition 2.1 must satisfy is intentionally slightly vague, in order to parallel Zagier's original definition of a quantum modular form [22].

(2) The ϵ_1, ϵ_2 are appropriate complex numbers, such as those appearing in the theory of half-integral weight modular (Jacobi) forms. We may also allow quantum Jacobi forms to transform on subgroups of $\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$, and on appropriate subsets of $\mathbb{Q} \times \mathbb{Q}$, which are in particular required to be closed under the action of the Jacobi group.

3 The group $\Gamma'_{b,1}$ and related functions

In this section, we define the group $\Gamma'_{b,1}$ on which the transformations of $G_{a,b}$ described in Theorem 1.1 hold. This group, as well as other matrices and related functions in this section, was originally defined in [17].

3.1 The group $\Gamma'_{b,1}$

To define $\Gamma'_{b,1}$, we require some intermediate definitions (see Chapter 2, [17]). (All other definitions in this section come from Chapter 3, [17].) As in Sect. 1, b is an even integer satisfying $b \geq 4$.

Definition 3.1 Define

$$\Gamma_{b,1} := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{matrix} \alpha \equiv 1 \pmod{2b} & \beta \equiv 0 \pmod{b} \\ \gamma \equiv 0 \pmod{4} & \delta \equiv 1 \pmod{2b} \end{matrix} \right\} \tag{3.1}$$

and

$$\Gamma_{b,2} := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{matrix} \alpha \equiv 1 \pmod{2b} & \beta \equiv 0 \pmod{2b} \\ \gamma \equiv 0 \pmod{2} & \delta \equiv 1 \pmod{2b} \end{matrix} \right\}.$$

It is straightforward to check that $\Gamma_{b,1}$ and $\Gamma_{b,2}$ are subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Note that the map $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \alpha & 2\beta \\ \frac{\gamma}{2} & \delta \end{smallmatrix}\right)$ is a bijection from $\Gamma_{b,1}$ to $\Gamma_{b,2}$. The group $\Gamma'_{b,1}$ under which $G_{a,b}$ transforms is a subgroup of $\Gamma_{b,1}$. More precisely, $\Gamma'_{b,1}$ is defined in terms of a subgroup $\Gamma'_{b,2} \subseteq \Gamma_{b,2}$, which is constructed using the standard generators $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 3.2 For $N \in 2\mathbb{N}_0$ and a sequence $\{m_j\}_{j=0}^N$ where

$$m_j = m_j(b,N) \in \begin{cases} 2\mathbb{Z}\backslash\{0\} & \text{if } j \text{ odd}, \\ b\mathbb{Z}\backslash\{0\} & \text{if } j \text{ even}, \ j \notin \{0,N\}, \\ b\mathbb{Z} & \text{if } j \in \{0,N\}, \end{cases} \tag{3.2}$$

let

$$B = \begin{pmatrix} \alpha & 2\beta \\ \gamma/2 & \delta \end{pmatrix} = T^{m_0} \left[\prod_{j=1}^{N} ST^{m_j} \right] (-I)^{N/2}, \tag{3.3}$$

where here and throughout, the product of matrices denotes right multiplication; that is,

$$\prod_{j=1}^{N} ST^{m_j} = \left[\prod_{j=1}^{N-1} ST^{m_j} \right] ST^{m_N}.$$

Finally, we define

$$\Gamma'_{b,2} := \left\{ B \in \Gamma_{b,2} : B \text{ as in (3.3)}, N \in 2\mathbb{N}_0, \{m_j\}_{j=0}^N \text{ as in (3.2)} \right\}. \tag{3.4}$$

We are now ready to define $\Gamma'_{b,1}$, the group on which our functions $G_{a,b}$ transform.

Definition 3.3 With notation and hypotheses as above, we define

$$\Gamma'_{b,1} := \left\{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{b,1} : B = \begin{pmatrix} \alpha & 2\beta \\ \gamma/2 & \delta \end{pmatrix} \in \Gamma'_{b,2} \right\}. \tag{3.5}$$

Remark $\Gamma'_{b,2}$ (and therefore $\Gamma'_{b,1}$) contains infinitely many distinct matrices. For instance, $\Gamma'_{b,2}$ contains the subgroup generated by T^{2b}, the subgroup generated by $ST^2S(-I)$, and the subgroup generated by $T^bST^2ST^b(-I)$.

3.2 Auxiliary definitions and results

Towards the proof of Theorem 1.1 part (i), we require a number of definitions. For the remainder of this section, fix $A = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma'_{b,1}$ and $B = T^{m_0}\left[\prod_{j=1}^{N} ST^{m_j}\right](-I)^{N/2} = \left(\begin{smallmatrix}\alpha & 2\beta \\ \gamma/2 & \delta\end{smallmatrix}\right) \in \Gamma'_{b,2}$.

First, we define for $0 \le \ell \le N-1$,

$$B_\ell = \left[\prod_{j=\ell+1}^{N} ST^{m_j}\right](-I)^{N/2}. \tag{3.6}$$

Furthermore, if we write $M = \left(\begin{smallmatrix}c_1 & c_2 \\ c_3 & c_4\end{smallmatrix}\right)$, we define for $z \in \mathbb{C}$, $\tau \in \mathbb{H} \cup \mathbb{Q}$,

$$M_\tau(z) := \frac{z}{c_3\tau + c_4} \quad\text{and}\quad M(\tau) := \frac{c_1\tau + c_2}{c_3\tau + c_4}.$$

We use the functions above to define one more function used in our proof of Theorem 1.1. For integers ℓ satisfying $1 \le \ell \le N$, define

$$\Lambda_\ell := e^{\frac{-\pi i(n_\ell(B_\ell(2\tau)+m_\ell)+n_{\ell+1}-B_{\ell,2\tau}(z))^2}{B_\ell(2\tau)+m_\ell}}(-1)^{n_\ell m_\ell} e^{\frac{-\pi i m_\ell}{4}}\sqrt{-i(B_\ell(2\tau)+m_\ell)}. \tag{3.7}$$

Now if $B_\ell = \left(\begin{smallmatrix}c_1 & c_2 \\ c_3 & c_4\end{smallmatrix}\right)$, we note that when $\tau = \frac{-c_4}{2c_3}$, we have $B_\ell(2\tau) = B_{\ell,2\tau}(z) = \infty$. As a result, we must exclude certain points from consideration throughout. To this end, we define the following set:

$$Z_A := \bigcup_{\ell=0}^{N}\left(\frac{1}{2}\right)B_\ell^{-1}(\infty) \subseteq \mathbb{Q}. \tag{3.8}$$

4 Modularity and $G_{a,b}$

Rewriting the errors to modularity of the functions $G_{a,b}$ is a key result used towards our proof of Theorem 1.1. To state this, for $A = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_{b,1}$, we define the errors to modularity by

$$\begin{aligned}\mathcal{M}_A(z;\tau) &= \mathcal{M}_{A,a,b}(z;\tau)\\ &:= G_{a,b}(z;\tau) - \Psi_{a,b}^{-1}\begin{pmatrix}\alpha & 2\beta \\ \gamma/2 & \delta\end{pmatrix}(\gamma\tau+\delta)^{-1/2}e^{\pi i\gamma z^2/(2(\gamma\tau+\delta))}G_{a,b}\left(\frac{z}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right),\end{aligned}$$

where the character $\Psi_{a,b}$ is defined in [17, equation (1.5.8)]. We also require a sequence $\{n_\ell\}_{\ell\in\mathbb{N}_0}$ defined by

$$n_\ell = n_{\ell,a,b} := \begin{cases}\frac{a}{b} - \frac{1}{2} & \text{if } \ell \equiv 0 \pmod 4,\\ -\frac{1}{2} & \text{if } \ell \equiv 1 \pmod 4,\\ -(\frac{a}{b} - \frac{1}{2}) & \text{if } \ell \equiv 2 \pmod 4,\\ \frac{1}{2} & \text{if } \ell \equiv 3 \pmod 4.\end{cases} \tag{4.1}$$

It turns out that the functions $\mathcal{M}_A$ can be rewritten in terms of the Mordell integrals h (see (1.2)). To be precise, for $A = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma'_{b,1}$, and $B = \left(\begin{smallmatrix}\alpha & 2\beta \\ \gamma/2 & \delta\end{smallmatrix}\right) \in \Gamma'_{b,2}$, we let

$$H_A(z;\tau) := \frac{i}{2} e^{-n_0{}^2\pi i A(\tau)} e^{2\pi i n_0 A_\tau(z)} \Psi_{a,b}^{-1}(B)(\gamma\tau+\delta)^{-1/2} e^{\pi i\gamma z^2/(2(\gamma\tau+\delta))} e^{m_0(-\pi i/4)} \times (-1)^{n_0 m_0} \sum_{\ell=0}^{N-1} (-1)^\ell \left[\prod_{j=1}^{\ell} \Lambda_j\right] h\big(n_\ell B_\ell(2\tau) - B_{\ell,2\tau}(z) + n_{\ell+1}; B_\ell(2\tau)\big), \tag{4.2}$$

where n_ℓ is defined in (4.1), m_0 comes from the S-and-T decomposition of B as in (3.3), and all other functions are defined in Sect. 3. In our proof of Theorem 1.1, we require that the function $H_A(z;\tau)$, when viewed as a function of $\tau \in \mathbb{R}$, is defined outside the set Z_A (see (3.8)). This is due to the following lemma [17, Lemma 3.2.9].

Lemma 4.1 *Let N, ℓ, m_ℓ, and B_ℓ be as in Sect. 3. Then we have that*

$$Z_A = \bigcup_{\ell=1}^{N} \{x \in \mathbb{R} : B_\ell(2x) + m_l = 0\},$$

and

$$\bigcup_{\ell=0}^{N-1} \{x \in \mathbb{R} : B_\ell(2x) = 0\} \subseteq Z_A.$$

As shown in [17, Theorem 3.3.1], we have Theorem 4.2, which holds away from the lattice

$$\Lambda_{a,b} := \left\{(z,\tau) \in \mathbb{C}\times\mathbb{H} : z \in \left(\frac{1}{2} \pm \frac{2a}{b}\tau + \mathbb{Z} + 2\mathbb{Z}\tau\right) \cup \left(\frac{1}{2} + \tau + \mathbb{Z} + 2\mathbb{Z}\tau\right)\right\}. \tag{4.3}$$

In the statement of Theorem 4.2, we write $\mathcal{Q}^*_{a,b}$ for the set

$$\{\tau \in \mathbb{Q} : \text{there is } z \in \mathbb{Q} \text{ such that } (z,\tau) \in \mathcal{Q}_{a,b}\},$$

where $\mathcal{Q}_{a,b}$ is as in (5.1).

Theorem 4.2 *Let $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma'_{b,1}$, and $B = \left(\begin{smallmatrix}\alpha & 2\beta\\ \gamma/2 & \delta\end{smallmatrix}\right) \in \Gamma'_{b,2}$. Then for $\tau \in \mathbb{H} \cup \mathcal{Q}^*_{a,b}$ and $z \in \mathbb{C}$ such that $(z,\tau) \notin \Lambda_{a,b}$, we have that $\mathcal{M}_A(z;\tau) = H_A(z;\tau)$.*

Remarks (1) Theorem 4.2 is almost identical to [17, Theorem 3.3.1], except that we have stated the result in Theorem 4.2 for $\tau \in \mathcal{Q}^*_{a,b}$ rather than for τ in the set $Q_{a,b}$ defined in (5.2) on which [17, Theorem 3.3.1] holds. It is not difficult to check that the proof of Theorem 3.3.1 in [17] holds for $\tau \in \mathcal{Q}^*_{a,b}$ as well.

(2) Starting points towards the proof of [17, Theorem 3.3.1] and hence Theorem 4.2 are results from [3,15,23].

A key result derived from [15] that was used in the proof of Theorem 4.2 (mentioned in the remark above), and which we use in the proof of part (ii) of Theorem 1.1, is the following.

Lemma 4.3 *For $(z,\tau) \in (\mathbb{C}\times\mathbb{H})\backslash\Lambda_{a,b}$, we have that*

$$G_{a,b}(z;\tau) = q^{-a^2/b^2+a/b-1/4} w^{a/b-1/2} \mu\left(\tfrac{2a}{b}\tau - z - \tfrac{1}{2}, \tau; 2\tau\right) + \frac{i q^{-a^2/b^2} w^{a/b} \eta^4(2\tau)}{\eta^2(\tau)\, \vartheta\left(\frac{2a}{b}\tau - z - \frac{1}{2}; 2\tau\right)}. \tag{4.4}$$

Recall that Definition 2.1 requires that the modular and elliptic transformation error functions (2.6) and (2.7) extend to suitably continuous or analytic functions in $\mathbb{R} \times \mathbb{R}$. To this end, we use the following Lemma, proved in [17, Corollary 3.5.4], extending [20,23].

Lemma 4.4 *For all $a_1, a_2 \in \mathbb{Q}$ and $\mu, \nu \in \mathbb{N}_0$, we have that*

$$\frac{\partial^\mu}{\partial\tau^\mu}\frac{\partial^\nu}{\partial z^\nu} h(a_1\tau + a_2 + z; \tau)$$

is defined for all $z \in \mathbb{C}$ and $\tau \in \mathbb{R}\backslash\{0\}$.

5 The quantum set $\mathcal{Q}_{a,b}$

In this section, we will prove various statements about the set $\mathcal{Q}_{a,b} \subseteq \mathbb{Q} \times \mathbb{Q}$, which we refer to as the "quantum set" because it is the set on which our quantum Jacobi form is defined and transforms. First, we give the definition:

$$\mathcal{Q}_{a,b} := \left\{ \left(\frac{r}{s}, \frac{h}{k}\right) \in \mathbb{Q} \times \mathbb{Q} : \begin{array}{l} \text{(i) } s > 0, k > 0, \gcd(r,s) = \gcd(h,k) = 1, \text{ and } k \text{ even} \\ \text{(ii) for all } j \bmod k, hs(a+bj) \not\equiv \frac{bk}{4}(2r+s) \pmod{bks} \\ \text{(iii) if } k \equiv 0 \pmod 4 \text{ then } h \not\equiv \pm 1 \pmod{2b} \end{array} \right\}. \tag{5.1}$$

Further, we define

$$Q_{a,b} := \left\{ \frac{h}{k} \in \mathbb{Q} : \left(\frac{0}{1}, \frac{h}{k}\right) \in \mathcal{Q}_{a,b} \right\}. \tag{5.2}$$

Note that the set $Q_{a,b}$ defined above is equal to the set $Q_{a,b}$ defined in [17].

From the definition of $\mathcal{Q}_{a,b}$ alone, it is arguably unclear how large (or small) the set is—and in order to have a meaningful quantum set, it must be large in some sense. We first prove that our quantum set $\mathcal{Q}_{a,b}$ is in fact dense in $\mathbb{R} \times \mathbb{R}$.

Proposition 5.1 *$\mathcal{Q}_{a,b}$ is dense in $\mathbb{R} \times \mathbb{R}$.*

Remark It follows immediately from this proposition that the image of the map $(z, \tau) \mapsto (e^{2\pi i z}, e^{2\pi i \tau})$ contains infinitely many distinct pairs of roots of unity.

Proof We will show that for all $h/k \in Q_{a,b}$, the set of $r/s \in \mathbb{Q}$ such that $(r/s, h/k) \in \mathcal{Q}_{a,b}$ is dense in $\mathbb{R}$. Since $Q_{a,b}$ is dense in $\mathbb{R}$ by [17, Proposition 2.2.6], Proposition 5.1 will follow. We break up the proof of this fact into two parts.

Part 1. Let $h/k \in Q_{a,b}$, and $r, s \in \mathbb{Z}$ with $s > 0$, $s \equiv 1 \pmod{2b}$, r even, and $\gcd(r, s) = 1$. We claim that $(r/s, h/k) \in \mathcal{Q}_{a,b}$. Clearly $(r/s, h/k)$ satisfies (i) and (iii) in the definition of $\mathcal{Q}_{a,b}$, so we need only show that $(r/s, h/k)$ satisfies (ii). Suppose not. Let $j \in \mathbb{Z}$ such that

$$hs(a + bj) \equiv \frac{bk}{4}(2r + s) \pmod{bks}.$$

Using the facts that $s \equiv 1 \pmod{2b}$ and r and b are even, we find that

$$h(a + bj') \equiv \frac{bk}{4} \pmod{bk}$$

for some $j' \pmod{bk}$. This contradicts the fact that $h/k \in Q_{a,b}$, completing part 1.

Part 2. By part 1, it suffices to show that the set of r/s with r and s satisfying the conditions of part 1 is dense in $\mathbb{R}$, for which it suffices in turn to show that the set of such r/s is dense in $\mathbb{Q}$. So let $p/q \in \mathbb{Q}$ with $\gcd(p,q) = 1$ and $q > 0$, and fix $\epsilon > 0$. Let $g := \gcd(q, 2b)$ and $q' := q/g$. Since $\gcd(q', 2bp) = 1$, we can let $q'' \in \mathbb{N}$ such that $q'q'' \equiv 1 \pmod{2bp}$. Let N be a multiple of $2bp$ that is relatively prime to q' and to q'', with N sufficiently large such that

$$\left|\frac{p}{q} - \frac{p}{q}\left(\frac{gN}{gN+1}\right)\right| < \frac{\epsilon}{2}.$$

Let M be $gN + 1$ times a sufficiently large power of $q'q''$ such that

$$\left|\frac{p}{q}\left(\frac{gN}{gN+1}\right) - \frac{p}{q}\left(\frac{gN}{gN+1}\right)\left(\frac{q''M+1}{q''M}\right)\right| < \frac{\epsilon}{2}.$$

Finally, let $r = pN(q''M + 1)$ and $s = q'(gN + 1)q''M$. Then

$$\left|\frac{p}{q} - \frac{r}{s}\right| = \left|\frac{p}{q} - \frac{p}{q}\left(\frac{gN}{gN+1}\right)\left(\frac{q''M+1}{q''M}\right)\right| < \epsilon,$$

so we need only check that r and s satisfy the conditions of part 1. For this, it suffices to check that the following four conditions hold:

1. $\gcd(r, s) = 1$;
2. N is even;
3. $gN + 1 \equiv 1 \pmod{2b}$;
4. $M \equiv 1 \pmod{2b}$.

Conditions (2), (3), and (4) follow immediately from our choices of N and M. Condition (1) is straightforward to verify by checking that each of the factors p, N, and $q''M + 1$ of r is relatively prime to each of the factors q', $gN + 1$, q'', and M of s. □

Next, we prove that $G_{a,b}$ is well defined on $\mathcal{Q}_{a,b}$.

Lemma 5.2 *$G_{a,b}$ is well defined on $\mathcal{Q}_{a,b}$.*

Proof We need only show that the terms of the sum

$$\sum_{n\geq 0} \frac{(-q;q)_n q^{n(n+1)/2}}{(-iq^{a/b}w^{-1/2};q)_{n+1}(iq^{1-a/b}w^{1/2};q)_{n+1}} = g_2(-iq^{\frac{a}{b}}w^{-\frac{1}{2}};q) \tag{5.3}$$

are well defined on $\mathcal{Q}_{a,b}$, and that the sum converges on $\mathcal{Q}_{a,b}$. For the former, we have that condition (ii) in the definition of $\mathcal{Q}_{a,b}$ ensures that the denominator of each summand in (5.3) is never zero on $\mathcal{Q}_{a,b}$. For the latter, because h is odd and k is a positive even integer, we have for all $n \geq k/2$ that $(-q;q)_n = 0$. □

Our final lemma in this section shows that usual Jacobi action (per [9], see also [4]) of $\Gamma'_{b,1} \ltimes (2\mathbb{Z}\times 2\mathbb{Z})$ on $\mathcal{Q}_{a,b}$ is well defined. In particular, the expressions $G_{a,b}\left(\gamma\tau+\delta; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$, $G_{a,b}(z+2m;\tau)$, and $G_{a,b}(z+2m\tau;\tau)$ in Theorem 1.1 are well defined.

Lemma 5.3 *For $(\frac{r}{s}, \frac{h}{k}) \in \mathcal{Q}_{a,b}$ and $\left(\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right), (\lambda, \mu)\right) \in \Gamma_{b,1} \ltimes (2\mathbb{Z}\times 2\mathbb{Z})$, let $r', s', h', k' \in \mathbb{Z}$ with $s' > 0$, $k' > 0$, and $\gcd(r', s') = \gcd(h', k') = 1$ such that*

$$\frac{r'}{s'} = \frac{rk + \lambda hs + \mu ks}{\gamma hs + \delta ks} \quad \textit{and} \quad \frac{h'}{k'} = \frac{\alpha h + \beta k}{\gamma h + \delta k}.$$

Then the multiplication

$$\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, (\lambda,\mu)\right)\cdot\left(\frac{r}{s},\frac{h}{k}\right) := \left(\frac{r'}{s'},\frac{h'}{k'}\right)$$

defines an action of $\Gamma_{b,1} \ltimes (2\mathbb{Z}\times 2\mathbb{Z})$ *on* $\mathcal{Q}_{a,b}$.

Proof We break up the proof into four parts: part 0 shows that this multiplication is well defined, and parts 1, 2, and 3 show that $(r'/s', h'/k')$ satisfies (i), (ii), and (iii) in the definition of $\mathcal{Q}_{a,b}$, respectively.

Part 0. To show that this multiplication is well defined, we need only show that $\gamma h+\delta k \neq 0$. So suppose towards a contradiction that $\gamma h+\delta k = 0$. Then since $\gcd(\gamma,\delta) = \gcd(h,k) = 1$, we have $\gamma = \pm k$ and $\delta = \mp h$. Therefore, since $\gamma \equiv 0 \pmod 4$ and $\delta \equiv 1 \pmod{2b}$, we have $k \equiv 0 \pmod 4$ and $h \equiv \pm 1 \pmod{2b}$. This contradicts the fact that $(r/s, h/k)$ satisfies (iii) in the definition of $\mathcal{Q}_{a,b}$.

Part 1. By the first paragraph of part 3 below and the evenness of γ and k, we have that k' is even. The pair $(r'/s', h'/k')$ satisfies the rest of (i) in the definition of $\mathcal{Q}_{a,b}$ by definition.

Part 2. To show that $(r'/s', h'/k')$ satisfies (ii) in the definition of $\mathcal{Q}_{a,b}$, it suffices to show that the elliptically transformed pair $\left(\frac{rk+\lambda hs+\mu ks}{ks}, \frac{h}{k}\right)$ and the modularly transformed pair $\left(\frac{rk}{\gamma hs+\delta ks}, \frac{\alpha h+\beta k}{\gamma h+\delta k}\right)$ both satisfy (ii) in the definition of $\mathcal{Q}_{a,b}$.

Suppose first that the elliptically transformed pair fails (ii). Let $j \in \mathbb{Z}$ such that

$$hks(a+bj) \equiv \frac{bk}{4}(2rk+2\lambda hs+2\mu ks+ks) \pmod{bk^2s}.$$

Cancelling a factor of k and using the fact that λ and μ are even, we get

$$hs\left(a+b\left(j-\frac{\lambda}{2}\right)\right) \equiv \frac{bk}{4}(2r+s) \pmod{bks},$$

contradicting the fact that $\left(\frac{r}{s},\frac{h}{k}\right)$ satisfies (ii).

Now suppose that the modularly transformed pair fails (ii). Let $j, N \in \mathbb{Z}$ such that

$$\begin{aligned}&(\alpha h+\beta k)(\gamma hs+\delta ks)(a+bj)\\ &\quad= \frac{b(\gamma h+\delta k)}{4}(2rk+\gamma hs+\delta ks)+Nb(\gamma h+\delta k)(\gamma hs+\delta ks).\end{aligned}$$

Since $\gamma h+\delta k \neq 0$ and $\beta \equiv 0 \pmod b$, this gives that

$$hs\left(\alpha a+b\left(\alpha j-\frac{\gamma}{4}-N\gamma\right)\right) \equiv \frac{bk}{4}2r+\frac{bk\delta}{4}s \pmod{bks}.$$

Writing $\alpha = 1+2b\alpha'$ and $\delta = 1+2b\delta'$ and using the fact that b is even, we get

$$hs(a+bj') \equiv \frac{bk}{4}(2r+s) \pmod{bks}$$

where $j' := 2\alpha' a+\alpha j-\frac{\gamma}{4}-N\gamma \mod k$. This contradicts the fact that $\left(\frac{r}{s},\frac{h}{k}\right)$ satisfies (ii).

Part 3. We first claim that $h' = \operatorname{sgn}(\gamma h+\delta k)\cdot(\alpha h+\beta k)$ and $k' = |\gamma h+\delta k|$. Indeed, let $d = \gcd(\alpha h+\beta k, \gamma h+\delta k)$. Then since $(\alpha h+\beta k)/d$ and $(\gamma h+\delta k)/d \in \mathbb{Z}$, we have

$$\frac{h}{d} = \delta\left(\frac{\alpha h+\beta k}{d}\right)-\beta\left(\frac{\gamma h+\delta k}{d}\right) \in \mathbb{Z}$$

and

$$\frac{k}{d} = -\gamma\left(\frac{\alpha h + \beta k}{d}\right) + \alpha\left(\frac{\gamma h + \delta k}{d}\right) \in \mathbb{Z}.$$

Thus, d divides both h and k, so $d = 1$. The claim follows.

Suppose now that $k' \equiv 0 \pmod 4$. Then by the claim, $\gamma h + \delta k \equiv 0 \pmod 4$. Therefore, since $\gamma \equiv 0 \pmod 4$ and $\delta \equiv 1 \pmod 4$, we have $k \equiv 0 \pmod 4$. Thus, $h \not\equiv \pm 1 \pmod{2b}$. Now because $h' = \pm(\alpha h + \beta k)$, $\beta \equiv 0 \pmod b$, and k is even, we have that $h' \equiv \pm\alpha h \pmod{2b}$. Since $\alpha \equiv 1 \pmod{2b}$, this implies that $h' \equiv \pm h \not\equiv \pm 1 \pmod{2b}$, as desired. □

6 Proofs

6.1 Proof of part (i) of Theorem 1.1

We first argue that Theorem 4.2 holds for the larger set $((\mathbb{C} \times \mathbb{H})\backslash\Lambda_{a,b}) \cup \mathcal{Q}_{a,b}$. To begin, we fix $(z_0, \tau_0) \in \mathcal{Q}_{a,b}$, and we set $\mathcal{M}_A(z) := \mathcal{M}_A(z, \tau_0)$ and $H_A(z) := H_A(z, \tau_0)$.

By Lemma 4.4, setting $\mu = a_1 = a_2 = 0$, we have that for all $\nu \geq 0$, $\frac{\partial^\nu}{\partial z^\nu} h(z;\tau)$ is defined for all $z \in \mathbb{C}$ and $\tau \in \mathbb{R}\backslash\{0\}$. So, in particular, $h(z;\tau_0)$ is a continuous function of z. From this it is not difficult to see from the definition of H_A (given by Eq. (4.2)) that $H_A(z)$ is continuous on $\mathbb{C}$.

On the other hand, it is also not difficult to check that the map $z \mapsto G_{a,b}(z, \tau_0)$ is meromorphic on $\mathbb{C}$. In particular, this map is continuous at z_0, and thus so is $\mathcal{M}_A$. Now clearly z_0 is a limit point of some subset of $\mathbb{C}\backslash\{z \in \mathbb{C} : (z, \tau_0) \in \Lambda_{a,b}\}$. Therefore, by Theorem 4.2 and the continuity of $\mathcal{M}_A$ and H_A at z_0, we have $\mathcal{M}_A(z_0, \tau_0) = H_A(z_0, \tau_0)$.

6.2 Proof of part (ii) of Theorem 1.1

We give a proof for $m = 1$; the general result follows by straightforward induction on m.

First, let $(z, \tau) \in (\mathbb{C} \times \mathbb{H})\backslash\Lambda_{a,b}$. The proof proceeds by taking advantage of Eq. (4.4) and the transformations of μ and ϑ. By Eq. (2.3), we have

$$\vartheta\left(\frac{2a}{b}\tau - z - 2\tau - \frac{1}{2}; 2\tau\right) = q^{\frac{2a}{b}-1} w^{-1}\, \vartheta\left(\frac{2a}{b}\tau - z - \frac{1}{2}; 2\tau\right).$$

This, along with the fact that $z \mapsto z + 2\tau$ means $w^{a/b} \mapsto w^{a/b} q^{2a/b}$, gives

$$\frac{i q^{-a^2/b^2} w^{a/b} q^{2a/b} \eta^4(2\tau)}{\eta^2(\tau)\, \vartheta(\frac{2a}{b}\tau - z - 2\tau - \frac{1}{2}; 2\tau)} = wq \frac{i q^{-a^2/b^2} w^{a/b} \eta^4(2\tau)}{\eta^2(\tau)\, \vartheta(\frac{2a}{b}\tau - z - \frac{1}{2}; 2\tau)}. \tag{6.1}$$

Furthermore, by Eq. (2.5), we have

$$\mu\left(\frac{2a}{b}\tau - z - 2\tau - \frac{1}{2}, \tau; 2\tau\right) = wq^{2(1-\frac{a}{b})} \mu\left(\frac{2a}{b}\tau - z - \frac{1}{2}, \tau; 2\tau\right) + q^{-\frac{a}{b}+\frac{5}{4}} w^{\frac{1}{2}}. \tag{6.2}$$

So, by Eqs. (6.1) and (6.2), together with (4.4), we obtain

$$G_{a,b}(z + 2\tau; \tau) = wq\, G_{a,b}(z;\tau) + q^{\frac{-a^2}{b^2} + \frac{2a}{b}}\, w^{\frac{a}{b}} \tag{6.3}$$

for $(z, \tau) \in (\mathbb{C} \times \mathbb{H})\backslash\Lambda_{a,b}$, which yields the desired form for $m = 1$ after dividing by wq and subtracting.

We now show that (6.3) also holds on $\mathcal{Q}_{a,b}$. Let $(z_0, \tau_0) \in \mathcal{Q}_{a,b}$. By Abel's Theorem, viewing $G_{a,b}$ as a q-series, the one-variable functions $\tau \mapsto G_{a,b}(z_0 + 2\tau; \tau)$ and $\tau \mapsto wqG_{a,b}(z_0; \tau) + q^{\frac{-a^2}{b^2}+\frac{2a}{b}} w^{\frac{a}{b}}$ are continuous at τ_0 when τ approaches τ_0 from the upper half-plane. Therefore, since τ_0 is a limit point of some subset of $\mathbb{H}\backslash\{\tau \in \mathbb{H} : (z_0, \tau) \in \Lambda_{a,b}\}$, Eq. (6.3) gives the desired result.

6.3 Proof of Part (iii) of Theorem 1.1

This follows immediately from the definition of $G_{a,b}$.

6.4 Proof that H_A is C^∞ on $\mathbb{R} \times (\mathbb{R}\backslash Z_A)$

By Lemma 4.4, we have that $h(z; \tau)$ is a C^∞ function on $\mathbb{C} \times (\mathbb{R}\backslash\{0\})$. The chain rule yields that all the partial derivatives of $h(n_\ell B_\ell(2\tau) - B_{\ell,2\tau}(z) + n_{\ell+1}; B_\ell(2\tau))$ are defined when $\tau \notin Z_A$.

Therefore it is evident that $H_A(z; \tau)$ is a finite product of constants, exponentials, and other C^∞ functions, and by Lemma 4.1, all partial derivatives of these functions exist for $(z, \tau) \in \mathbb{R} \times (\mathbb{R}\backslash Z_A)$.

6.5 Proof that the $G_{a,b}$ are quantum Jacobi forms

From Theorem 1.1 parts (i), (ii) and (iii), we have that on the subsets $\mathcal{Q}_{a,b} \subseteq \mathbb{Q}\times\mathbb{Q}$ (which are dense in $\mathbb{R} \times \mathbb{R}$ by Proposition 5.1), the functions $G_{a,b}$ transform on $\Gamma'_{b,1} \ltimes (2\mathbb{Z} \times 2\mathbb{Z})$ with weight $1/2$ and index $-1/4$ according to the definition of a quantum Jacobi form, up to the fact that the functions on the right-hand sides of (i), (ii), and (iii) in Theorem 1.1 are suitably continuous or analytic in $\mathbb{R} \times \mathbb{R}$. Indeed, the function on the right-hand side of (iii) is identically zero, and the function on the right-hand side of (ii) is a finite sum of products of exponential functions, so it is analytic in $\mathbb{R} \times \mathbb{R}$. Finally, from the proof given in Sect. 6.4, we have that H_A on the right-hand side of (i) is C^∞ in $\mathbb{R} \times (\mathbb{R}\backslash Z_A)$. Together, these things prove that the $G_{a,b}$ are quantum Jacobi forms.

6.6 Proof of Theorem 1.2

Let $(z, \tau) = \left(\frac{r}{s}, \frac{h}{k}\right) \in \mathcal{Q}_{a,b}$. From Theorem 1.1 (ii) with $m = k$, we find that

$$G_{a,b}(z + 2h; \tau)\Big|_{(z,\tau)=\left(\frac{r}{s},\frac{h}{k}\right)\in\mathcal{Q}_{a,b}} = \left(w^k\, G_{a,b}(z; \tau) + q^{\frac{-a^2}{b^2}} w^{\frac{a}{b}} \sum_{j=1}^{k} q^{-j^2+k^2} w^{k-j} q^{\frac{2aj}{b}}\right)\Bigg|_{(z,\tau)=\left(\frac{r}{s},\frac{h}{k}\right)\in\mathcal{Q}_{a,b}}. \tag{6.4}$$

On the other hand, from Theorem 1.1 part (iii) with $m = h$, we obtain

$$G_{a,b}(z + 2h; \tau) = e^{4\pi i\frac{ah}{b}}\, G_{a,b}(z; \tau). \tag{6.5}$$

Combining (6.4) and (6.5), we have that

$$(e^{\frac{4\pi iah}{b}} - w^k)G_{a,b}(z; \tau)\Big|_{(z,\tau)=\left(\frac{r}{s},\frac{h}{k}\right)\in\mathcal{Q}_{a,b}} = \left(q^{\frac{-a^2}{b^2}} w^{\frac{a}{b}} \sum_{j=1}^{k} q^{-j^2+k^2} w^{k-j} q^{\frac{2aj}{b}}\right)\Bigg|_{(z,\tau)=\left(\frac{r}{s},\frac{h}{k}\right)\in\mathcal{Q}_{a,b}}. \tag{6.6}$$

We have that $2ah/b - rk/s \notin \mathbb{Z}$, so we divide both sides of (6.6) by $(e^{\frac{4\pi iah}{b}} - w^k)$ (where $w = e^{\frac{2\pi ir}{s}}$), substitute $(z, \tau) = (\frac{r}{s}, \frac{h}{k})$, and simplify, to conclude the proof of Theorem 1.2.

6.7 Proof of Corollary 1.3

Briefly speaking, Corollary 1.3 follows from part (i) of Theorems 1.1 and 1.2. In particular, we wish to apply Theorem 1.2 to the functions $G_{a,b}(z;\tau)$ and $G_{a,b}\left(\frac{z}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$ appearing in part (i) of Theorem 1.1. It is clear that we may do so for the function $G_{a,b}(z;\tau)$. For the function $G_{a,b}\left(\frac{z}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$, we first note that

$$\left(\frac{r'}{s'}, \frac{h'}{k'}\right) = \left(\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}, (0,0)\right) \cdot \left(\frac{r}{s}, \frac{h}{k}\right),$$

and hence by the definitions of h', k', r', and s', as well as Lemma 5.3, we have that $\left(\frac{r'}{s'}, \frac{h'}{k'}\right) \in \mathcal{Q}_{a,b}$. We now need only check that $2ah'/b - r'k'/s' \notin \mathbb{Z}$. This follows after a short calculation using the fact that $2ah/b - rk/s \notin \mathbb{Z}$, the fact that $h' = \operatorname{sgn}(\gamma h + \delta k) \cdot (\alpha h + \beta k)$ and $k' = |\gamma h + \delta k|$ (as established in the proof of Lemma 5.3), and the definition of the larger group $\Gamma_{b,1}$ containing $\Gamma'_{b,1}$.

Acknowledgements

The authors are grateful for support from National Science Foundation CAREER Grant DMS-1449679 (third author), Amherst College, and the Simons Foundation Fellows Program in Mathematics (third author).

Received: 14 August 2018 Accepted: 10 November 2018
Published online: 26 November 2018

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