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Overpartition ranks and quantum modular forms

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Abstract

For each $d \in \mathbb{N}$, we establish an infinite family of weight 1/2 quantum modular forms from the overpartition M_d-rank generating function. Infinite quantum families from both the Dyson rank overpartition generating function and the overpartition M_2-rank generating function appear as special cases of our work. As a corollary, we obtain explicit closed expressions which may be used to evaluate Eichler integrals of certain weight 3/2 theta functions.

1 Introduction and statement of results

A *partition* of a positive integer n is any non-increasing sum of positive integers whose sum is n, and the *partition function* $p(n)$ counts the number of partitions of n. This combinatorial function turns out to play an important role in the theory of modular forms, stemming from the following relationship between its generating function and the Dedekind η-function $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty$, a well-known weight 1/2 modular form:

$$1 + \sum_{n=1}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = q^{\frac{1}{24}}\eta(\tau)^{-1}. \tag{1.1}$$

Here, $q := e^{2\pi i\tau}$, $\tau \in \mathbb{H} := \{x + iy \mid x \in \mathbb{R}, y \in \mathbb{R}^+\}$ the upper half of the complex plane, and the q-Pochhammer symbol is defined for $n \in \mathbb{N}_0 \cup \{\infty\}$ by

$$(a;q)_n := \prod_{j=1}^{n}(1 - aq^{j-1}).$$

It is natural to ask whether other combinatorial partition-type generating functions give rise to examples of modular or modular-type forms – and the answer is yes. As an example, consider the *Dyson rank* of a partition, defined to be its largest part minus its number of parts [10]. The *Dyson partition rank function* is defined for $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ by

$$N(m,n) := p(n \mid \text{Dyson rank } m),$$

where $N(m,0) = \delta_{m0}$, with δ_{ij} the Kronecker delta. This function has been historically important for the roles it plays in combinatorics and number theory, e.g., Atkin and Swinnerton-Dyer [2] proved Dyson's conjecture that the Dyson rank could be used to combinatorially explain Ramanujan's famous partition congruences modulo 5 and 7. The associated two-variable generating function for $N(m,n)$ may be expressed as the q-hypergeometric series

$$R(z;q) := \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n}. \tag{1.2}$$

Specializing in z has proved fruitful in the modular world. For example when $z=1$, we have that

$$R(1;q) = 1 + \sum_{n=1}^{\infty} p(n)q^n = q^{\frac{1}{24}}\eta^{-1}(\tau) \tag{1.3}$$

recovering (1.1), (essentially[1]) a modular form. When $z=-1$, we have that

$$R(-1;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} =: f(q), \tag{1.4}$$

Ramanujan's third order mock theta function. Thanks to the influential thesis of Zwegers [25], we now know that $f(q)$, along with Ramanujan's other mock theta functions, are holomorphic parts of *harmonic Maass forms* as originally defined by Bruinier and Funke [9]. Such holomorphic parts are now referred to as *mock modular forms* after Zagier [23]. While mock modular forms do not transform like true modular forms, they can be completed by adding suitable non-holomorphic functions, such that the resulting non-holomorphic sums do transform (and moreover, obey the rest of the parts of the definition of a harmonic Maass form). Extending Zwegers' work on $R(1;q) = f(q)$, Bringmann and Ono [7] later established an infinite family of mock modular forms by specializing the Dyson rank generating function at $z = \zeta_c^a \neq 1$, complex roots of unity not equal to 1 (where $\zeta_n := e^{2\pi i/n}$). See also [23] for related work. More precisely, they showed in [7, Theorem 1.1] that (under certain hypotheses), the functions

$$q^{-\frac{m_c}{24}} R(\zeta_c^a; q^{m_c}) + \frac{i\sin\left(\frac{\pi a}{c}\right) m_c^{\frac{1}{2}}}{\sqrt{3}} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta\left(\frac{a}{c}; m_c\rho\right)}{\sqrt{-i(\tau+\rho)}} d\rho$$

are harmonic Maass forms of weight $\frac{1}{2}$. Here, $\Theta\left(\frac{a}{c}; m_c\tau\right)$ is a certain weight 3/2 cusp form.

Other partition rank functions have been of interest for both combinatorial and modular-like properties (to name a few, see [1,3,6,12–14]). Here, we focus our attention on certain *overpartition* ranks. Precisely, an *overpartition* λ of a positive integer n is defined to be a partition of n in which the first instance of a part may or may not be overlined. For example, three of the 64 possible overpartitions of 7 are $3+2+1+1$, $3+\overline{2}+2$, and $\overline{3}+2+\overline{1}+1$. Of particular interest are the M_d-ranks of overpartitions, where d is a positive integer. The M_1 or Dyson rank of an overpartition λ is defined to be $\ell(\lambda) - \#(\lambda)$, where $\ell(\lambda)$ is the largest part of λ, overlined or not, and $\#(\lambda)$ is the number of parts of λ,

[1] Here and throughout, as is standard in this subject for simplicity's sake, we may slightly abuse terminology and refer to a function as a modular form or other modular object when in reality it must first be multiplied by a suitable power of q to transform appropriately.

exactly as the Dyson rank of a partition is defined above. The M_2 rank of an overpartition λ is defined by the slightly more complicated formula

$$\left\lceil \frac{\ell(\lambda)}{2} \right\rceil - \#(\lambda) - \#(\lambda_O) - \chi(\lambda).$$

Here λ_O is the sub-partition of λ consisting of non-overlined odd parts, and $\chi(\lambda)$ is 1 if the largest part of λ is non-overlined and odd, and 0 otherwise. Lovejoy [17,18] gave the generating functions for the overpartition (M_1) Dyson and M_2-ranks; they can be generalized to the following function of interest. For $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $m \in \mathbb{Z}$, we let $M_d(m,n) := \overline{p}(n \mid M_d\text{-rank } m)$. The M_d-rank overpartition generating function is given by

$$\mathcal{O}_d(z;q) = \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} M_d(m,n) z^m q^n$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+dn}}{(1-zq^{dn})(1-z^{-1}q^{dn})}\right).$$

Upon specializing $d = 1$ we recover the overpartition Dyson rank generating function, and with $d = 2$ we recover the overpartition M_2-rank generating function. Morrill in [19] gives combinatorial interpretations of the $\mathcal{O}_d$ series coefficients in terms of overpartition rank functions when $d > 2$. Here we focus our attention on certain analytic properties of $\mathcal{O}_d$. To describe this, throughout, we use parameters a, b, c, d satisfying the following conditions:

$$\begin{aligned} &a, b \in \mathbb{Z}, \\ &c \in \mathbb{N}, c > 1, \\ &\gcd(a,c) = \gcd(b,c) = \gcd(c,d) = 1. \end{aligned} \tag{1.5}$$

Using $\mathcal{O}_d$ above, define

$$\mathcal{O}_d^+(\tau) = \mathcal{O}_d^+(a,b,c;\tau) := \frac{(1+\zeta_c^a q^{\frac{b}{c}})}{(1-\zeta_c^a q^{\frac{b}{c}})} q^{-\frac{b^2}{c^2d^2}} \mathcal{O}_d(\zeta_c^a q^{\frac{b}{c}};q),$$

where $q = e^{2\pi i\tau}$ is the usual modular variable. We point out that the functions $\mathcal{O}_d^+(a,b,c;\tau)$ depend on parameters a, b, c and d, but for brevity's sake, we sometimes suppress the dependence on a, b, c and may write $\mathcal{O}_d^+(\tau)$. Analogous to the mock modularity established by Bringmann and Ono for the Dyson rank generating function $R(z;q)$ for roots of unity z as described above, Jennings-Shaffer and Swisher established the mock modular properties of $\mathcal{O}_d^+(\tau)$ in [16, Theorem 1.1, Theorem 1.2] (see also Theorem 2.3 below for specific transformation properties).

In this paper, we establish the quantum modularity of the M_d-rank overpartition generating functions $\mathcal{O}_d^+(a,b,c;\tau)$. Loosely speaking, a quantum modular form exhibits a modular-like transformation with respect to the action of a suitable subgroup of $\mathrm{SL}_2(\mathbb{Z})$; however, the domain of a quantum modular form is not the upper half-plane $\mathbb{H}$, but rather the set of rationals $\mathbb{Q}$ or an appropriate subset, and the functions transform with a "well-behaved" error to modularity on $\mathbb{Q}$, as explained in Definition 1.1 below. The formal definition of a quantum modular form was originally introduced by Zagier in [24] and has been slightly modified since then to allow for half-integral weights, subgroups of $\mathrm{SL}_2(\mathbb{Z})$, etc. (see [5]).

Definition 1.1 A weight $k \in \frac{1}{2}\mathbb{Z}$ quantum modular form is a complex-valued function f on $\mathbb{Q}$, such that for all $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the functions $h_A : \mathbb{Q} \setminus A^{-1}(i\infty) \to \mathbb{C}$ defined by

$$h_A(x) := f(x) - \varepsilon^{-1}(A)(\gamma x + \delta)^{-k} f\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

satisfy a "suitable" property of continuity or analyticity in a subset of $\mathbb{R}$.

Remark The complex numbers $\varepsilon(\gamma)$, which satisfy $|\varepsilon(\gamma)| = 1$, are such as those appearing in the theory of half-integral weight modular forms.

Remark We may modify Definition 1.1 appropriately to allow transformations on subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domains of the functions h_A to be suitable subsets of $\mathbb{Q}$.

The subject of quantum modular forms has been well-studied since the time of their definition roughly 10 years ago. For example, quantum modular forms have been shown to be related to the diverse areas of harmonic Maass forms, partial theta functions, colored Jones polynomials, meromorphic Jacobi forms, and vertex algebras, among other things (see, e.g., [5] and references therein). We also now know that the notion of a quantum modular form is related to Ramanujan's original notion of a mock theta function (see, e.g., [5,8,15]).

We now state our main result, which proves that the set of $\mathcal{O}_d^+(a,b,c;x)$ forms an infinite family of quantum modular forms, with quantum sets $Q_{a,b,c,d} \subseteq \mathbb{Q}$, with respect to the groups $G_{a,b,c,d} \subseteq \mathrm{SL}_2(\mathbb{Z})$, and with character χ_A (see Section 3 and (2.5) for precise definitions).

Theorem 1.2 *For any a, b, c, d as in (1.5), the function $\mathcal{O}_d^+(a,b,c;x)$ is a quantum modular form of weight $1/2$ on $Q_{a,b,c,d} \subseteq \mathbb{Q}$ with respect to $G_{a,b,c,d} \subseteq \mathrm{SL}_2(\mathbb{Z})$. That is, for all $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G_{a,b,c,d}$, and $x \in Q_{a,b,c,d}$, we have that*

$$H_{d,A}(x) := \mathcal{O}_d^+(a,b,c;x) - \chi_A^{-1}(\gamma x + \delta)^{-\frac{1}{2}} \mathcal{O}_d^+(a,b,c;Ax)$$

is defined, and extends to an analytic function in x on $\mathbb{R} \setminus \{\frac{-\delta}{\gamma}\}$. In particular, we have that

$$H_{d,A}(x) = m_{a,b,c,d} \int_{\frac{\delta}{\gamma}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d) - \frac{2a\kappa_d}{c}}(2d^2\kappa_d^2 w)}{\sqrt{-i(u+x)}} du + P_{d,A}(x), \tag{1.6}$$

where the weight $3/2$ theta functions g_{a_1,b_1} are defined in (2.2), the polynomial $P_{d,A}$ is defined in (2.6), and the constants κ_d, λ_d and $m_{a,b,c,d}$ are defined (2.4) and (5.3), respectively.

Interestingly, Theorem 1.2 (and its proof) reveal closed expressions which may be used to evaluate the Eichler integrals appearing on the right-hand side of (1.6), as we show in Corollary 1.3 below. To state it, we define the following finite q-hypergeometric series for $x = s/t \in Q_{a,b,c,d}$:

$$F_{a,b,c,d}(x) := \frac{1+\zeta_c^a e\left(\frac{xb}{c}\right)}{1-\zeta_c^a e\left(\frac{xb}{c}\right)} e\left(\frac{-xb^2}{c^2d^2}\right) \sum_{0\le n_1,n_2,\dots,n_d\le \frac{t}{2}} (-1;e(x))_{N_d} e\left(\frac{x}{2}(N_d^2-N_d)\right)$$
$$\times \prod_{j=1}^{d} \frac{(1-x_{d-j+1})(1-x_{d-j+1}^{-1})e(xN_j)}{(x_{d-j+1}e(xN_{j-1});e(x))_{n_j+1}(x_{d-j+1}^{-1}e(xN_{j-1});e(x))_{n_j+1}},$$

where $x_j = \zeta_d^{j-1}\zeta_{cd}^a e\left(\frac{xb}{cd}\right)$ $(1 \le j \le d)$, $N_0 := 0$, and $N_j = n_1 + n_2 + \cdots + n_j$ for $j \ge 1$. Here and throughout we also use the notation $e(u) := e^{2\pi i u}$.

Corollary 1.3 *For $x = s/t \in Q_{a,b,c,d}$, and any $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in G_{a,b,c,d}$, we have that*

$$m_{a,b,c,d}\int_{\frac{\delta}{\gamma}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2 w)}{\sqrt{-i(u+x)}}du$$
$$= F_{a,b,c,d}(x) - \chi_A^{-1}(\gamma x+\delta)^{-\frac{1}{2}}F_{a,b,c,d}(Ax) - P_{d,A}(x).$$

Remark A more direct or elementary proof of Corollary 1.3 (avoiding the use of quantum or mock modular forms) would be of interest. We pose this as an open problem.

The remainder of the paper is structured as follows. In Section 2 we record some known definitions and properties of certain modular and mock modular forms, which we use in our proof of Theorem 1.2. In Section 3, we define the sets $Q_{a,b,c,d} \subseteq \mathbb{Q}$ and groups $G_{a,b,c,d} \subseteq \mathrm{SL}_2(\mathbb{Z})$ with respect to which our quantum modular forms in Theorem 1.2 transform. We also establish density results for our sets $Q_{a,b,c,d}$, show that they are closed under the relevant group actions, and that they form appropriate domains for our quantum modular forms. In Section 4, we establish some additional lemmas used to prove Theorem 1.2. Finally in Section 5, we prove Theorem 1.2 and Corollary 1.3. Unless otherwise specified, throughout, the variable $\tau \in \mathbb{H}$ and the variable $x \in \mathbb{Q}$.

2 Some modular and mock modular forms

In this section, we record some known definitions and properties of certain modular and mock modular forms, which we use in our proof of Theorem 1.2.

2.1 Modular forms

A special ordinary modular form we make use of is Dedekind's η-function, defined in Section 1. This function is well known to satisfy the following transformation law [20].

Lemma 2.1 *For $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have that*

$$\eta(A\tau) = \nu_A(\gamma\tau+\delta)^{\frac{1}{2}}\eta(\tau),$$

where

$$\nu_A = \begin{cases} \left(\frac{\delta}{|\gamma|}\right)e\left(\frac{1}{24}\left((\alpha+\delta)\gamma - \beta\delta(\gamma^2-1)-3\gamma\right)\right) & \text{if } \gamma \equiv 1 \pmod 2,\\ \left(\frac{\gamma}{\delta}\right)e\left(\frac{1}{24}\left((\alpha+\delta)\gamma - \beta\delta(\gamma^2-1)+3\delta-3-3\gamma\delta\right)\right) & \text{if } \delta \equiv 1 \pmod 2, \end{cases} \tag{2.1}$$

and $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol.

We next introduce two families of weight 3/2 modular theta functions. The first is defined for $a_1, b_1 \in \mathbb{R}$ by

$$g_{a_1,b_1}(\tau) := \sum_{\omega\in a_1+\mathbb{Z}} \omega q^{\frac{\omega^2}{2}} e(\omega b_1) \tag{2.2}$$

(see [25]). The second is defined for $N \in \mathbb{N}$ and $h \in \mathbb{Z}$ as

$$\theta(\tau; h, N) := \sum_{n\equiv h \pmod N} nq^{\frac{n^2}{2N}}.$$

This function exhibits the modular transformation property shown in Lemma 2.2. The character $\psi_{A,N}$ is defined for $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ (with $\delta \equiv 1 \pmod 2$) and $N \in \mathbb{Z}$ by

$$\psi_{A,N} := \left(\frac{N}{\delta}\right)\left(\frac{2\gamma}{\delta}\right)\epsilon_\delta^{-1}, \tag{2.3}$$

where $\epsilon_\delta := \begin{cases} 1, & \delta \equiv 1 \pmod 4, \\ i, & \delta \equiv -1 \pmod 4. \end{cases}$

Lemma 2.2 *Let $N \in \mathbb{N}$ and let $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(2N) \cap \Gamma^0(2N) \cap \Gamma_1(N)$. Then we have that*

$$\theta(A\tau; h, N) = \psi_{A,N}(\gamma\tau + \delta)^{\frac{3}{2}}\theta(\tau; h, N).$$

Proof of Lemma 2.2 Note that $\theta(\tau; h, N) = \theta(\tau; h, N, N, \mathrm{id})$ (with $n = 1$) as defined by Shimura in [21, (2.0)]. Using [21, Proposition 2.1],

$$\begin{aligned} \theta(A\tau; h, N) &= e^{\frac{2\pi i\alpha\beta h^2}{2N}}\left(\frac{N}{\delta}\right)\left(\frac{2\gamma}{\delta}\right)\epsilon_d^{-1}(\gamma\tau + \delta)^{\frac{3}{2}}\theta(\tau; \alpha h, N) \\ &= \left(\frac{N}{\delta}\right)\left(\frac{2\gamma}{\delta}\right)\epsilon_\delta^{-1}(\gamma\tau + \delta)^{\frac{3}{2}}\theta(\tau; \alpha h, N) \\ &= \psi_{A,N}(\gamma\tau + \delta)^{\frac{3}{2}}\theta(\tau; h, N), \end{aligned}$$

where we have also used that $\alpha \equiv 1 \pmod N$ and $\beta \equiv 0 \pmod{2N}$ in simplifying the above.

□

2.2 Mock modular forms

Let

$$\kappa_d := \begin{cases} 1, & \text{if } d \text{ odd}, \\ \frac{1}{2}, & \text{if } d \text{ even}, \end{cases} \quad \text{and} \quad \lambda_d := \begin{cases} 0, & \text{if } d \text{ odd}, \\ -1, & \text{if } d \equiv 2 \pmod 4, \\ -\frac{1}{2}, & \text{if } d \equiv 0 \pmod 4. \end{cases} \tag{2.4}$$

From [16], we define

$$\widetilde{\mathcal{O}}_d(a, b, c; \tau) := \mathcal{O}_d^+(\tau) + \mathcal{O}_d^-(\tau),$$

where

$$\begin{aligned}\mathcal{O}_d^-(\tau) &= \mathcal{O}_d^-(a,b,c;\tau)\\ &:= -(-1)^d \left(e(\lambda_d^2)\zeta_c^{a\kappa_d} q^{-\frac{d^2\kappa_d^2}{4} - \frac{b^2}{c^2d^2} + \frac{b\kappa_d}{c}} R\left(\frac{2a\kappa_d}{c} + \lambda_d \right.\right.\\ &\qquad \left.\left. + \left(\frac{2b\kappa_d}{c} - d^2\kappa_d^2\right)\tau; 2d^2\kappa_d^2\tau\right) + (-1)^d q^{-\frac{b^2}{c^2d^2}}\right).\end{aligned}$$

The nonholomorphic function $R(u;\tau)$ is defined in [25] by

$$R(u;\tau) := \sum_{\omega\in\frac{1}{2}+\mathbb{Z}} \left\{\mathrm{sgn}(\omega) - E\left(\left(\omega + \frac{\mathrm{Im}(u)}{\mathrm{Im}(\tau)}\right)\sqrt{2\,\mathrm{Im}(\tau)}\right)\right\}(-1)^{\omega-\frac{1}{2}} e^{-\pi i\omega^2\tau - 2\pi i\omega u},$$

with

$$E(z) := 2\int_0^z e^{-\pi t^2}\,dt.$$

We next state a result from [16]. The character χ_A is defined for matrices $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ by

$$\chi_A := (-1)^{\beta + \frac{\alpha-1}{2}} i^{-\alpha\beta}\nu({}^2A)^{-3}, \tag{2.5}$$

where here and throughout, for a matrix $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, and $m \in \mathbb{Q}\setminus\{0\}$, we define ${}^mA := \left(\begin{smallmatrix}\alpha & m\beta\\ \gamma/m & \delta\end{smallmatrix}\right)$. The group referred to in Theorem 2.3 is explicitly defined in Section 3.

Theorem 2.3 *For any* $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in G'_{a,b,c,d}$, *we have that*

$$\widetilde{O}_d(A\tau) = \chi_A(\gamma\tau+\delta)^{\frac{1}{2}}\widetilde{O}_d(\tau).$$

Remark Theorem 1.1 and Theorem 1.2 in [16] use the transformation given above to ultimately establish that $\widetilde{O}_d$ is a harmonic Maass form of weight 1/2.

Let

$$\begin{aligned}p_d(\tau) &= p_{a,b,c,d}(\tau)\\ &:= -q^{-\frac{b^2}{c^2d^2}} + 2e\left((2n-1)\lambda_d^2\right)\\ &\times \sum_{m=1}^{|n|}(-1)^{m+n} e\left((2m-1)\mathrm{sgn}(n)\lambda_d^2\right)\zeta_c^{(2n+1-\mathrm{sgn}(n)(2m-1))a\kappa_d}\\ &\times q^{n\left(\frac{2b\kappa_d}{c} - d^2\kappa_d^2(n+1)\right) - (2m-1)\mathrm{sgn}(n)\left(\frac{b\kappa_d}{c} - d^2\kappa_d^2(n+\frac{1}{2})\right) - m(m-1)d^2\kappa_d^2 - \frac{d^2\kappa_d^2}{2} + \frac{b\kappa_d}{c} - \frac{b^2}{c^2d^2}},\end{aligned}$$

where $n = n_{b,c,d} := \left\lfloor \frac{b}{cd^2\kappa_d}\right\rfloor$. Using p_d we also define $P_{d,A}$ (dependent on matrices $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$) as in Theorem 1.2 by

$$P_{d,A}(\tau) := -p_d(\tau) + \chi_A^{-1}(\gamma\tau+\delta)^{-1/2}p_d(A\tau). \tag{2.6}$$

For simplicity's sake, we slightly abuse notation and refer to p_d and $P_{d,A}$ as *polynomials*, due to the fact that they are defined by finite sums in powers of q. From [16], we have the following result, which gives an alternative expression for $\mathcal{O}_d^-$.

Proposition 2.4 *For any a, b, c, d as in* (1.5)*, we have that*

$$\mathcal{O}_d^-(\tau) = p_d(\tau) - i\sqrt{2}d\kappa_d e\left(\frac{2ab}{c^2d^2} - \frac{b(1-2\lambda_d)}{2cd^2}\right) \times \int_{-\overline{\tau}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2w\kappa_d^2)}{\sqrt{-i(\tau+w)}}dw.$$

Remark The theta function in the integrand in Proposition 2.4 is written in a slightly different way than as given in [16], as we have applied the easily verifiable property that $g_{a_1+n,b_1} = g_{a_1,b_1}$ for any $n \in \mathbb{Z}$.

We also make use of the following result, which is a direct consequence of [19, Theorem 4.1], and gives an alternative expression for $\mathcal{O}_d^+$.

Proposition 2.5 *For any a, b, c, d as in* (1.5)*, and let* $x_j = \zeta_d^{j-1}\zeta_{cd}^a q^{\frac{b}{cd}}$, $1 \le j \le d$. *Then we have that*

$$\mathcal{O}_{a,b,c,d}^+(\tau) = \frac{1+\zeta_c^a q^{b/c}}{1-\zeta_c^a q^{b/c}} q^{\frac{-b^2}{c^2d^2}} \times \sum_{n_1,n_2,\dots,n_d \ge 0} (-1;q)_{N_d} q^{\frac{N_d^2-N_d}{2}} \prod_{j=1}^{d} \frac{(1-x_{d-j+1})(1-x_{d-j+1}^{-1})q^{N_j}}{(x_{d-j+1}q^{N_{j-1}};q)_{n_j+1}(x_{d-j+1}^{-1}q^{N_{j-1}};q)_{n_j+1}},$$

where $N_0 := 0$*, and* $N_j = n_1 + n_2 + \cdots + n_j$ *for* $j \ge 1$.

3 Quantum sets and groups

We call a subset $S \subseteq \mathbb{Q}$ a *quantum set* for a function F with respect to the group $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ if both $F(x)$ and $F(Mx)$ exist (are non-singular) for all $x \in S$ and $M \in G$. In this section, we will show that $Q_{a,b,c,d}$ as defined in (3.2) is a quantum set for $\mathcal{O}_d^+(a,b,c;x)$ with respect to the group $G_{a,b,c,d}$, defined below. We first recall the definitions of the congruence subgroups (defined for $N \in \mathbb{N}$):

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \gamma \equiv 0 \pmod{N} \right\},$$
$$\Gamma^0(N) := \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \beta \equiv 0 \pmod{N} \right\},$$
$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\}.$$

We define the groups

$$G'_{a,b,c,d} := \begin{cases} \Gamma_0(2d^2) \cap \Gamma_{a,b,c,d} \cap \Gamma'_{a,b,c,d} \cap \Gamma_1(d), & \text{if } d \text{ odd}, \\ \Gamma_0(d^2/2) \cap \Gamma_{a,b,c,d} \cap \Gamma'_{a,b,c,d} \cap \Gamma_1(2d), & \text{if } d \equiv 2 \pmod{4}, \\ \Gamma_0(4d^2) \cap \Gamma_{a,b,c,d} \cap \Gamma'_{a,b,c,d} \cap \Gamma_1(2d), & \text{if } d \equiv 0 \pmod{4}, \end{cases}$$

$$\Gamma_{a,b,c,d} := \Gamma_0\left(\frac{c^2d^2}{\gcd(a^2,c^2d^2)}\right) \cap \Gamma(c) \cap \Gamma^0\left(\frac{c^2d^2}{\gcd(b^2,c^2d^2)}\right),$$

$$\Gamma'_{a,b,c,d} := \begin{cases} \Gamma_0\left(\frac{2cd^2}{\gcd(a,2cd^2)}\right) \cap \Gamma_1\left(\frac{c^2d^2}{\gcd(ab,c^2d^2)}\right) \cap \Gamma_1\left(\frac{2cd^2}{\gcd(b,2cd^2)}\right), & \text{if } d \text{ odd,} \\ \Gamma_0\left(\frac{cd^2}{\gcd(a,cd^2)}\right) \cap \Gamma_1\left(\frac{c^2d^2}{\gcd(2ab,c^2d^2)}\right) \cap \Gamma_1\left(\frac{cd^2}{\gcd(b,cd^2)}\right), & \text{if } d \equiv 2 \pmod 4, \\ \Gamma_0\left(\frac{cd^2}{2\gcd(a,cd^2/2)}\right) \cap \Gamma_1\left(\frac{c^2d^2}{\gcd(2ab,c^2d^2)}\right) \cap \Gamma_1\left(\frac{cd^2}{2\gcd(b,cd^2/2)}\right), & \text{if } d \equiv 0 \pmod 4, \end{cases}$$

$$G''_{c,d} := \Gamma_0(4\ell_{c,d}^2 c^2 d^2) \cap \Gamma^0(c^2d^2/\kappa_d^2) \cap \Gamma_1(\ell'_{c,d} c^2 d^2),$$

and

$$G_{a,b,c,d} := G'_{a,b,c,d} \cap G''_{c,d}.$$

We also let

$$\ell_{c,d} := \begin{cases} 2, & \text{if } d \text{ odd and } c \text{ odd,} \\ 1, & \text{if } d \text{ odd and } c \text{ even, or } d \text{ even,} \end{cases}, \quad \ell'_{c,d} := \begin{cases} 2, & \text{if } d \text{ even and } c \text{ odd,} \\ \ell_{c,d}, & \text{else.} \end{cases} \tag{3.1}$$

Remark The groups $\Gamma_{a,b,c,d}$ and $\Gamma'_{a,b,c,d}$ are as in [16]; we have slightly simplified the definition of $\Gamma_{a,b,c,d}$ using that $\gcd(a,c) = \gcd(b,c) = 1$.

For a, b, c, d as above, we next define the sets $Q_{a,b,c,d}$ by

$$Q_{a,b,c,d} := \left\{ \frac{s}{t} \in \mathbb{Q} \;\middle|\; \begin{array}{l} (i)\ t \geq 2 \text{ even} \\ (ii)\ \gcd(s,t) = 1 \\ (iii)\ at + bs \not\equiv 0 \pmod c \\ (iv)\ \gamma s + \delta t \neq 0 \text{ for any } \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G_{a,b,c,d} \end{array} \right\}. \tag{3.2}$$

Proposition 3.1 *For any a, b, c, d as in (1.5), the set $Q_{a,b,c,d}$ is dense in $\mathbb{R}$.*

Proof of Proposition 3.1 We split the proof into two cases, depending on whether c is even or odd. For c even, define

$$S_e := \left\{ \frac{n}{c^2m+2} \;\middle|\; m \in \mathbb{N}, n \in \mathbb{Z}, \gcd(n, c^2m+2) = 1 \right\}.$$

We first show that $S_e \subseteq Q_{a,b,c,d}$. Let $n/(c^2m+2) \in S_e$. Then, conditions (i) and (ii) defining $Q_{a,b,c,d}$ are satisfied by definition. For condition (iii), we have that

$$a(c^2m+2) + bn \equiv 2a + bn \pmod c.$$

If $2a+bn \equiv 0 \pmod c$, then $2a+bn$ would be even in this case. But because $\gcd(b,c) = 1$, b is odd; moreover, because $\gcd(n, c^2m+2) = 1$ and c^2m+2 is even, n is also odd. Thus $2a + bn$ is odd, contradicting the assumption above. Hence, we must have that $a(c^2m+2) + bn \equiv 2a + bn \not\equiv 0 \pmod c$.

To prove condition (iv), assume for the sake of contradiction that for some $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G_{a,b,c,d}$, we have $\gamma n + \delta(c^2m+2) = 0$. This implies that $|\gamma n| = |(c^2m+2)\delta|$. Since $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(4\ell_{c,d}^2 c^2 d^2)$, we have that $|\gamma n| \equiv 0 \pmod 4$ and δ is odd. Thus (also using that c is even in this case), we have that $|(c^2m+2)\delta| \equiv 2 \pmod 4$, which is a contradiction.

For c odd, define

$$S_o := \left\{ \frac{n}{4cm+2c} \;\middle|\; m \in \mathbb{N}, n \in \mathbb{Z}, \gcd(n, 4cm+2c) = 1 \right\}.$$

The proof that $S_o \subseteq Q_{a,b,c,d}$ proceeds almost identically to the even case, so we omit details for brevity's sake.

Thus, since $S_e, S_o \subseteq Q_{a,b,c,d}$ are both dense, $Q_{a,b,c,d}$ is dense in $\mathbb{R}$ for all values a, b, c, d (with $\gcd(a,c) = \gcd(b,c) = \gcd(c,d) = 1, c > 1$). □

Lemma 3.2 *The sets $Q_{a,b,c,d}$ are closed under the group action of $G_{a,b,c,d}$.*

Proof of Lemma 3.2 Let $s/t \in Q_{a,b,c,d}$ and let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_{a,b,c,d}$. We will show that

$$A \cdot \frac{s}{t} = \frac{\alpha s + \beta t}{\gamma s + \delta t} = \frac{\operatorname{sgn}(\gamma s + \delta t)(\alpha s + \beta t)}{|\gamma s + \delta t|}$$

satisfies conditions $(i) - (iv)$ in the definition of $Q_{a,b,c,d}$. We begin with condition (ii).

Let $g = \gcd(\alpha s + \beta t, \gamma s + \delta t)$. Then $\alpha s + \beta t = gs'$ and $\gamma s + \delta t = gt'$ for some $s', t' \in \mathbb{Z}$ with $\gcd(s', t') = 1$. Recalling that $G_{a,b,c,d} \subseteq SL_2(\mathbb{Z})$ and considering the expressions

$$\delta s' - \beta t' = (\alpha\delta - \beta\gamma)\frac{s}{g} = \frac{s}{g} \in \mathbb{Z},$$
$$\alpha t' - \gamma s' = (\alpha\delta - \beta\gamma)\frac{t}{g} = \frac{t}{g} \in \mathbb{Z},$$

we see that $g = 1$ since $\gcd(s,t) = 1$. Hence, condition (ii) is satisfied.

To check condition (i), note that $2 \mid \gamma$ and $2 \mid t$, and therefore $2 \mid \gamma s + \delta t$. Moreover, since $s/t \in Q_{a,b,c,d}$, by condition (iv), we have that $\gamma s + \delta t \neq 0$.

To check condition (iii), note that $a(\gamma s + \delta t) + b(\alpha s + \beta t) \equiv at + bs \not\equiv 0 \mod c$.

To check condition (iv), we have for any $B \in G_{a,b,c,d}$ that $B \cdot (A \cdot \frac{s}{t}) = BA \cdot \frac{s}{t}$. Since $s/t \in Q_{a,b,c,d}$ and $BA \in G_{a,b,c,d}$, we have (by condition (iv) applied to s/t with the matrix BA) that $BA \cdot \frac{s}{t}$ (hence $B \cdot (A \cdot \frac{s}{t})$) has non-zero denominator. □

Proposition 3.3 *For any a, b, c, d as in (1.5), the set $Q_{a,b,c,d}$ is a quantum set for $\mathcal{O}_d^+$ with respect to the group $G_{a,b,c,d}$.*

Proof of Proposition 3.3 To establish well-definedness we employ the expression for O_d^+ given by Morrill in Proposition 2.5:

$$\mathcal{O}^+_{a,b,c,d}(x) = \frac{1+z}{1-z} q^{\frac{-b^2}{c^2d^2}} \sum_{n_1,n_2,\ldots,n_d \geq 0} (-1;q)_{N_d} q^{\frac{N_d^2 - N_d}{2}} \prod_{j=1}^{d} \frac{(1 - x_{d-j+1})(1 - x_{d-j+1}^{-1}) q^{N_j}}{(x_{d-j+1} q^{N_{j-1}}; q)_{n_j+1} (x_{d-j+1}^{-1} q^{N_{j-1}}; q)_{n_j+1}}, \tag{3.3}$$

where $N_0 = 0$, $N_k = n_1 + n_2 + \cdots + n_k$ for each positive integer k, $x_l = \zeta_d^{l-1} z^{1/d}$, $z = \zeta_c^a q^{b/c}$, and $q = e^{2\pi i x}$ where $x \in Q_{a,b,c,d}$ as defined above. We first simplify the numerators in the summands:

$$\prod_{j=1}^{d} (1 - x_{d-j+1})(1 - x_{d-j+1}^{-1}) = \prod_{k=1}^{d} (1 - \zeta_d^{k-1} z^{1/d})(1 - \zeta_d^{-(k-1)} z^{-1/d}) = (1-z)(1-z^{-1}). \tag{3.4}$$

We now cancel the $(1-z)$ from (3.4) with the same prefactor appearing in the denominator in (3.3). Next we will show that $z \neq \pm 1$, which guarantees that the remaining factor

$(1+z)(1-z^{-1})$ in front of the sum in (3.3) is non-zero. Note that $z=\zeta_c^a\zeta_{ct}^{sb}=\zeta_{ct}^{at+sb}$, and this equals 1 if and only if $at+sb\equiv 0 \pmod{ct}$, which implies $at+sb\equiv 0 \pmod{c}$, contradicting condition (*iii*) from our definition of $Q_{a,b,c,d}$ in (3.2). Moreover, $z=\zeta_{ct}^{at+sb}=-1$ if and only if $at+sb\equiv ct/2 \pmod{ct}$, but since t is even, this again implies that $at+sb\equiv 0 \pmod{c}$, contradicting condition (*iii*) from our definition of $Q_{a,b,c,d}$.

Next we will show that the denominators of the summands in (3.3) are non-zero for $x=s/t\in Q_{a,b,c,d}$. We compute

$$\begin{aligned}&(x_{d-j+1}q^{N_{j-1}};q)_{n_j+1}(x_{d-j+1}^{-1}q^{N_{j-1}};q)_{n_j+1}\\&\quad=\prod_{r=0}^{n_j}(1-\zeta_d^{d-j}\zeta_{cd}^a q^{b/cd}q^{N_{j-1}}q^r)(1-\zeta_d^{j-d}\zeta_{cd}^{-a}q^{-b/cd}q^{N_{j-1}}q^r)\\&\quad=\prod_{r=0}^{n_j}\left(1-\zeta_d^{d-j}\zeta_{cd}^a e\left(\tfrac{s}{t}(N_{j-1}+\tfrac{b}{cd}+r)\right)\right)\left(1-\zeta_d^{j-d}\zeta_{cd}^{-a} e\left(\tfrac{s}{t}(N_{j-1}-\tfrac{b}{cd}+r)\right)\right).\end{aligned}$$

Singularities occur only if

$$\zeta_d^{\pm(d-j)}\zeta_{cd}^{\pm a}e\left(\tfrac{s}{t}(N_{j-1}\pm\tfrac{b}{cd}+r)\right)=1$$

for some $1\le j\le d$, $0\le r\le n_j$, which implies that

$$\pm\frac{d-j}{d}\pm\frac{a}{cd}+\frac{s}{t}(N_{j-1}\pm\tfrac{b}{cd}+r)$$

is an integer. Equivalently, $\pm ct(d-j)\pm at+sdc(N_{j-1}\pm\frac{b}{cd}+r)\equiv 0 \pmod{dct}$. Either equation implies $at+sb\equiv 0 \pmod{c}$, which contradicts the conditions imposed by $Q_{a,b,c,d}$.

Finally, we note that whenever $N_d>t/2$, for any $x\in Q_{a,b,c,d}$, we have that $(-1;q)_{N_d}=0$. Hence, we have shown that the series in (3.3) converges for $x\in Q_{a,b,c,d}$. The proposition now follows after applying Lemma 3.2. □

4 Additional Lemmas

In this section, we state and prove some additional lemmas used in the proof of Theorem 1.2. We recall that κ_d and λ_d are as defined in (2.4), and $\ell_{c,d}$, $\ell'_{c,d}$ as in (3.1). The characters $\psi_{A,N}$ and χ_A are defined in (2.3) and (2.5) respectively.

Lemma 4.1 *Let a,b,c,d be as in (1.5). For $A=\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\in G''_{c,d}$, we have that*

$$g_{\frac{b}{cd^2\kappa_d},\frac12(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2{}^{-1}A\tau)=\psi_{-2\ell'_{c,d}\kappa_d^2 d_A,\ell'_{c,d}c^2d^2}(-\gamma\tau+\delta)^{\frac32}g_{\frac{b}{cd^2\kappa_d},\frac12(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2\tau).$$

To prove Lemma 4.1, we make use of the next lemma.

Lemma 4.2 *Let a,b,c,d be as in (1.5). We have that*

$$\begin{aligned}&g_{\frac{b}{cd^2\kappa_d},\frac12(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(\tau)\\&\quad=\frac{e\left(\frac{b}{cd^2\kappa_d}\left(\frac12(1-2\lambda_d)-\frac{2a\kappa_d}{c}\right)\right)}{cd^2}\sum_{j=0}^{\ell'_{c,d}c-1}\zeta_{2c}^{j(c(1-2\lambda_d)-4a\kappa_d)}\theta\left(\frac{\ell'_{c,d}\tau}{d^2};\frac{b}{\kappa_d}+jcd^2,\ell'_{c,d}c^2d^2\right).\end{aligned}$$

Proof of Lemma 4.2 First taking advantage of definition (2.2) and reindexing, we have

$$\begin{aligned}
&g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(\tau)\\
&\quad=\sum_{\omega\in\frac{b}{cd^2\kappa_d}+\mathbb{Z}}\omega e^{\pi i\omega^2\tau+2\pi i\omega\left(\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}\right)}\\
&\quad=\sum_{n\in\mathbb{Z}}\left(\frac{b}{cd^2\kappa_d}+n\right)e^{\pi i\left(\frac{b}{cd^2\kappa_d}+n\right)^2\tau+2\pi i\left(\frac{b}{cd^2\kappa_d}+n\right)\left(\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}\right)}\\
&\quad=\frac{e\left(\frac{b}{cd^2\kappa_d}\left(\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}\right)\right)}{cd^2}\sum_{n\in\mathbb{Z}}\zeta_{2c}^{n(c(1-2\lambda_d)-4a\kappa_d)}\left(\frac{b}{\kappa_d}+ncd^2\right)e^{\frac{\pi i\left(\frac{b}{\kappa_d}+ncd^2\right)^2\tau}{c^2d^4}}.
\end{aligned}$$

Reindexing, this becomes

$$\begin{aligned}
&\frac{e\left(\frac{b}{cd^2\kappa_d}\left(\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}\right)\right)}{cd^2}\sum_{j=0}^{\ell'_{c,d}c-1}\zeta_{2c}^{j(c(1-2\lambda_d)-4a\kappa_d)}\sum_{m\equiv\frac{b}{\kappa_d}+jcd^2\pmod{\ell'_{c,d}c^2d^2}}me^{\frac{\pi im^2\tau}{c^2d^4}}\\
&\quad=\frac{e\left(\frac{b}{cd^2\kappa_d}\left(\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}\right)\right)}{cd^2}\sum_{j=0}^{\ell'_{c,d}c-1}\zeta_{2c}^{j(c(1-2\lambda_d)-4a\kappa_d)}\theta\left(\frac{\ell'_{c,d}\tau}{d^2};\frac{b}{\kappa_d}+jcd^2,\ell'_{c,d}c^2d^2\right).
\end{aligned}$$

□

Proof of Lemma 4.1 By Lemma 4.2, it is sufficient to compute $\theta\left(\frac{\ell'_{c,d}\tau}{d^2};\frac{b}{\kappa_d}+jcd^2,\ell'_{c,d}c^2d^2\right)$, with $\tau\mapsto 2d^2\kappa_d^{2\ -1}A\tau$. That is, we will compute $\theta\left(2\ell'_{c,d}\kappa_d^{2\ -1}A\tau;\frac{b}{\kappa_d}+jcd^2,\ell'_{c,d}c^2d^2\right)$. We first note that

$$2\ell'_{c,d}\kappa_d^{2\ -1}A\tau={}^{-2\ell'_{c,d}\kappa_d^2}A(2\ell'_{c,d}\kappa_d^2\tau).$$

To use Lemma 2.2, we must show that ${}^{-2\ell'_{c,d}\kappa_d^2}A\in\Gamma_0(2\ell'_{c,d}c^2d^2)\cap\Gamma^0(2\ell'_{c,d}c^2d^2)\cap\Gamma_1(\ell'_{c,d}c^2d^2)$. Indeed, because $A\in G''_{c,d}$, we have that $\alpha,\delta\equiv 1\pmod{\ell'_{c,d}c^2d^2}$. Similarly, $-2\ell'_{c,d}\kappa_d^2\beta\equiv 0\pmod{2\ell'_{c,d}c^2d^2}$, and $\gamma/(-2\ell'_{c,d}\kappa_d^2)\equiv 0\pmod{2\ell'_{c,d}c^2d^2}$. Thus, by Lemma 2.2, we have that

$$\begin{aligned}
&\theta\left(2\ell'_{c,d}\kappa_d^{2\ -1}A\tau;\frac{b}{\kappa_d}+jcd^2,\ell'_{c,d}c^2d^2\right)\\
&\quad=\psi_{-2\ell'_{c,d}\kappa_d^2A,\ell'_{c,d}c^2d^2}(-\gamma\tau+\delta)^{\frac{3}{2}}\theta\left(2\ell'_{c,d}\kappa_d^2\tau;\frac{b}{\kappa_d}+jcd^2,\ell'_{c,d}c^2d^2\right).
\end{aligned}$$

Combining this with Lemma 4.2 finishes the proof. □

Next we establish a connection between the characters $\psi_{A,N}$ and χ_A, which will prove important in the next section.

Lemma 4.3 *With hypotheses as above, and* $A=\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\in G_{a,b,c,d}$, *we have that*

$$\psi_{-2\ell'_{c,d}\kappa_d^2A,\ell'_{c,d}c^2d^2}=\chi_A.$$

Proof of Lemma 4.3 We will show $\chi_A^{-1}\psi_{-2\ell'_{c,d}\kappa_d^2 A,\ell'_{c,d}c^2d^2} = 1$. By definition of $G_{a,b,c,d}$, we have $\delta \equiv 1 \pmod 2$ and $\gamma \equiv 0 \pmod{16}$. So,

$$\nu(^2A)^3 = \left(\frac{\frac{1}{2}\gamma}{\delta}\right)^3 \exp\left(\frac{\pi i}{4}\left((\alpha+\delta)\left(\frac{1}{2}\gamma\right) - 2\beta\delta\left(\left(\frac{1}{2}\gamma\right)^2 - 1\right) + 3\delta - 3 - \frac{3}{2}\gamma\delta\right)\right)$$
$$= \left(\frac{\frac{1}{2}\gamma}{\delta}\right) e^{\frac{\pi i\beta\delta}{2}} e^{\frac{3\pi i(\delta-1)}{4}}. \tag{4.1}$$

Next, we simplify

$$(-1)^{-\beta-\frac{\alpha-1}{2}} i^{\alpha\beta} = e^{-\pi i\left(-\beta-\frac{\alpha-1}{2}\right)} e^{\frac{\pi i\alpha\beta}{2}} = e^{\pi i\left(\beta+\frac{\alpha-1}{2}+\frac{\alpha\beta}{2}\right)} = (-1)^{\frac{\alpha-1}{2}} e^{\frac{\pi i\beta(2+\alpha)}{2}}.$$

Note that because $\gamma \equiv 0 \pmod 4$ and $\alpha\delta - \beta\gamma = 1$, we have that $(\alpha,\delta) \equiv (1,1) \pmod 4$ or $(\alpha,\delta) \equiv (3,3) \pmod 4$. So, $\alpha + \delta + 2 \equiv 0 \pmod 4$. Therefore, after combining the above expression with (4.1) we have

$$\chi_A^{-1} = (-1)^{-\beta-\frac{\alpha-1}{2}} i^{\alpha\beta}\nu(^2A)^3 = \left(\frac{\frac{1}{2}\gamma}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}} (-1)^{\frac{\alpha-1}{2}} e^{\frac{\pi i\beta(\delta+2+\alpha)}{2}} = \left(\frac{\frac{1}{2}\gamma}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}} (-1)^{\frac{\alpha-1}{2}}. \tag{4.2}$$

Using that $\left(\frac{x^2}{y}\right) = 1$ if $\gcd(x,y) = 1$,

$$\psi_{-2\ell'_{c,d}\kappa_d^2 A,\ell_{c,d}c^2d^2} = \left(\frac{\ell'_{c,d}c^2d^2}{\delta}\right)\left(\frac{-\gamma/(\ell'_{c,d}\kappa_d^2)}{\delta}\right)\epsilon_\delta^{-1} = \left(\frac{c^2d^2}{\delta}\right)\left(\frac{-\gamma/\kappa_d^2}{\delta}\right)\epsilon_\delta^{-1} = \left(\frac{-\gamma/\kappa_d^2}{\delta}\right)\epsilon_\delta^{-1}. \tag{4.3}$$

Here, $\left(\frac{-\gamma/\kappa_d^2}{\delta}\right)$ can be rewritten as $\left(\frac{-2/\kappa_d^2}{\delta}\right)\left(\frac{\frac{1}{2}\gamma}{\delta}\right)$. So, combining (4.2) and (4.3), we have

$$\chi_A^{-1}\cdot\psi_{-2\ell'_{c,d}\kappa_d^2 A,\ell_{c,d}c^2d^2} = \left(\frac{-\gamma/\kappa_d^2}{\delta}\right)\left(\frac{\frac{1}{2}\gamma}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}}(-1)^{\frac{\alpha-1}{2}}\epsilon_\delta^{-1} = \left(\frac{-2/\kappa_d^2}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}}(-1)^{\frac{\alpha-1}{2}}\epsilon_\delta^{-1}.$$

To continue simplifying, we split the rest of the proof into two cases. When $(\alpha,\delta) \equiv (1,1) \pmod 4$, then $\epsilon_\delta^{-1} = 1$ and $(-1)^{\frac{\alpha-1}{2}} = 1$. So, we are left to simplify

$$\left(\frac{-2/\kappa_d^2}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}}.$$

If $\delta \equiv 1 \pmod 8$, then $e^{\frac{3\pi i(\delta-1)}{4}} = 1 = \left(\frac{-1}{\delta}\right)\left(\frac{2/\kappa_d^2}{\delta}\right)$. Similarly, when $\delta \equiv 5 \pmod 8$, then $e^{\frac{3\pi i(\delta-1)}{4}} = -1 = \left(\frac{-1}{\delta}\right)\left(\frac{2/\kappa_d^2}{\delta}\right)$.

On the other hand, when $(\alpha,\delta) \equiv (3,3) \pmod 4$, we have $\epsilon_\delta^{-1} = -i$ and $(-1)^{\frac{\alpha-1}{2}} = -1$. Then, in this case we must simplify

$$i\left(\frac{-2/\kappa_d^2}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}}. \tag{4.4}$$

When $\delta \equiv 3 \pmod 8$, then $e^{\frac{3\pi i(\delta-1)}{4}} = -i$. And, $\left(\frac{-2/\kappa_d}{\delta}\right) = \left(\frac{-1}{\delta}\right)\left(\frac{2/\kappa_d}{\delta}\right) = 1$. So,

$$i\left(\frac{-2/\kappa_d^2}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}} = -i\cdot i = 1.$$

Similarly, when $\delta \equiv 7 \pmod 8$, we have $e^{\frac{3\pi i(\delta-1)}{4}} = i$ and $\left(\frac{-2/\kappa_d}{\delta}\right) = -1$, which also reduces (4.4) to 1. Therefore, we can see that in either of the two cases

$$\chi_A^{-1} \cdot \psi_{-2\ell'_{c,d}\kappa_d^2 A, \ell'_{c,d}c^2d^2} = \left(\frac{-2/\kappa_d^2}{\delta}\right) e^{\frac{3\pi i(\delta-1)}{4}} = 1.$$

□

5 Proofs of Theorem 1.2 and Corollary 1.3

Corollary 1.3 follows from equation (1.6) in Theorem 1.2, Proposition 2.5, and the proof of Proposition 3.3, where it is revealed that the summands appearing in Proposition 2.5 evaluate to 0 whenever $N_d > t/2$.

The remainder of this section is devoted to the proof of Theorem 1.2. From Theorem 2.3 and Proposition 2.4, for any $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G'_{a,b,c,d}$, we have that

$$\mathcal{O}_d^+(\tau) - \chi_A^{-1}(\gamma\tau+\delta)^{-\frac{1}{2}}\mathcal{O}_d^+(A\tau) = -\mathcal{O}_d^-(\tau) + \chi_A^{-1}(\gamma\tau+\delta)^{-\frac{1}{2}}\mathcal{O}_d^-(A\tau) \tag{5.1}$$

$$= m_{a,b,c,d}\left(\int_{-\overline{\tau}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2 w)}{\sqrt{-i(\tau+w)}}dw \right.$$

$$\left. - \chi_A^{-1}(\gamma\tau+\delta)^{-\frac{1}{2}} \int_{-\overline{A\tau}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2 w)}{\sqrt{-i(A\tau+w)}}dw\right) + P_{d,A}(\tau), \tag{5.2}$$

where

$$m_{a,b,c,d} := i\sqrt{2}d\kappa_d\, e\left(\frac{2ab}{c^2d^2} - \frac{b(1-2\lambda_d)}{2cd^2}\right). \tag{5.3}$$

We make the change of variable $w = {}^{-1}Au$, and obtain

$$\int_{-\overline{A\tau}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2 w)}{\sqrt{-i(A\tau+w)}}dw = \int_{-\overline{A\tau}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2\, {}^{-1}Au)}{\sqrt{-i(A\tau+{}^{-1}Au)}}d({}^{-1}Au).$$

A direct calculation using that $\det(A) = 1$ reveals that

$$A\tau + {}^{-1}Au = \frac{(\tau+u)}{(\gamma\tau+\delta)(-\gamma u+\delta)}.$$

Again after a short calculation using that $\det(A) = 1$, we find that

$$d({}^{-1}Au) = (-\gamma u+\delta)^{-2}du.$$

Next, we use that $u = \left(\begin{smallmatrix} \delta & \beta \\ \gamma & \alpha \end{smallmatrix}\right) w$ to determine the new bounds of integration after the change of variable. Combining all of the above, we have that

$$\int_{-\overline{A\tau}}^{i\infty} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2 w)}{\sqrt{-i(A\tau+w)}}dw$$

$$= \sqrt{\gamma\tau+\delta}\int_{-\overline{\tau}}^{\frac{\delta}{\gamma}} \frac{g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2\, {}^{-1}Au)}{(-\gamma u+\delta)^{3/2}\sqrt{-i(\tau+u)}}du. \tag{5.4}$$

By Lemma 4.1, for $A \in G''_{c,d}$, we have that

$$g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2\, {}^{-1}Au) = \psi_{-2\ell_{c,d}\kappa_d^2 A, \ell_{c,d}c^2d^2}(-\gamma u+\delta)^{\frac{3}{2}} g_{\frac{b}{cd^2\kappa_d}, \frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2 u).$$

Thus, we have that (5.4) equals

$$\psi_{-2\ell_{c,d}\kappa_d^2 A,\ell_{c,d}c^2d^2}\sqrt{\gamma\tau+\delta}\int_{-\overline{\tau}}^{\frac{\delta}{\gamma}}\frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2u)}{\sqrt{-i(\tau+u)}}du$$

$$\chi_A\sqrt{\gamma\tau+\delta}\int_{-\overline{\tau}}^{\frac{\delta}{\gamma}}\frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2u)}{\sqrt{-i(\tau+u)}}du,$$

where we have also used Lemma 4.3. Thus, (5.1) becomes

$$m_{a,b,c,d}\int_{\frac{\delta}{\gamma}}^{i\infty}\frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2w\kappa_d^2)}{\sqrt{-i(\tau+w)}}dw+P_{d,A}(\tau). \tag{5.5}$$

From Proposition 3.3, we have that $Q_{a,b,c,d}$ is a quantum set for $\mathcal{O}_d^+$ with respect to the group $G_{a,b,c,d}$. Thus, it remains to be seen that (5.5) is real analytic.

We integrate first on the vertical path from δ/γ to $\delta/\gamma+i\infty$ and then on the horizontal path from $\delta/\gamma+i\infty$ to $i\infty$. Recall from (2.2) that

$$g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(\tau)=\sum_{\omega\in\frac{b}{cd^2\kappa_d}+\mathbb{Z}}\omega q^{\frac{\omega^2}{2}}e\left(\omega\left(\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}\right)\right). \tag{5.6}$$

Writing $\tau=u+iv$ and letting $v\to\infty$, since $\omega\neq 0$, we can see that $q^{\frac{\omega^2}{2}}\to 0$. Thus, the integrand vanishes on the horizontal path, and we only need to consider the vertical path. After a change of variable $w=\delta/\gamma+it$, we must show that

$$i\cdot m_{a,b,c,d}\int_0^{\infty}\frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2(\frac{\delta}{\gamma}+it))}{\sqrt{-i(x+\frac{\delta}{\gamma}+it)}}dt+P_{d,A}(x)$$

is real analytic. The function $P_{d,A}(x)$ defined in (2.6) is clearly analytic, so, it suffices to show the real analyticity of

$$\int_0^{\infty}\frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2(\frac{\delta}{\gamma}+it))}{\sqrt{-i(x+\frac{\delta}{\gamma}+it)}}dt. \tag{5.7}$$

After the change of variable, we have a singularity in the denominator of the integrand when $t=0$ at $x=-\delta/\gamma$, however this point is excluded from consideration in the (real analytic) statement of Theorem 1.2. Following a similar strategy to the one established in [4] (see also [11,15,22,24] for similar arguments), we have (using the definition of g_{a_1,b_1}) that

$$\left|g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(\tau)\right|\ll\left|n_{b,c,d}\right|e\left(-n_{b,c,d}^2v\right)$$

for some $n_{b,c,d}\in\mathbb{Q}\setminus\{0\}$ as $v\to\infty$ (more specifically, we note that the decay of this theta function is minimized when $\omega=\frac{b}{cd^2\kappa_d}-\lfloor\frac{b}{cd^2\kappa_d}\rfloor$). That (5.7) is analytic on $\mathbb{R}\setminus\{\frac{-\delta}{\gamma}\}$ now follows from the Leibniz rule. In particular,

$$\frac{d}{dx}\int_0^{\infty}\frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2(\frac{\delta}{\gamma}+it))}{\sqrt{-i(x+\frac{\delta}{\gamma}+it)}}dt=\int_0^{\infty}\frac{g_{\frac{b}{cd^2\kappa_d},\frac{1}{2}(1-2\lambda_d)-\frac{2a\kappa_d}{c}}(2d^2\kappa_d^2(\frac{\delta}{\gamma}+it))}{\frac{d}{dx}\sqrt{-i(x+\frac{\delta}{\gamma}+it)}}dt.$$

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