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On a general class of non-squashing partitions

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ABSTRACT

We define M-sequence non-squashing partitions, which specialize to m-ary partitions (studied by Andrews, Churchhouse, Erdös, Hirschhorn, Knuth, Mahler, Rødseth, Sellers, and Sloane, among others), factorial partitions, and numerous other general partition families of interest. We establish an exact formula, various combinatorial interpretations, as well as the asymptotic growth of M-sequence non-squashing partition functions, functions whose associated generating functions are non-modular. In particular, we obtain an exact formula for the m-ary partition function, and by new methods, we recover Mahler's and Erdös' asymptotic for the m-ary partition function. We also establish new results on factorial partitions, colored m-ary partitions, and many other general families which have not been well understood or systematically studied. Finally, we conjecture Ramanujan-like congruences for the M-sequence non-squashing partition functions.

1. Introduction and statement of results

1.1. Introduction and motivation

Integer partitions have been a vibrant topic of study in number theory for almost three centuries. Among the many notable mathematicians who have studied integer partitions, Ramanujan made a tremendous impact in the subject by uncovering his celebrated congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \bmod 5, \\ p(7n+5) &\equiv 0 \bmod 7, \\ p(11n+6) &\equiv 0 \bmod 11, \end{aligned} \tag{1.1}$$

for the partition function $p(n) := \#\{\text{partitions of } n\}$, which hold for all $n \in \mathbb{N}_0$. Together, Hardy and Ramanujan further advanced the theory of partitions by introducing the "Circle Method" in analytic number theory, which led to their discovery of the asymptotic growth of the partition function. Namely, as $n \to \infty$, Hardy and Ramanujan showed that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{1.2}$$

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Hardy and Wright found an entire asymptotic expansion for $p(n)$, and later Rademacher and independently Selberg found an exact formula for $p(n)$ as an infinite series.

In addition to ordinary partitions, the so-called m-ary partitions, partitions into parts that are powers of some integer $m \geq 2$, have been a subject of study since the early 20th century. In 1969, Churchhouse [6] initiated the study of congruence properties for the m-ary partitions by stating the following conjecture on binary (2-ary) partitions. Here and throughout, we let $\beta_m(n) := \#\{m\text{-ary partitions of } n\}$.

Conjecture 1.1 (*Churchhouse [6]*). *For all integers $k \geq 1$ and odd integers $n \geq 1$, we have that*

$$\beta_2(2^{2k+2}n) \equiv \beta_2(2^{2k}n) \bmod 2^{3k+2},$$
$$\beta_2(2^{2k+1}n) \equiv \beta_2(2^{2k-1}n) \bmod 2^{3k}.$$

Conjecture 1.1 was first proved by Rødseth [18]. Related work of Hirschhorn and Loxton [12] later determined all "admissible" congruences satisfied by the binary partition function. The Churchhouse conjectures were later extended to m-ary partitions ($m \geq 2$) by Andrews [2], Gupta [10], and Rødseth and Sellers [19]. In particular, more analogous to the Ramanujan congruences (1.1), Rødseth and Sellers proved the following congruence properties of the m-ary partition function on certain arithmetic progressions modulo powers of m. In what follows, the $\mathbb{N}$-valued function $c_r = c_{r,m}$ equals 2^{r-1} if m is even, and equals 1 if m is odd. The function $\sigma_r = \sigma_{r,m} := \sigma_{r,M}(\epsilon)$ is as defined in (1.6) with $M = \{m, m, m, \ldots\}$.

Theorem 1.2 (*Rødseth–Sellers [19]*). *For any integers $r, n \geq 1$, we have that*

$$\beta_m(m^{r+1}n - \sigma_r - m) \equiv 0 \bmod \frac{m^r}{c_r}.$$

Prior to the Churchhouse conjectures, in 1940, Mahler [16] determined the asymptotic growth of the m-ary partition function as $n \to \infty$, namely,

$$\beta_m(n) \sim \exp\left(\frac{(\ln n)^2}{2\ln m}\right), \tag{1.3}$$

which naturally grows more slowly than $p(n)$ in (1.2). As we shall see, the generating function for m-ary partitions is not a modular form, while it is well known that the generating function for $p(n)$ is essentially modular. Mahler used methods other than the Circle Method to determine his asymptotic (1.3). By still different analytic methods, De Bruijn [7] and Pennington [17] were able to improve and generalize "Mahler's problem" on the asymptotic growth of $\beta_m(n)$. Other mathematicians have also studied m-ary partition function asymptotics by more elementary methods, such as Knuth [15] in the case $m = 2$ and Erdös [9], who in the same paper nearly recovered the Hardy–Ramanujan asymptotic for $p(n)$ in (1.2).

In 2004, Hirschhorn and Sellers [13] studied m-ary partitions through an entirely new lens, as they related m-ary partitions to *m-non-squashing partitions*, defined as follows.

Definition 1.3 (*Hirschhorn–Sellers [13]*). For any integer $m \geq 2$, an *m-non-squashing partition* of a positive integer n is a set of positive integers, $\{p_1, \ldots, p_k\}$, such that

(1) $p_1 + \cdots + p_k = n$,
(2) $p_1 \leq p_2 \leq \cdots \leq p_k$, and
(3) $(m-1)(p_1 + \cdots + p_{j-1}) \leq p_j$ for $j = 2, \ldots, k$.

Hirschhorn and Sellers beautifully showed that for any integers $n \geq 0$ and $m \geq 2$ the number of m-ary partitions of n is equal to the number of m-non-squashing partitions of n, that is,

$$\alpha_m(n) = \beta_m(n), \tag{1.4}$$

where we let $\alpha_m(n) := \#\{m\text{-non-squashing partitions of } n\}$. (Note. As usual, $\alpha_m(0) = \beta_m(0) := 1$.) The combinatorial interpretation of m-ary partitions due to Hirschhorn and Sellers given by (1.4), as well as subsequent elaborations by Sellers and Sloane [20] have led to new methods by which the m-ary partitions can be studied. Non-squashing partitions have been linked to other interesting problems as well, including the box stacking problem [20], which we will elaborate upon in Section 1.2.4, double binary number systems [8], and recursively self-conjugating partitions [14]. As noted by Andrews and Sellers [5], the study of partitions using factorial numbers as parts also served to motivate the study of m-non-squashing partitions and the box stacking problem; however, the presumed relationship ultimately proved to be false. We address this question on factorial partitions in more detail in the following sections.

1.2. Statement of results

With the context as described in Section 1.1, in this paper, we are motivated by the following questions:

(Q1) What can be said if the so-called "weight constant" $m-1$ in (3) of Definition 1.3 changes with the number of parts of the partition?

(Q2) In particular, can we relate m-ary partitions, factorial partitions, and other partitions of interest to this new type of restricted partition studied in (Q1)?

(Q3) Can we determine exact formulas for these partition functions? Inspired by Andrews, Churchhouse, Rødseth, and Sellers (among others), what can be said about Ramanujan-like congruence properties?

(Q4) In [3], Andrews deems the problem of understanding the asymptotics of non-modular partition functions one of five central problems at the interface of q-series and modular forms. Inspired by this problem, can we determine the asymptotic growth of the (non-modular) partition functions studied in [1] and [3]? If so, can we do so by (elementary) methods à la Erdös and Knuth?

To this end, we define *M-sequence non-squashing partitions* as follows.

Definition 1.4. For any sequence of integers $M=\{m_j\}_{j=0}^{\infty}$ with each $m_j\geq 2$, an *M-sequence non-squashing partition* of a positive integer n is a set of positive integers, $\{p_1,\ldots,p_k\}$, such that

(1) $p_1+\cdots+p_k=n$,
(2) $p_1\leq p_2\leq\cdots\leq p_k$, and
(3) $(m_{k-j}-1)(p_1+\cdots+p_{j-1})\leq p_j$ for $j=2,\ldots,k$.

Definition 1.4 is a first step towards (Q1). We address (Q2) in Section 1.2.1 (and Section 1.2.5), (Q3) in Sections 1.2.1 and 1.2.2, and (Q4) in Sections 1.2.3 and 1.2.5.

1.2.1. Combinatorial properties of M-sequence non-squashing partitions

Towards (Q2) and (Q3), we first establish an exact formula for $\alpha_M(n):=\#\{M\text{-non-squashing partitions of }n\}$. (Note. As usual, we set $\alpha_M(0):=1$.)

Theorem 1.5. *Given a sequence of integers $M=\{m_j\}_{j=0}^{\infty}$ with each $m_j\geq 2$, for all non-negative integers n, we have that*

$$\alpha_M(n)=\sum_{k_1=0}^{\left\lfloor\frac{n}{m_0}\right\rfloor}\sum_{k_2=0}^{\left\lfloor\frac{k_1}{m_1}\right\rfloor}\cdots\sum_{k_N=0}^{\left\lfloor\frac{k_{N-1}}{m_{N-1}}\right\rfloor}1,$$

where $N=N(n,M)$ is the critical number with respect to n and M (see Definition 3.1).

In particular, if we take $M=\{m,m,m,\ldots\}$ in Theorem 1.5, we obtain an exact formula for the number of m-non-squashing partitions of n, and hence the number of m-ary partitions of n by (1.4).

Corollary 1.6. *For any integer $m\geq 2$ and all non-negative integers n, we have that*

$$\alpha_m(n)=\beta_m(n)=\sum_{k_1=0}^{\left\lfloor\frac{n}{m}\right\rfloor}\sum_{k_2=0}^{\left\lfloor\frac{k_1}{m}\right\rfloor}\cdots\sum_{k_N=0}^{\left\lfloor\frac{k_{N-1}}{m}\right\rfloor}1,$$

where $N=\max\{0,\lfloor\log_m n\rfloor\}$.

We also determine the generating function for the general M-sequence non-squashing partition function.

Theorem 1.7. *For any sequence of integers $M=\{m_j\}_{j=0}^{\infty}$ with each $m_j\geq 2$, we have that*

$$\sum_{n=0}^{\infty}\alpha_M(n)q^n=\frac{1}{(1-q)\prod_{\ell=0}^{\infty}\left(1-q^{\prod_{j=0}^{\ell}m_j}\right)}. \tag{1.5}$$

Remark 1.8. As is typical, we require $q\in\mathbb{C}$ with $|q|<1$ to ensure convergence in (1.5).

Theorem 1.7 immediately reveals the following generalization of Hirschhorn's and Sellers' result (1.4); we note in particular a new interpretation of the factorial partition function $f(n):=\#\{\text{partitions of }n\text{ into factorial parts}\}$ in terms of M_f-sequence non-squashing partitions, where $M_f:=\{2,3,4,\ldots\}$.

Corollary 1.9. *For all $n \in \mathbb{N}_0$, the number of M-sequence non-squashing partitions of n, $\alpha_M(n)$, is equal to the number of partitions of n from the set*

$$\{1, m_0, m_0m_1, m_0m_1m_2, \ldots\}.$$

In particular, we have that $\alpha_m(n) = \beta_m(n)$ and $\alpha_{M_f}(n) = f(n)$.

1.2.2. Congruence properties of M-sequence non-squashing partitions

Inspired by the Churchhouse conjectures (Conjecture 1.1), Theorem 1.2, and (1.4), towards (Q3), we conjecture Ramanujan-like congruence properties for general M-sequence non-squashing partitions on arithmetic progressions modulo products of members of the sequence M. To describe this, we first introduce some notation. For a sequence of integers $M = \{m_j\}_{j=0}^{\infty}$ with each $m_j \geq 2$, we let

$$\mu_j = \mu_{j,M} := \frac{m_j}{\gcd(m_j, P_j)}, \qquad \sigma = \sigma_{r,M}(\boldsymbol{\epsilon}) := \sum_{j=1}^{r-1} \epsilon_j \prod_{i=0}^{j} m_i, \tag{1.6}$$

where $P_j := \prod_{\text{prime } p \leq j} p$, and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(r) = (\epsilon_1, \ldots, \epsilon_{r-1}) \in \{0, 1\}^{r-1}$, $r \in \mathbb{N}$.

Conjecture 1.10. *Let $M = \{m_j\}_{j=0}^{\infty}$ be a sequence of integers with each $m_j \geq 2$. Assuming the notation and hypotheses as above, for all integers $r, n \geq 1$, $0 \leq c \leq m_0 - 1$, and $\epsilon_j \in \{0, 1\}$ for each j, $(1 \leq j \leq r - 1)$, we have that*

$$\alpha_M(n \cdot m_0m_1 \cdots m_r - \sigma - m_0 + c) \equiv 0 \bmod \mu_1\mu_2 \cdots \mu_r. \tag{1.7}$$

In particular, if we let $M = M_f$, Conjecture 1.10 yields congruence results for factorial partitions, which have not been well understood. In what follows, for ease of notation, we re-index, and let

$$\sigma_f := \sigma_{r,M_f}(\boldsymbol{\epsilon}) = \sum_{j=3}^{r-1} \epsilon_j \cdot j!, \qquad D_r := \prod_{\text{prime } p \leq r-2} p^{\left\lfloor \frac{r-2}{p} \right\rfloor},$$

with $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(r) := (\epsilon_3, \epsilon_4, \ldots, \epsilon_{r-1}) \in \{0, 1\}^{r-3}$, $r \geq 3$ an integer.

Conjecture 1.11. *Assuming the notation and hypotheses as above, for all integers $r \geq 3$, $n \geq 1$ and $c \in \{1, 2\}$, we have that*

$$f(r! \cdot n - \sigma_f - c) \equiv 0 \bmod \frac{r!}{D_r}.$$

Remark. Conjecture 1.11 follows immediately from (1.7).

1.2.3. Asymptotic properties of M-sequence non-squashing partitions

In [3], as mentioned in (Q4), we are motivated by Andrews' central problem of understanding the asymptotics of non-modular partition functions, as well as Mahler's asymptotic (1.3) for m-ary partitions. To this end, we work to find the asymptotic growth of our M-sequence non-squashing partition function as $n \to \infty$; towards this goal, we first establish the following bound.

Theorem 1.12. *Let $M = \{m_j\}_{j=0}^{\infty} \subset \mathbb{N}$ with each $m_j \geq 2$. For all nonnegative integers n, let $N = N(n, M)$ be the critical number with respect to n and M (see Definition 3.1). Then, we have that*

$$\frac{n^N}{N! \prod_{k=0}^{N-1} m_k^{N-k}} \leq \alpha_M(n) \leq \frac{(2n)^N}{\prod_{k=0}^{N-1} m_k^{N-k}}.$$

Choosing $M = \{m, m, m, \ldots\}$, Theorem 1.12 leads us to recover Mahler's asymptotic (1.3) for the m-ary partition function by new (elementary) methods.

Corollary 1.13. *For all integers $m \geq 2$, as $n \to \infty$, we have that*

$$\ln \beta_m(n) \sim \frac{(\ln n)^2}{2 \ln m}, \qquad \ln \alpha_m(n) \sim \frac{(\ln n)^2}{2 \ln m}.$$

Theorem 1.12 also eventually leads us to establish the asymptotic growth of the factorial partition function, which was not known previously.

Theorem 1.14. *As $n \to \infty$, we have that*

$$\ln f(n) \sim \frac{(\ln n)^2}{2 \ln \ln n}.$$

While we have highlighted the m-ary partition function $\beta_m(n)$ and the factorial partition function $f(n)$ in this paper as two special cases of our M-sequence non-squashing partition functions, the number of sequences to which our results apply is infinite. In addition to $\beta_m(n)$ and $f(n)$, in Section 4, we apply our M-sequence results to a number of special general sequences, which include geometric, stabilizing, periodic, and arithmetic sequences. (See Propositions 4.4–4.8.)

1.2.4. M-sequence box stacking

Another problem closely related to non-squashing partitions is the so-called "box stacking problem", originally published by Sloane and Sellers [20]. Sloane's version of the problem is stated as follows [5]: Given n boxes, labeled $1, 2, \ldots, n$, suppose box i weighs $(m-1)i$ grams (where $m \geq 2$ is a fixed integer), and box i can support a total weight of i grams. What is the number, $a_m(n)$, of different ways to build a single stack of boxes in which no box will be squashed by the weight of the boxes above it?

Sloane and Sellers [20] showed that when $m = 2$, the number of ways to 2-box stack with n boxes is equal to the number of non-squashing partitions of the numbers $0, 1, \ldots, n$ with distinct parts and no part larger than n. This is easily generalized for any m as well, but for $m \geq 3$, distinctness is no longer a separate requirement because the non-squashing condition from box stacking encompasses this requirement (as explained by Andrews and Sellers [5]).

Given this relationship between m-box stacking and m-non-squashing partitions, as well as our generalized M-sequence non-squashing partitions (see Sections 1.2.1–1.2.3), we make the following definition.

Definition 1.15. For any sequence of integers $M = \{m_j\}_{j=0}^{\infty}$ with each $m_j \geq 2$, an *M-sequence box stacking partition* from n boxes ($n \in \mathbb{N}$) is a subset $\{p_1, \ldots, p_k\} \subseteq \{1, 2, \ldots, n\}$ ($1 \leq k \leq n$) such that

(1) $p_1 < p_2 < \cdots < p_k$,
(2) $(m_{k-j} - 1)(p_1 + p_2 + \cdots + p_{j-1}) \leq p_j$ for $j = 2, \ldots, k$.

Throughout, we let $a_M(n) := \#\{M\text{-sequence box stacking partitions from } n \text{ boxes}\}$ denote the M-sequence box stacking partition function. Extending the relationship between m-non-squashing and m-box stacking partitions described above, we analogously relate M-sequence non-squashing and M-sequence box stacking partitions as follows.

Theorem 1.16. *For any sequence of integers $M = \{m_0, m_1, m_2, \ldots\}$ with each $m_j \geq 3$ and any integer $c \geq 2$, let $M^+ = \{m_0 + 1, m_1, m_2, \ldots\}$ and $M^c = \{c, m_0, m_1, m_2, \ldots\}$. Then, we have that*

$$a_{M^+}(n) = \alpha_{M^c}(cn).$$

With this relationship, we are able to prove M-sequence box stacking asymptotic results similar to Theorem 1.12 and conjecture congruence properties similar to Conjecture 1.10; these M-sequence box stacking results are given in Corollary 5.3 and Conjecture 5.4, respectively.

1.2.5. P-colored m-ary partitions

Our results thus far have pertained to M-sequence non-squashing partitions, which generalize m-ary partitions. We also extend m-ary partitions in another direction, much in the same way that one may view "n-color partitions" as extending ordinary partitions. To be more precise, n-color partitions are partitions in which a part of size n may be used to partition an integer in n different (colored) ways. For example, we may represent the three n-color partitions of 2 by $2_1, 2_2$, and $1_1 + 1_1$, and say $p_n(2) = 3$, where $p_n(k) := \#\{n\text{-color partitions of } k\}(k \in \mathbb{N}_0)$. Agarwal and Andrews [1] (among others) studied p_n, and noted that p_n has the following generating function

$$\sum_{m=0}^{\infty} p_n(m) q^m = \prod_{n=1}^{\infty} (1 - q^n)^{-n}, \tag{1.8}$$

to which one may compare the generating function for the ordinary partition function $p(k)$ (in which each part of size n may be colored in only one way) given by $\prod_{n=1}^{\infty}(1 - q^n)^{-1}$. Here, we consider colored m-ary partitions in the following sense.

Definition 1.17. Let $c \in \mathbb{N}_0$, and let $P(x) = P_{\ell,a}(x)$ be any polynomial of degree $\ell - 2$ ($\ell \geq 2$ an integer) such that $P(x)$ has leading coefficient $a \in \mathbb{Q}^+$, and for all positive integers n, $P(n) \in \mathbb{N}_0$. A *P-c-colored m-ary partition* of a non-negative integer k is a partition of k into m-ary parts in which any part of size m^n ($n \in \mathbb{N}$) may be used in $P(n)$ different (colored) ways, and the part $m^0 = 1$ may be used in c different ways.

With this definition, we let $\beta_m^{P,c}(k) := \#\{P\text{-}c\text{-colored } m\text{-ary partitions of } k\}(k \in \mathbb{N}_0)$, e.g. $\beta_m^{1,1}(k) = \beta_m(k)$ is the m-ary partition function, and if $P(x) = x$, then $\beta_m^{x,1}(k) = \#\{n\text{-color } m\text{-ary partitions of } k\}$, that is, m-ary partitions in which the part m^n comes in n colors. We establish the following asymptotic result for P-c-colored m-ary partitions, from which we recover Mahler's asymptotic (1.3) a second time (see Corollary 1.13), again by different methods.

Theorem 1.18. *With notation and hypotheses as above, for all integers $\ell \geq 2$, $m \geq 2$, and $c \geq 0$, as $k \to \infty$, we have that*

$$\ln \beta_m^{P,c}(k) \sim \frac{a \ln^\ell k}{\ell(\ell-1)\ln^{\ell-1} m}.$$

In particular, we have that $\ln \beta_m(k) \sim \frac{(\ln k)^2}{2\ln m}$, and $\ln \beta_m^{x,1}(k) \sim \frac{(\ln k)^3}{6(\ln m)^2}$.

It is interesting to compare our asymptotic for the n-color m-ary partition function $\beta_m^{x,1}(k)$ in Theorem 1.18 to the asymptotic growth of the n-color partition function $p_n(k)$ as $n \to \infty$, namely,

$$p_n(k) \sim \frac{\zeta(3)^{\frac{7}{36}}}{\sqrt{12\pi}} \left(\frac{2}{k}\right)^{\frac{25}{36}} \exp\left(3\zeta(3)^{\frac{1}{3}}\left(\frac{k}{2}\right)^{\frac{2}{3}} + \zeta'(-1)\right). \tag{1.9}$$

The asymptotic in (1.9) for n-color partitions follows from Wright's celebrated asymptotic for the "plane partition" function [21], combined with work of MacMahon [4] and (1.8) (which reveal that $p_n(k)$ equals the number of "plane partitions" of k). Wright essentially employs the "saddlepoint method" to establish the asymptotic given in (1.9), a method which has since been used to establish the asymptotic properties of many other non-modular partition functions (see [3] for a discussion). Here, we are able to circumvent this method, and prove Theorem 1.18 by more elementary methods, inspired by Knuth [15].

The remainder of this paper is structured as follows. In Section 2, we discuss some preliminary functions and results used towards the proofs of our main theorems. In Section 3, we discuss M-sequence non-squashing and prove Theorem 1.5 in Section 3.2, Theorem 1.7 in Section 3.3, and Theorem 1.12 in Section 3.4. In Section 4, we apply our M-sequence results to a number of special general sequences, i.e. geometric, stabilizing, periodic, and arithmetic sequences. We prove Corollary 1.13 in Section 4.1 and Theorem 1.14 in Section 4.2. In Section 5, we discuss the M-sequence box stacking problem, establish Theorem 1.16, and offer a congruence conjecture. Finally, in Section 6, we discuss P-c-colored m-ary partitions, and prove Theorem 1.18.

2. Tools

In this section, we provide some tools that we will use throughout the paper. In Section 2.1, we discuss MacMahon's *Omega operator*. We analyze partition generating functions using this operator, much like in Andrews and Sellers [5]. In Section 2.2, we state some known results on special functions. These include Stirling's approximation and the asymptotic for the Barnes G-function. We also describe some properties of the Bernoulli numbers, which are useful in proving our asymptotic results. In Section 2.3, we extend tools found in Knuth [15] (i.e. related to harmonic numbers) which will be used to find the asymptotics for colored m-ary partitions.

2.1. The omega operator

MacMahon's Omega operator is defined as follows [5].

Definition 2.1. The Omega operator $\underset{\geqslant}{\Omega}$ is defined by

$$\underset{\geqslant}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_j=-\infty}^{\infty} A_{s_1,\ldots,s_j} \lambda_1^{s_1} \cdots \lambda_j^{s_j} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_j=0}^{\infty} A_{s_1,\ldots,s_j}, \tag{2.1}$$

where the domain of $A_{s_1,\ldots,s_j}$ is the field of rational functions over $\mathbb{C}$ in several complex variables restricted to a small neighborhood of the origin, and the λ_i are restricted to annuli of the form $1-\epsilon < |\lambda_i| < 1+\epsilon$.

Andrews and Sellers proved the following lemma which we will use to analyze our generating functions.

Lemma 2.2 (*Andrews–Sellers [5]*). *For nonnegative integers $s_1, s_2, \ldots, s_r$, and for complex numbers x, λ, y_i $(1 \leq i \leq r)$, we have that*

$$\underset{\geqslant}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y_1}{\lambda^{s_1}}\right)\left(1-\frac{y_2}{\lambda^{s_2}}\right)\cdots\left(1-\frac{y_r}{\lambda^{s_r}}\right)} = \frac{1}{(1-x)(1-x^{s_1}y_1)(1-x^{s_2}y_2)\cdots(1-x^{s_r}y_r)}.$$

2.2. Special functions

2.2.1. Factorial asymptotics

Stirling's Approximation is a well known result describing the asymptotic nature of factorial numbers. It is crucial to the proof of several of our asymptotic results. Stirling's Approximation in its entirety is an asymptotic series for $\ln n!$; here, we state a weaker version, which is sufficient for our purposes.

Lemma 2.3 (*Stirling's Approximation*)**.** *For positive integers n, as $n \to \infty$,*

$$\ln n! = n \ln n - n + O(\ln n).$$

The following corollary to Stirling's Approximation describes the asymptotic growth of a type of "inverse" to the factorial function. The proof follows by a direct calculation (see Zhang [23], for example, for one method of proof).

Lemma 2.4. *For any $x \geq 2$ and $b \in \mathbb{R}$, let n be an integer such that $\ln x = n \ln n + bn + O(\ln n)$. Then, as $x \to \infty$, we have that*

$$n = \frac{\ln x}{\ln \ln x} + O\left(\frac{\ln x \ln \ln \ln x}{(\ln \ln x)^2}\right).$$

2.2.2. The Barnes G-function

We use an approximation of the Barnes G-function to derive some of our asymptotic results.

Definition 2.5. The Barnes G-function, $G(z)$, is a function defined on the complex numbers by

$$G(1+z) := (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z + z^2(1+\gamma)}{2}\right) \prod_{k=1}^{\infty} \left\{\left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right)\right\}.$$

In particular, for any integer $n \geq 2$, the Barnes G-function satisfies

$$G(n) = \prod_{j=0}^{n-2} j!.$$

The following asymptotic for $G(z)$ is well known. In particular, Hirschhorn [11] determined a complete asymptotic expansion of $\prod_{k=0}^{n} \binom{n}{k}$, which when combined with Stirling's asymptotic series may be used to establish the following lemma.

Lemma 2.6. *As $z \to \infty$, we have that*

$$\ln G(z+1) = \frac{z \ln(2\pi)}{2} + \left(\frac{z^2}{2} - \frac{1}{12}\right) \ln z - \frac{3z^2}{4} + O(1).$$

2.2.3. Bernoulli numbers

The Bernoulli numbers B_n ($n \in \mathbb{N}_0$) are defined by the exponential generating function

$$\frac{t}{e^t - 1} := \sum_{n \geq 0} B_n \frac{t^n}{n!}.$$

One way in which the Bernoulli numbers appear is in Bernoulli's famous formula for sums of powers of integers.

Lemma 2.7 (*Bernoulli's Formula*)**.** *For all positive integers n and r,*

$$\sum_{k=1}^{n} k^r = \frac{1}{r+1} \sum_{k=0}^{r} \binom{r+1}{k} B_k \cdot n^{r+1-k}.$$

2.3. Harmonic numbers

In this subsection, we extend some results of Knuth [15], most of which pertain to harmonic numbers. The ideas presented here will be used in Section 6. First, we need a definition relating to the power series expansion of the exponent function.

Definition 2.8. For all polynomials $P(x)$, real numbers $a > 0$, integers $\ell \geq 2$ and integers $n \geq 0$, let $C_n = C_n(P, a, \ell)$, where

$$\sum_{n=0}^{\infty} C_n x^n = e^{a \ln^{\ell}(1-x)+P(x)}.$$

For all real numbers $a > 0$ and integers $\ell \geq 2$, let $c_n = c_n(a, \ell) := C_n(0, a, \ell)$.

Theorem 2.9. *For all real numbers $a > 0$, positive integers $\ell \geq 2$, and non-negative integers n,*

$$\ln c_n(a, \ell) = a \ln^{\ell} n + O((\ln^{\ell-1} n)(\ln \ln n)).$$

Proof. For all integers $m \geq 2$ and $n \geq 0$, let

$$H_{m,n} = \begin{cases} \displaystyle\sum_{\substack{1 \leq a_i < n \\ a_i \text{distinct}}} \frac{1}{a_1 a_2 \cdots a_{m-1}} & \text{if } m \leq n \\ 0 & \text{if } m > n. \end{cases}$$

From Knuth [15], we know that since $\ell \geq 2$ is an integer,

$$\ln^{\ell}(1-x) = \sum_{n=\ell}^{\infty} \frac{\ell}{n} H_{\ell,n} x^n.$$

Thus, from the power series expansion of the exponent function, we have that

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} \frac{a^k \ln^{\ell k}(1-x)}{k!} = 1 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a^k}{k!} \frac{\ell k}{n} H_{\ell k,n} x^n.$$

Knuth [15] also tells us that if h_k is the kth harmonic number and $n \geq \ell$, then

$$(h_{n-1} - h_{\ell-1})^{\ell-1} \leq H_{\ell,n} \leq h_{n-1}^{\ell-1}. \tag{2.2}$$

Hence, for $n \geq 1$,

$$c_n = \frac{\ell}{n} \sum_{k=1}^{\infty} \frac{a^k}{(k-1)!} H_{\ell k,n} \leq \frac{\ell}{n} \sum_{k=1}^{\infty} \frac{a^k}{(k-1)!} h_{n-1}^{\ell k-1} = \frac{\ell a h_{n-1}^{\ell-1}}{n} \sum_{k=1}^{\infty} \frac{a^{k-1} h_{n-1}^{\ell(k-1)}}{(k-1)!} = \frac{\ell a h_{n-1}^{\ell-1}}{n} e^{a h_{n-1}^{\ell}}.$$

Now, using the fact that $h_n = \ln n + O(1)$ from Knuth [15], we can write more simply

$$\ln c_n \leq a h_{n-1}^{\ell} + \ln\left(\frac{\ell a h_{n-1}^{\ell-1}}{n}\right) = a \ln^{\ell} n + O(\ln^{\ell-1} n). \tag{2.3}$$

For the lower bound, we notice that for all integers $K \geq 1$,

$$c_n = \frac{\ell}{n} \sum_{k=1}^{\infty} \frac{a^k}{(k-1)!} H_{\ell k,n} \geq \frac{\ell}{n} \frac{a^K}{(K-1)!} H_{\ell K,n}.$$

Let $K = \lceil a h_{n-1}^{\ell} + 1 \rceil$ and n large enough such that $n > \ell K$. Now, using the left hand side of (2.2) and Stirling's Approximation (Lemma 2.3), we find that

$$\begin{aligned}
\ln c_n &\geq \ln\left(\frac{\ell a}{n} \frac{a^{K-1}}{(K-1)!} (h_{n-1} - h_{\ell K-1})^{\ell K-1}\right) \\
&= (K-1)\ln a + (\ell K - 1)\ln(h_{n-1} - h_{\ell K-1}) - (K-1)(\ln(K-1) - 1) + O(\ln n) \\
&\geq a h_{n-1}^{\ell} \ln a + \ell a h_{n-1}^{\ell} \ln(h_{n-1} - h_{\ell K-1}) - (a h_{n-1}^{\ell} + 1)(\ln(a h_{n-1}^{\ell} + 1) - 1) + O(\ln n) \\
&= a h_{n-1}^{\ell} + \ell a h_{n-1}^{\ell} \ln\left(1 - \frac{h_{\ell K-1}}{h_{n-1}}\right) + O(\ln n) \\
&= a h_{n-1}^{\ell} - \ell a h_{n-1}^{\ell-1} h_{\ell K-1} + O((\ln^{\ell-2} n)(\ln \ln n)) + O(\ln n) \\
&= a \ln^{\ell} n + O((\ln^{\ell-1} n)(\ln \ln n)).
\end{aligned} \tag{2.4}$$

The result now follows from (2.3) and (2.4). □

Theorem 2.10. *For all real $a > 0$ and integers $\ell \geq 2$,*

$$\lim_{n\to\infty} \frac{c_{n+1}(a,\ell)}{c_n(a,\ell)} = 1.$$

Proof. Given a and ℓ as in the hypotheses, recall that for all non-negative integers n,

$$c_n = \frac{\ell}{n}\sum_{k=1}^{\infty}\frac{a^k}{(k-1)!}H_{\ell k,n}.$$

Since $H_{m,n+1} \geq H_{m,n}$ for all integers $m \geq 2$ and $n \geq 0$, we have that

$$\frac{c_{n+1}}{c_n} \geq \frac{n}{n+1}.$$

From Knuth [15], we know that for all integers $m \geq 2$ and $n \geq 0$,

$$H_{m,n} = H_{m,n-1} + \frac{m-1}{n-1}H_{m-1,n-1},$$

and

$$H_{m,n} \leq h_{n-1}H_{m-1,n}.$$

Therefore,

$$H_{m,n+1} = H_{m,n} + \frac{m-1}{n}H_{m-1,n} \leq H_{m,n} + \frac{m-1}{n}h_{n-1}^{\ell-1}H_{m-\ell,n}. \tag{2.5}$$

Now we obtain an upper bound using (2.5):

$$\begin{aligned} c_{n+1} &= \frac{\ell}{n+1}\sum_{k=1}^{\infty}\frac{a^k}{(k-1)!}H_{\ell k,n+1} \\ &\leq \frac{\ell}{n+1}\sum_{k=1}^{\infty}\frac{a^k}{(k-1)!}\left(H_{\ell k,n} + \frac{\ell k-1}{n}h_{n-1}^{\ell-1}H_{\ell(k-1),n}\right) \\ &= \frac{\ell}{n+1}\sum_{k=1}^{\infty}\frac{a^k}{(k-1)!}H_{\ell k,n} + \frac{\ell a h_{n-1}^{\ell-1}}{n+1}\sum_{k=1}^{\infty}\frac{a^{k-1}}{(k-1)!}\left(\frac{\ell k-1}{\ell k-\ell}\right)\left(\frac{\ell k-\ell}{n}\right)H_{\ell(k-1),n} \\ &\leq \frac{n}{n+1}c_n + \frac{2\ell a h_{n-1}^{\ell-1}}{n+1}c_n. \end{aligned}$$

Therefore, we have that

$$\frac{n}{n+1} \leq \frac{c_{n+1}}{c_n} \leq \frac{n}{n+1} + \frac{2\ell a h_{n-1}^{\ell-1}}{n+1},$$

and taking the limit as $n \to \infty$ gives us the desired result. □

Corollary 2.11. *For all real $a > 0$, integers $\ell \geq 2$, polynomials $P(x)$, and integers $n \geq 0$,*

$$\ln C_n(P,a,\ell) = \ln c_n(a,\ell) + O(1).$$

Proof. Let $e^{P(x)} = a_0 + a_1x + a_2x^2 + \cdots$. Then,

$$\frac{C_n}{c_n} = \frac{a_0c_n + a_1c_{n-1} + \cdots + a_nc_0}{c_n}.$$

If we take the limit of the above equation as $n \to \infty$, by Theorem 2.10,

$$\lim_{n\to\infty}\frac{C_n}{c_n} = a_0 + a_1 + \cdots = e^{P(1)}.$$

Since $e^{P(1)}$ is a constant, we can take the natural log of both sides and the result follows. □

Finally, we state a key lemma as proved in [15].

Lemma 2.12 (*Knuth [15]*). *Let*

$$A(x) = \exp \int_0^x (1-t)a(t)dt$$
$$B(x) = \exp \int_0^x (1-t)b(t)dt$$

where

$$A(x) = \sum_{k=1}^{\infty} A_k x^k, \qquad a(x) = \sum_{k=1}^{\infty} a_k x^{k-1}, \qquad B(x) = \sum_{k=1}^{\infty} B_k x^k, \qquad b(x) = \sum_{k=1}^{\infty} b_k x^{k-1}.$$

Assume the coefficients of $A(x)$ and $b(x)$ are non-negative and non-decreasing. If $a_k \leq b_k$ for all k, then $A_k \leq B_k$. If $a_k \geq b_k$ for all k, then $A_k \geq B_k$.

3. Sequence non-squashing partitions

In this section, we present our findings on the properties of the M-sequence non-squashing partition function. In Section 3.1, we provide some useful definitions related to M-sequence non-squashing partitions (see Definition 1.4). In Section 3.2, we discuss the recursive relationship the function satisfies as well as prove its exact formula given by Theorem 1.5. In Section 3.3, we give two proofs for the generating function of the sequence non-squashing partition function (Theorem 1.7) and also discuss its implications. In Section 3.4, we give the proof for Theorem 1.12, which is our main asymptotic result for the general sequence non-squashing partition function.

3.1. Preliminaries

Definition 3.1. Given a nonnegative integer n and a sequence of integers $M = \{m_j\}_{j=0}^{\infty}$ with all $m_j \geq 2$, define the *critical number* $N := N(n, M)$ with respect to n and M as the unique integer such that if $n \geq m_0$, then

$$m_0 m_1 \cdots m_{N-1} \leq n < m_0 m_1 \cdots m_N;$$

otherwise, we set $N := 0$.

To denote certain subsequences of a sequence M, we will use the following definition.

Definition 3.2. For any sequence $M = \{m_j\}_{j=0}^{\infty}$ and for any nonnegative integer i, we define $M_i := \{m_j\}_{j=i}^{\infty}$.

3.2. Recursion and summation formula

We begin our discussion of the sequence non-squashing partition function by discussing the recurrence relation it satisfies. This recurrence relation is similar to the recurrence relation for the m-non-squashing partition function given by Sloane and Sellers in [20].

Theorem 3.3. *Given a sequence of integers $M = \{m_j\}_{j=0}^{\infty}$ with each $m_j \geq 2$, for all positive integers n, we have that*

$$\alpha_M(n) = \begin{cases} \alpha_M(n-1) & \text{if } n \not\equiv 0 \bmod m_0, \\ \alpha_M(n-1) + \alpha_{M_1}\left(\dfrac{n}{m_0}\right) & \text{if } n \equiv 0 \bmod m_0, \end{cases}$$

where $\alpha_M(0) = 1$.

Proof. We proceed in a manner similar to [20]. The initial conditions $\alpha_M(0) = \alpha_M(1) = \cdots = \alpha_M(m_0 - 1) = 1$ are clear from the fact that the smallest M-sequence non-squashing partition that has more than one part occurs when $n = m_0$. Now, note that given any M-sequence non-squashing partition of n with parts $p_1, \ldots, p_k$, we have that $p_k \geq (m_0 - 1)\frac{n}{m_0}$. If not, then $(m_0 - 1)(p_1 + \cdots + p_{k-1}) > (m_0 - 1)\frac{n}{m_0} > p_k$, violating (3) in Definition 1.4. We now consider two cases.

Case 1: Suppose n is not divisible by m_0. Since p_k is an integer, $p_k > (m_0 - 1)\frac{n}{m_0}$. Therefore, since $p_1 + \cdots + p_k = n$, we have $p_1 + \cdots + p_{k-1} < \frac{n}{m_0}$. Hence

$$(m_0 - 1)(p_1 + \cdots + p_{k-1}) < (m_0 - 1)\frac{n}{m_0} < p_k,$$

so

$$(m_0 - 1)(p_1 + \cdots + p_{k-1}) \leq p_k - 1.$$

Thus, $\{p_1, \ldots, p_{k-1}, p_k - 1\}$ is an M-sequence non-squashing partition of $n - 1$.

Case 2: Suppose n is divisible by m_0. If $p_k > (m_0 - 1)\frac{n}{m_0}$, then we can argue as in Case 1 to obtain an M-sequence non-squashing partition of $n-1$. If $p_k = (m_0 - 1)\frac{n}{m_0}$, then we can remove p_k to obtain $\{p_1, \ldots, p_{k-1}\}$, an M_1-sequence non-squashing partition of $\frac{n}{m_0}$. $\square$

From Theorem 3.3, it is easy to derive the following proposition which connects M-sequence non-squashing partitions to M_1-sequence non-squashing partitions.

Proposition 3.4. *Given a sequence of integers $M = \{m_j\}_{j=0}^{\infty}$ with each $m_j \geq 2$, for all non-negative integers n, we have that*

$$\alpha_M(n) = \sum_{k=0}^{\left\lfloor \frac{n}{m_0} \right\rfloor} \alpha_{M_1}(k).$$

Proof. For any $n \in \mathbb{N}$, let $y = y_{n,M}$ be the unique integer satisfying $0 \leq y \leq m_0 - 1$ such that $n = xm_0 + y$, where $x = x_{n,M} := \left\lfloor \frac{n}{m_0} \right\rfloor$. Then, using the recurrence from Theorem 3.3,

$$\begin{aligned}
\alpha_M(n) &= \alpha_M(xm_0 + y) = \alpha_M(xm_0) = \alpha_{M_1}(x) + \alpha_M(xm_0 - 1) \\
&= \alpha_{M_1}(x) + \alpha_M((x-1)m_0) = \alpha_{M_1}(x) + \alpha_{M_1}(x-1) + \alpha_M((x-1)m_0 - 1) \\
&\ \ \vdots \\
&= \sum_{k=0}^{x} \alpha_{M_1}(k) = \sum_{k=0}^{\left\lfloor \frac{n}{m_0} \right\rfloor} \alpha_{M_1}(k)
\end{aligned}$$

as claimed. $\square$

With Proposition 3.4 and Definition 3.1, we prove our exact "nested sum" formula (Theorem 1.5) for the M-sequence non-squashing partition function.

Proof of Theorem 1.5. Let $n \in \mathbb{N}$. We apply Proposition 3.4 N times (where $N = N(n, M)$ as in Definition 3.1), and find that

$$\alpha_M(n) = \sum_{k=0}^{\left\lfloor \frac{n}{m_0} \right\rfloor} \alpha_{M_1}(k_1) = \sum_{k_1=0}^{\left\lfloor \frac{n}{m_0} \right\rfloor} \sum_{k_2=0}^{\left\lfloor \frac{k_1}{m_1} \right\rfloor} \alpha_{M_2}(k_2) = \cdots = \sum_{k_1=0}^{\left\lfloor \frac{n}{m_0} \right\rfloor} \sum_{k_2=0}^{\left\lfloor \frac{k_1}{m_1} \right\rfloor} \cdots \sum_{k_N=0}^{\left\lfloor \frac{k_{N-1}}{m_{N-1}} \right\rfloor} \alpha_{M_N}(k_N).$$

Since N is the critical number, $\alpha_{M_N}(k_N) = 1$. $\square$

3.3. Generating function

Here, we prove Theorem 1.7 for the M-sequence non-squashing partition generating function in two ways, one with the Omega operator defined in Definition 2.1, the other using the recurrence from Theorem 3.3. Our first proof is inspired by [5], the second is inspired by [20].

First proof of Theorem 1.7. By Definition 1.4, the generating function for M-sequence non-squashing partitions is

$$\sum_{n=0}^{\infty} \alpha_M(n) q^n = \lim_{k \to \infty} F_{M,k}(q),$$

where

$$F_{M,k}(q) := \underset{\geqslant}{\Omega} \sum_{p_1, p_2, \ldots, p_k \geq 0} q^{p_1 + p_2 + \cdots + p_k} \prod_{j=2}^{k} \lambda_j^{p_j - (m_{k-j} - 1)(p_1 + p_2 + \cdots + p_{j-1})}$$

and $\underset{\geqslant}{\Omega}$ is the Omega operator as defined in Definition 2.1. We can write $F_{M,k}(q)$ as the Omega operator applied to a product of geometric series:

$$F_{M,k}(q) = \underset{\geqslant}{\Omega} \frac{1}{(1 - q\lambda_k) \left[\prod_{j=2}^{k-1} \left(1 - \frac{q\lambda_j}{\prod_{\ell=j+1}^{k} \lambda_\ell^{m_{k-\ell} - 1}} \right) \right] \left(1 - \frac{q}{\prod_{j=2}^{k} \lambda_j^{m_{k-j} - 1}} \right)}.$$

We now proceed by applying the Omega operator on $\lambda_k, \lambda_{k-1}, \ldots, \lambda_2$, in that order. We have

$$\begin{aligned}
F_{M,k}(q) &= \underset{\geqslant}{\Omega} \frac{1}{(1-q\lambda_k)\left[\prod_{j=2}^{k-1}\left(1-\frac{q\lambda_j}{\prod_{\ell=j+1}^{k}\lambda_\ell^{m_{k-\ell-1}}}\right)\right]\left(1-\frac{q}{\prod_{j=2}^{k}\lambda_j^{m_{k-j-1}}}\right)} \\
&= \frac{1}{1-q}\times \underset{\geqslant}{\Omega} \frac{1}{(1-q^{m_0}\lambda_{k-1})\left[\prod_{j=2}^{k-2}\left(1-\frac{q^{m_0}\lambda_j}{\prod_{\ell=j+1}^{k-1}\lambda_\ell^{m_{k-\ell-1}}}\right)\right]\left(1-\frac{q^{m_0}}{\prod_{j=2}^{k-1}\lambda_j^{m_{k-j-1}}}\right)} \\
&= \frac{1}{(1-q)(1-q^{m_0})}\times \underset{\geqslant}{\Omega} \frac{1}{(1-q^{m_0m_1}\lambda_{k-2})\left[\prod_{j=2}^{k-3}\left(1-\frac{q^{m_0m_1}\lambda_j}{\prod_{\ell=j+1}^{k-2}\lambda_\ell^{m_{k-\ell-1}}}\right)\right]\left(1-\frac{q^{m_0m_1}}{\prod_{j=2}^{k-2}\lambda_j^{m_{k-j-1}}}\right)} \\
&\;\vdots \\
&= \frac{1}{(1-q)\prod_{\ell=0}^{k-4}\left(1-q^{\prod_{j=0}^{\ell}m_j}\right)}\times \underset{\geqslant}{\Omega} \frac{1}{(1-q^{m_0m_1\cdots m_{k-3}}\lambda_2)\left(1-\frac{q^{m_0m_1\cdots m_{k-3}}}{\lambda_2^{m_{k-2-1}}}\right)} \\
&= \frac{1}{(1-q)\prod_{\ell=0}^{k-2}\left(1-q^{\prod_{j=0}^{\ell}m_j}\right)}.
\end{aligned}$$

Now, taking the limit as $k \to \infty$, we see that

$$\sum_{n=0}^{\infty}\alpha_M(n)q^n = \lim_{k\to\infty}F_{M,k}(q) = \frac{1}{(1-q)\prod_{\ell=0}^{\infty}\left(1-q^{\prod_{j=0}^{\ell}m_j}\right)},$$

as desired. □

Second proof of Theorem 1.7. Let $\beta_M(n)$ be the number of partitions of n with parts in the set

$$S_M = \{1, m_0, m_0m_1, m_0m_1m_2, \ldots\}.$$

Then, clearly the generating function for $\beta_M(n)$ is

$$\sum_{n=0}^{\infty}\beta_M(n) = \frac{1}{(1-q)\prod_{\ell=0}^{\infty}\left(1-q^{\prod_{j=0}^{\ell}m_j}\right)}.$$

We will show that $\beta_M(n)$ satisfies the same recurrence and initial condition as $\alpha_M(n)$, as stated in Theorem 3.3. The initial condition is the same, since $\beta_M(0) = 1$. Write $n = b_{-1} + b_0m_0 + b_1m_0m_1 + b_2m_0m_1m_2 + \cdots + b_rm_0m_1\cdots m_r$, where integers $r, b_{-1}, b_0, \ldots, b_r$ are such that $m_0m_1\cdots m_r \le n < m_0m_1\cdots m_{r+1}$ and $0 \le b_j < m_{j+1}$ for all $-1 \le j \le r$. Any partition of n with parts in S_M either has no 1's or at least one 1. If it has at least one 1, then the same partition with one less 1 is a partition of $n-1$. If it has no 1's, then $m_0|n$, and that partition of n with each part divided by m_0 is a partition of $\frac{n}{m_0}$ with parts in S_{M_1}. Thus, $\beta_M(n)$ satisfies the recurrence

$$\beta_M(n) = \begin{cases}\beta_M(n-1) & \text{if } n \not\equiv 0 \bmod m_0,\\ \beta_M(n-1) + \beta_{M_1}\left(\dfrac{n}{m_0}\right) & \text{if } n \equiv 0 \bmod m_0.\end{cases}$$

Since $\alpha_M(n)$ and $\beta_M(n)$ satisfy the same recurrence and initial condition, $\alpha_M(n) = \beta_M(n)$ for all $n \geq 0$, their generating functions are equal. □

Remark. As pointed out by the referee with respect to the above proof, that $\alpha_M(n) = \beta_M(n)$ can also be seen by direct bijection, by analyzing as follows. Pick a largest part p_k, then pick a second largest part p_{k-1} and add $(m_0 - 1)p_{k-1}$ to the largest part. Then pick a third largest part p_{k-2} and add $(m_1 - 1)p_{k-2}$ to the second largest part and $m_1(m_0 - 1)p_{k-2}$ to the third largest part. In general, for the ℓth largest part, pick it (p_ℓ), and then add $(m_{n-2}m_{n-3}\cdots m_j)(m_{j-1} - 1)p_n$ to the jth largest part, for $1 \leq j < \ell$. In this way, we create an M-non-squashing partition, and if we combine the coefficients for each p_i together, we will have also obtained a partition with parts in the set S_M.

As we see from this second proof of Theorem 1.7, the number of M-sequence non-squashing partitions of n is equal to the number of partitions of n whose parts are in the set $S_M = \{1, m_0, m_0m_1, m_0m_1m_2, \ldots\}$, proving Corollary 1.9.

3.4. Asymptotic bounds

In this section, we prove Theorem 1.12 using Theorem 1.5. We will later apply Theorem 1.12 to specific (families of) sequences of interest, and subsequently derive the asymptotic growth of their associated partition functions. First, we establish a short lemma that will be used in proving the lower bound of Theorem 1.12.

Lemma 3.5. *Let $M = \{m_j\}_{j=0}^{\infty}$ be a sequence of integers with each $m_j \geq 2$. Then, for all integers $A \geq 1$ and $n \geq 0$, we have that*

$$\sum_{k=0}^{\lfloor \frac{n}{A} \rfloor} \alpha_M(k) \geq \frac{1}{A} \sum_{k=0}^{n} \alpha_M\left(\left\lfloor \frac{k}{A} \right\rfloor\right).$$

Proof. By the division algorithm, there exist unique integers q and r such that $n = qA + r$, where $0 \leq r < A$. Then,

$$A \sum_{k=0}^{\lfloor \frac{n}{A} \rfloor} \alpha_M(k) = \sum_{k=0}^{(q+1)A-1} \alpha_M\left(\left\lfloor \frac{k}{A} \right\rfloor\right) \geq \sum_{k=0}^{n} \alpha_M\left(\left\lfloor \frac{k}{A} \right\rfloor\right)$$

because each term is positive and $n \leq (q + 1)A - 1$. Dividing both sides by A gives us our desired result. □

We are now ready to prove Theorem 1.12.

Proof of Theorem 1.12. To establish the lower bound in Theorem 1.12, we use Lemma 3.5 repeatedly to find that

$$\begin{aligned}
\alpha_{M_0}(n) &= \sum_{k_1=0}^{\lfloor \frac{n}{m_0} \rfloor} \alpha_{M_1}(k_1) \\
&\geq \frac{1}{m_0} \sum_{k_1=0}^{n} \alpha_{M_1}\left(\left\lfloor \frac{k_1}{m_0} \right\rfloor\right) = \frac{1}{m_0} \sum_{k_1=0}^{n} \sum_{k_2=0}^{\lfloor \frac{k_1}{m_0m_1} \rfloor} \alpha_{M_2}(k_2) \\
&\geq \frac{1}{m_0^2 m_1} \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \alpha_{M_2}\left(\left\lfloor \frac{k_2}{m_0m_1} \right\rfloor\right) = \frac{1}{m_0^2 m_1} \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{\lfloor \frac{k_2}{m_0m_1m_2} \rfloor} \alpha_{M_3}(k_3) \\
&\geq \frac{1}{m_0^3 m_1^2 m_2} \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \alpha_{M_3}\left(\left\lfloor \frac{k_3}{m_0m_1m_2} \right\rfloor\right) \\
&\;\;\vdots \\
&\geq \frac{1}{\prod_{k=0}^{N-1} m_k^{N-k}} \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_N=0}^{k_{N-1}} 1 = \frac{1}{\prod_{k=0}^{N-1} m_k^{N-k}} \binom{n+N}{N} \\
&\geq \frac{n^N}{N! \prod_{k=0}^{N-1} m_k^{N-k}},
\end{aligned}$$

as claimed. To establish the upper bound, we use Theorem 1.5:

$$\begin{aligned}
\alpha_M(n) &= \sum_{k_1=0}^{\lfloor \frac{n}{m_0} \rfloor} \sum_{k_2=0}^{\lfloor \frac{k_1}{m_1} \rfloor} \cdots \sum_{k_N=0}^{\lfloor \frac{k_{N-1}}{m_{N-1}} \rfloor} 1 \le \sum_{k_1=0}^{\lfloor \frac{n}{m_0} \rfloor} \sum_{k_2=0}^{\lfloor \frac{n}{m_0 m_1} \rfloor} \cdots \sum_{k_N=0}^{\lfloor \frac{n}{m_0 m_1 \cdots m_{N-1}} \rfloor} 1 \\
&= \left(\left\lfloor \frac{n}{m_0} \right\rfloor + 1 \right) \left(\left\lfloor \frac{n}{m_0 m_1} \right\rfloor + 1 \right) \cdots \left(\left\lfloor \frac{n}{m_0 m_1 \cdots m_{N-1}} \right\rfloor + 1 \right) \\
&\le \left(\frac{n}{m_0} + 1 \right) \left(\frac{n}{m_0 m_1} + 1 \right) \cdots \left(\frac{n}{m_0 m_1 \cdots m_{N-1}} + 1 \right) \\
&\le \frac{(2n)^N}{\prod_{k=0}^{N-1} m_k^{N-k}},
\end{aligned}$$

as claimed. □

4. Applications to other partition functions

In this section, we demonstrate how many of our results for the M-sequence non-squashing partition function can be applied to interesting partition families. In Section 4.1, by new methods, we recover some asymptotic results pertaining to the m-ary and m-non-squashing partition functions, as well as establish some new results. In Section 4.2, we give new asymptotic results for the factorial partition function, which has not been well understood. In Section 4.3, we establish the asymptotics of general families of well characterized sequences M from Theorem 1.12, namely geometric, stabilizing, periodic, and arithmetic sequences. We also briefly discuss implications of Conjecture 1.10 for m-ary partitions, m-non-squashing partitions and the factorial partitions.

4.1. Application to m-non-squashing

Our generalization of m-non-squashing to M-sequence non-squashing can be reverted by setting all $m_j = m$ in the sequence $M = \{m_j\}_{j=0}^{\infty}$. By doing so, we can obtain some new results about m-ary partitions and recover others. First, we recover the generating function for m-non-squashing partitions from Theorem 1.7, as originally proved in [13].

Corollary 4.1 (*Corollary of* Theorem 1.7). *For any integer $m \ge 2$ and all non-negative integers n,*

$$\sum_{n=0}^{\infty} \alpha_m(n) q^n = \frac{1}{\prod_{i=0}^{\infty} \left(1 - q^{m^i}\right)}.$$

Our work also provides an elementary proof of the asymptotics of the m-ary (and m-non-squashing) partition functions from Theorem 1.12, thereby recovering Mahler's asymptotic [16]. We re-state Corollary 1.13 here for convenience.

Corollary (Corollary 1.13). *For all integers $m \ge 2$, as $n \to \infty$, we have that*

$$\ln \beta_m(n) \sim \frac{(\ln n)^2}{2 \ln m}, \qquad \ln \alpha_m(n) \sim \frac{(\ln n)^2}{2 \ln m}.$$

Proof. Letting $j = j_{m,n} := \max\{0, \lfloor \log_m n \rfloor\}$, and setting $m_i = m$ for all $i \ge 0$ in Theorem 1.12, we have that

$$\frac{n^j}{j! m^{j(j+1)/2}} \le \alpha_m(n) \le \frac{(2n)^j}{m^{j(j+1)/2}}.$$

Now, by taking the logarithm, using that $\log_m n - 1 < j \le \log_m n$, and applying Stirling's approximation (Lemma 2.3), we obtain the following lower bound:

$$\begin{aligned}
\ln \alpha_m(n) &\ge j \ln n - j \ln j + j - \frac{j(j+1)}{2} \ln m + O(\ln j) \\
&\ge (\log_m n - 1) \ln n - \log_m n \ln \log_m n + \log_m n - 1 - \frac{(\log_m n)(\log_m n + 1)}{2} \ln m + O(\ln \ln n) \\
&= \frac{(\ln n)^2}{2 \ln m} + O((\ln n)(\ln \ln n)).
\end{aligned}$$

Similarly, for the upper bound, we have that

$$\ln\alpha_m(n) \le j\ln 2 + j\ln n - \frac{j(j+1)}{2}\ln m \le \frac{(\ln n)^2}{2\ln m} + O(\ln n).$$

Therefore,

$$\ln\alpha_m(n) = \frac{(\ln n)^2}{2\ln m} + O((\ln n)(\ln\ln n)),$$

and the asymptotic for $\alpha_m(n)$ follows. The asymptotic stated for the m-ary partition function $\beta_m(n)$ now follows from (1.4). □

Finally, it is easy to see that Conjecture 1.10 is consistent with Theorem 1.2 due to Rødseth–Sellers [19], when each m_j is taken to be m. However, it should also be noted that Conjecture 4.2 is weaker than its Rødseth–Sellers counterpart, in that it is only equally strong when $r < d$, where d is the smallest factor of m that is greater than 2.

Conjecture 4.2. *For any integers $m \ge 2$ and $j \ge 1$, let*

$$\mu_j = \mu_{m,j} := \frac{m}{\gcd(m, P_j)}.$$

For all integers $r, n \ge 1$, $1 \le c \le m$, and σ defined as in (1.6), we have that

$$\alpha_m(m^{r+1}n - \sigma - c) \equiv 0 \bmod \mu_1\mu_2\cdots\mu_r.$$

4.2. Application to factorial partitions

The following proposition shows a relationship between factorial partitions and M-sequence non-squashing partitions.

Proposition 4.3. *Let $M_f = \{2, 3, 4, \ldots\}$. Then,*

$$f(n) = \alpha_{M_f}(n).$$

Proof. By the definition of factorial partitions, it is clear that

$$\sum_{n=0}^{\infty} f(n)q^n = \frac{1}{\prod_{j=1}^{\infty}(1-q^{j!})}. \tag{4.1}$$

By Theorem 1.7, we also have

$$\sum_{n=0}^{\infty} \alpha_{M_f}(n)q^n = \frac{1}{\prod_{j=1}^{\infty}(1-q^{j!})}. \tag{4.2}$$

Equating coefficients in (4.1) and (4.2) yields the result. □

Because factorial partitions arise as a special case of sequence non-squashing partitions, we can apply Theorem 1.12 to the factorial partition function, which will allow us to find its asymptotic growth.

Proof of Theorem 1.14. Let N be the integer such that $N! \le n < (N+1)!$. Then, by Lemma 2.4, we know that

$$N = \frac{\ln n}{\ln\ln n} + O\left(\frac{\ln n \ln\ln\ln n}{(\ln\ln n)^2}\right).$$

Now, by using Theorem 1.12 on the sequence $M_f = \{2, 3, 4, \ldots\}$, we find that

$$\frac{n^{N-1}}{(N-1)!G(N+2)} \le f(n) \le \frac{(2n)^{N-1}}{G(N+2)},$$

where $G(n) = \prod_{i=0}^{n-2} i!$ is the Barnes G-function. Using Stirling's approximation (Lemma 2.3) and the asymptotic for the Barnes G-function (Lemma 2.6), we find

$$\ln f(n) \le (N-1)\ln n - \ln G(N+2) + (N-1)\ln 2 = N\ln n - \ln G(N+2) + O(\ln n),$$

and

$$\begin{aligned}\ln f(n) &\geq (N-1)\ln n - \ln G(N+2) - (N-1)\ln(N-1) + N - 1 + O(\ln N)\\ &= N\ln n - \ln G(N+2) + O(\ln n).\end{aligned}$$

Therefore,

$$\begin{aligned}\ln f(n) &= N\ln n - \ln G(N+2) + O(\ln n)\\ &= N\ln n - \frac{(N+1)\ln(2\pi)}{2} - \frac{(N+1)^2\ln(N+1)}{2} + \frac{\ln(N+1)}{12} + \frac{3(N+1)^2}{4} + O(\ln n)\\ &= N\ln n - \frac{N^2}{2}\ln N + \frac{3N^2}{4} + O(\ln n)\\ &= \frac{(\ln n)^2}{2\ln\ln n} + O\left(\frac{(\ln n)^2(\ln\ln\ln n)}{(\ln\ln n)^2}\right),\end{aligned}$$

from which the result follows. □

Remark. As noted earlier, we can also apply Conjecture 1.10 to the factorial partition function, which leads to Conjecture 1.11. This follows directly from Conjecture 1.10 by setting $M = \{2, 3, 4, \ldots\}$ and the definition of μ_j in (1.6).

4.3. Asymptotics for special general sequences

In this section, using Theorem 1.12, we give the asymptotic growth of $\alpha_M(n)$ for multiple different general families of sequences M. The sequences we choose to illustrate here are meant to demonstrate the general method by which we can find exact asymptotics from Theorem 1.12 when we are able to characterize the sequence M even slightly. In particular, we examine geometric, stabilizing, periodic, and arithmetic sequences.

4.3.1. Geometric sequences

In this subsection, we consider the case when the sequence is a geometric sequence; $M = \{a, ab, ab^2, ab^3, \ldots\}$ $(a, b \in \mathbb{N})$.

Proposition 4.4. *Suppose $M = \{m_j\}_{j=0}^{\infty}$ is a sequence of integers and $a, b \geq 2$ are integers such that $m_j = ab^j$ for all $j \geq 0$. Then, as $n \to \infty$, we have that*

$$\ln \alpha_M(n) \sim \frac{2}{3}\sqrt{\frac{2}{\ln b}}\,(\ln n)^{3/2}.$$

Proof. First, we will find the asymptotic growth of the critical number N with respect to n. Note that for $n \geq a$,

$$m_0m_1\cdots m_{N-1} = a^N b^{T_{N-1}} \leq n < m_0m_1\cdots m_N = a^{N+1}b^{T_N},$$

where $T_n = \frac{n(n+1)}{2}$ is the nth triangular number.

Taking the logarithm of both sides and expanding gives us

$$\frac{\ln b}{2}N^2 + \left(\ln a - \frac{\ln b}{2}\right)N \leq \ln n < \frac{\ln b}{2}N^2 + \left(\ln a + \frac{\ln b}{2}\right)N + \ln a.$$

Solving these quadratics in N gives us the inequalities

$$N > \frac{-\ln a - \frac{\ln b}{2} + \sqrt{(\ln a)^2 + \left(\frac{\ln b}{2}\right)^2 - \ln a\ln b + 2\ln b\ln n}}{\ln b}$$

and

$$N \leq \frac{-\ln a + \frac{\ln b}{2} + \sqrt{(\ln a)^2 + \left(\frac{\ln b}{2}\right)^2 - \ln a\ln b + 2\ln b\ln n}}{\ln b}.$$

Thus,

$$N = \sqrt{\frac{2\ln n}{\ln b}} + O(1). \tag{4.3}$$

Now, we apply Theorem 1.12 and find that

$$\frac{n^N}{N!\,a^{1/2(N^2+N)}b^{1/6(N^3-N)}} \leq \alpha_M(n) \leq \frac{(2n)^N}{a^{1/2(N^2+N)}b^{1/6(N^3-N)}}.$$

By using Stirling's Approximation (Lemma 2.3), for the lower bound we have that

$$\ln \alpha_M(n) \geq N \ln n - N \ln N + N - \frac{N^2+N}{2} \ln a - \frac{N^3-N}{6} \ln b + O(\ln N)$$
$$= N \ln n - \frac{N^2+N}{2} \ln a - \frac{N^3-N}{6} \ln b + O(N \ln N),$$

and for the upper bound we have that

$$\ln \alpha_M(n) \leq N \ln n + N \ln 2 - \frac{N^2+N}{2} \ln a - \frac{N^3-N}{6} \ln b + O(\ln N)$$
$$= N \ln n - \frac{N^2+N}{2} \ln a - \frac{N^3-N}{6} \ln b + O(\ln N).$$

Then, using (4.3), we find that

$$\ln \alpha_M(n) = N \ln n - \frac{N^2+N}{2} \ln a - \frac{N^3-N}{6} \ln b + O(N \ln N)$$
$$= \frac{2}{3}\sqrt{\frac{2}{\ln b}}\,(\ln n)^{3/2} + O(\ln n).$$

Hence,

$$\ln \alpha_M(n) \sim \frac{2}{3}\sqrt{\frac{2}{\ln b}}\,(\ln n)^{3/2}$$

as claimed. □

Remark 4.5. As pointed out by the referee, it is also possible to prove Proposition 4.4 by an alternative method, which avoids solving the quadratics in N as in the above proof.

4.3.2. Stabilizing sequences

In this subsection, we consider the case when the sequence M eventually stabilizes to a constant value. For example, $\{2, 3, 3, 3, 3, \ldots\}$ is one such sequence, and $\{3, 6, 5, 8, 2, 7, 7, 7, 7, 7, \ldots\}$ is another.

Proposition 4.6. *Let $M = \{m_j\}_{j=0}^{\infty}$ be a sequence of integers with all $m_j \geq 2$. Suppose there exists some integer $K \geq 0$ such that for all $j > K$, $m_j = m_K = m$. Then, as $n \to \infty$,*

$$\ln \alpha_M(n) \sim \frac{(\ln n)^2}{2 \ln m}.$$

Proof. If $n \geq m_0 m_1 \cdots m_K$ and N is the critical number for n, then

$$m_0 m_1 \cdots m_K m^{N-K-1} \leq n < m_0 m_1 \cdots m_K m^{N-K}.$$

Hence, by taking the logarithm, we find that

$$\ln(m_0 m_1 \cdots m_K) + (N-K-1)\ln m \leq \ln n < \ln(m_0 m_1 \cdots m_K) + (N-K)\ln m.$$

Therefore,

$$N = \frac{\ln n}{\ln m} + O(1).$$

Now, we can use Theorem 1.12 as in the proof of Corollary 1.13 to obtain the desired result. □

4.3.3. Periodic sequence of weight constants

In this subsection, we consider the case when the sequence of weight constants, M, is a periodic sequence of the form

$$M = \{m_0, m_1, \ldots, m_{T-1}, m_0, m_1, \ldots, m_{T-1}, m_0, m_1, \ldots\},$$

where T is the period of the sequence. Then we have the following corollary to Theorem 1.12.

Proposition 4.7. *Let $M = \{m_j\}_{j=0}^{\infty}$ be a periodic sequence of integers with all $m_j \geq 2$ and period T. Let $\mathcal{M} = m_0 m_1 \cdots m_{T-1}$. Then, as $n \to \infty$,*

$$\ln \alpha_M(n) \sim \frac{T(\ln n)^2}{2 \ln \mathcal{M}}.$$

Proof. By Theorem 1.12, we know that

$$\frac{n^N}{N!\prod_{k=0}^{N-1} m_k^{N-k}} \le \alpha_M(n) \le \frac{(2n)^N}{\prod_{k=0}^{N-1} m_k^{N-k}}, \tag{4.4}$$

where $N = N(n, M)$. By the division algorithm, we can let $N = Tp + r$ for some integers p, r such that $0 \le r \le T - 1$. Then, we can rewrite (4.4) as

$$\frac{n^N}{N!\prod_{k=0}^{r-1} m_k^{\sum_{i=0}^{p}(Ti+r-k)} \prod_{k=r}^{T-1} m_k^{\sum_{i=1}^{p}(Ti+r-k)}} \le \alpha_M(n)$$

$$\le \frac{(2n)^N}{\prod_{k=0}^{r-1} m_k^{\sum_{i=0}^{p}(Ti+r-k)} \prod_{k=r}^{T-1} m_k^{\sum_{i=1}^{p}(Ti+r-k)}}.$$

This is the same as

$$\frac{n^N}{N!\prod_{k=0}^{r-1} m_k^{\frac{1}{2}(p+1)(Tp+2r-2k)} \prod_{k=r}^{T-1} m_k^{\frac{1}{2}(p)(T(p+1)+2r-2k)}} \le \alpha_M(n)$$

$$\le \frac{(2n)^N}{\prod_{k=0}^{r-1} m_k^{\frac{1}{2}(p+1)(Tp+2r-2k)} \prod_{k=r}^{T-1} m_k^{\frac{1}{2}(p)(T(p+1)+2r-2k)}}.$$

Note that $p \le \log_{\mathcal{M}} n < p + 1$. Using this along with Stirling's Approximation (Lemma 2.3), for the lower bound, we have that

$$\begin{aligned}
\ln \alpha_M(n) &\ge N \ln n - N \ln N + N \\
&\quad - \frac{1}{2}\left(\sum_{k=0}^{r-1}(p+1)(Tp+2r-2k)\ln m_k + \sum_{k=r}^{T-1} p(T(p+1)+2r-2k)\ln m_k\right) + O(\ln N) \\
&\ge \frac{T(\ln n)^2}{\ln \mathcal{M}} - \frac{T\ln n}{\ln \mathcal{M}} \ln\left(\frac{T\ln n}{\ln \mathcal{M}}\right) - \frac{1}{2}(T(p+1)+2r)(p+1)\ln \mathcal{M} + O(\ln n) \\
&\ge \frac{T(\ln n)^2}{\ln \mathcal{M}} - \frac{T(\ln n)^2}{2\ln \mathcal{M}} + O((\ln n)(\ln\ln n)) \\
&\ge \frac{T(\ln n)^2}{2\ln \mathcal{M}} + O((\ln n)(\ln\ln n)).
\end{aligned}$$

For the upper bound,

$$\begin{aligned}
\ln \alpha_M(n) &\le N \ln n + N \ln 2 \\
&\quad - \frac{1}{2}\left(\sum_{k=0}^{r-1}(p+1)(Tp+2r-2k)\ln m_k + \sum_{k=r}^{T-1} (p)((T+1)p+2r-2k)\ln m_k\right) \\
&\le \frac{T(\ln n)^2}{\ln \mathcal{M}} - \frac{1}{2}(Tp+2r-2T+2)(p)\ln \mathcal{M} + O(\ln n) \\
&\le \frac{T(\ln n)^2}{\ln \mathcal{M}} - \frac{T(\ln n)^2}{2\ln \mathcal{M}} + O(\ln n) \\
&\le \frac{T(\ln n)^2}{2\ln \mathcal{M}} + O(\ln n).
\end{aligned}$$

Combining the upper and lower bounds for $\ln \alpha_M(n)$ above we find that

$$\ln \alpha_M(n) = \frac{T(\ln n)^2}{2\ln \mathcal{M}} + O((\ln n)(\ln\ln n)),$$

which completes the proof. □

Interestingly, we can rewrite the asymptotic for periodic sequences in Proposition 4.7 as

$$\ln \alpha_M(n) = \frac{(\ln n)^2}{2 \ln \mathcal{M}_G},$$

where $\mathcal{M}_G$ is the geometric average of $m_0, m_1, \ldots, m_{T-1}$. In this form, it looks very similar to the asymptotic for the m-ary (m-non-squashing) partition function given by Mahler in (1.3), as we also recover in Corollary 1.13.

4.3.4. Arithmetic sequences

In this subsection, we consider the case when the sequence of weight constants, M, is an arithmetic sequence.

Proposition 4.8. *Suppose $M = \{m_j\}_{j=0}^{\infty}$ is a sequence of integers and a and b are integers with $a \geq 2$ and $b \geq 1$ such that $m_j = a + bj$ for each $j \geq 0$. Then, as $n \to \infty$, we have that*

$$\ln \alpha_M(n) \sim \frac{(\ln n)^2}{2 \ln \ln n}.$$

Proof. Let $k = \left\lfloor \frac{a}{b} \right\rfloor$. Then, we have that $kb \leq a < (k+1)b$. Choose $n \geq a$ and let N be the critical number with respect to n. Then,

$$\begin{aligned} &kb \cdot (k+1)b \cdot \cdots \cdot (k+N-1)b \leq a(a+b)\cdots(a+(N-1)b) \leq n \\ &\quad < a(a+b)\cdots(a+Nb) \leq (k+1)b \cdot (k+2)b \cdot \cdots \cdot (k+N+1)b. \end{aligned}$$

Hence, we have that

$$\begin{aligned} n &\geq \frac{(k+N-1)!}{(k-1)!} b^N, \quad \text{hence} \\ \ln n &\geq (k+N-1)\ln(k+N-1) - (k+N-1) + N \ln b + O(\ln N) \\ &\geq N \ln N + (\ln b - 1)N + O(\ln N). \end{aligned}$$

We also have that

$$\begin{aligned} n &< \frac{(k+N+1)!}{k!} b^{N+1}, \quad \text{hence} \\ \ln n &< (k+N+1)\ln(k+N+1) - (k+N+1) + (N+1)\ln b + O(\ln N) \\ &\leq N \ln N + (\ln b - 1)N + O(\ln N). \end{aligned}$$

Therefore, $\ln n = N \ln N + N(\ln b - 1) + O(\ln N)$. Hence, by Lemma 2.4, we have that

$$N = \frac{\ln n}{\ln \ln n} + O\left(\frac{(\ln n)(\ln \ln \ln n)}{(\ln \ln n)^2}\right). \tag{4.5}$$

Now, by Theorem 1.12, we have that

$$\frac{n^N}{N! \prod_{j=0}^{N-1} m_j^{N-j}} \leq \alpha_M(n) \leq \frac{(2n)^N}{\prod_{j=0}^{N-1} m_j^{N-j}}.$$

For the lower bound, we have that

$$\alpha_M(n) \geq \frac{n^N}{N! \prod_{j=0}^{N-1} (a+bj)^{N-j}} \geq \frac{n^N}{N! \prod_{j=0}^{N-1} ((k+j+1)b)^{N-j}} = \frac{n^N (k!)^N G(k+2)}{N! b^{T_N} G(k+N+2)},$$

where $G(n)$ is the Barnes G-function (Definition 2.5). For the upper bound, we have that

$$\alpha_M(n) \leq \frac{(2n)^N}{\prod_{j=0}^{N-1} (a+bj)^{N-j}} \leq \frac{(2n)^N}{\prod_{j=0}^{N-1} ((k+j)b)^{N-j}} = \frac{(2n)^N ((k-1)!)^N G(k+1)}{b^{T_N} G(k+N+1)}.$$

Now, by Lemma 2.6 and Stirling's Approximation (Lemma 2.3), we have that

$$\begin{aligned} \ln \alpha_M(n) &\geq N \ln n + N \ln(k!) + \ln G(k+2) - \ln(N!) - \frac{N(N+1)}{2} \ln b - \ln G(k+N+2) \\ &= N \ln n - \frac{N^2}{2} \ln b - \frac{N^2}{2} \ln N + \frac{3N^2}{4} + O(\ln n), \end{aligned}$$

and also

$$\ln \alpha_M(n) \le N \ln n + N \ln(k!) + \ln G(k+1) - \ln(N!) - \frac{N(N+1)}{2} \ln b - \ln G(k+N+1)$$
$$= N \ln n - \frac{N^2}{2} \ln b - \frac{N^2}{2} \ln N + \frac{3N^2}{4} + O(\ln n).$$

Therefore, by (4.5),

$$\ln \alpha_M(n) = N \ln n - \frac{N^2}{2} \ln b - \frac{N^2}{2} \ln N + \frac{3N^2}{4} + O(\ln n)$$
$$= \frac{(\ln n)^2}{\ln \ln n} - \frac{(\ln n)^2}{2 \ln \ln n} + O\left(\frac{(\ln n)^2 (\ln \ln \ln n)}{(\ln \ln n)^2}\right)$$
$$= \frac{(\ln n)^2}{2 \ln \ln n} + O\left(\frac{(\ln n)^2 (\ln \ln \ln n)}{(\ln \ln n)^2}\right),$$

which completes the proof. □

5. Sequence box stacking

In this section, we extend the concept of sequence non-squashing partitions (see Definition 1.4) to the box stacking problem studied by Andrews, Sellers, and Sloane [5,20] (as described in Section 1.2.4). In Section 5.1, we present the generating function for this more general problem. In Section 5.2, we show how the sequence box stacking problem and M-sequence non-squashing partitions are related. Finally, in Section 5.3, we demonstrate a method for finding asymptotics for the sequence box stacking problem, and give a congruence conjecture.

5.1. Generating function

We can use the Omega operator (see Section 2.1) to derive a generating function for sequence box stacking. Note that the generating function is only valid for weight constants each at least 3. Throughout, we assume the definitions and notations as given in Section 1.2.4.

Theorem 5.1. *For a sequence of integers $M = \{m_j\}_{j=0}^{\infty}$ with each $m_j \ge 3$, the generating function for $a_M(n)$ is*

$$\sum_{n=0}^{\infty} a_M(n) q^n = \frac{1}{(1-q)^2 \left(1 - q^{m_0 - 1}\right) \prod_{h=1}^{\infty} \left(1 - q^{(m_0-1) \prod_{\ell=1}^{h} m_\ell}\right)}.$$

In the special case $m_j = m$ for each integer $j \ge 0$, we recover from Theorem 5.1 the generating function for $a_m(n)$ as originally proved by Andrews and Sellers in [5]. We prove Theorem 5.1 using methods similar to their methods.

Corollary 5.2. *For integers $m \ge 3$, we have that*

$$\sum_{n=0}^{\infty} a_m(n) q^n = \frac{1}{(1-q)^2 \prod_{h=0}^{\infty} \left(1 - q^{(m-1)m^h}\right)}.$$

Proof of Theorem 5.1. By Definition 1.15, the generating function of $a_M(n)$ is

$$\sum_{n=0}^{\infty} a_M(n) q^n = \lim_{k \to \infty} F_{M,k}(q),$$

where

$$F_{M,k}(q) := \underset{\geqslant}{\Omega} \sum_{n=0}^{\infty} q^n \sum_{p_1, p_2, \ldots, p_k \ge 0} \lambda_{k+1}^{n-p_k} \prod_{\ell=2}^{k} \lambda_\ell^{p_\ell - (m_{k-\ell}-1)(p_1+p_2+\cdots+p_{\ell-1})},$$

and $\underset{\geqslant}{\Omega}$ is the Omega operator as defined in Definition 2.1. Now, we can rewrite $F_{M,k}(q)$ as the Omega operator applied to a product of geometric series:

$$F_{M,k}(q) = \underset{\geqslant}{\Omega} \frac{1}{(1-q\lambda_{k+1})\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\left[\prod_{\ell=2}^{k-1}\left(1-\frac{\lambda_\ell}{\prod_{h=\ell+1}^{k}\lambda_h^{m_{k-h}-1}}\right)\right]\left(1-\frac{1}{\prod_{h=2}^{k}\lambda_h^{m_{h-k}-1}}\right)}.$$

We now proceed by applying the Omega operator on $\lambda_{k+1}, \lambda_k, \ldots, \lambda_2$, in that order. Proceeding as in our first proof of Theorem 1.7 (in Section 3.3), we find that

$$F_{M,k}(q) = \frac{1}{(1-q)^2\left(1-q^{m_0-1}\right)\prod_{h=1}^{k-2}\left(1-q^{(m_0-1)\prod_{\ell=1}^{h}m_\ell}\right)}.$$

Now, by taking the limit as k goes to infinity, we find that

$$\sum_{n=0}^{\infty} a_M(n)q^n = \lim_{k\to\infty} F_{M,k}(q) = \frac{1}{(1-q)^2\left(1-q^{m_0-1}\right)\prod_{h=1}^{\infty}\left(1-q^{(m_0-1)\prod_{\ell=1}^{h}m_\ell}\right)},$$

as desired. □

5.2. Relationship with non-squashing partitions

We now relate the sequence box stacking problem back to sequence non-squashing partitions (see Definition 1.4), and prove Theorem 1.16. Using the generating functions from Theorems 3.3 and 5.1, the proof is immediate; however, we also offer a combinatorial proof here for the interested reader.

Proof of Theorem 1.16. For all integers $n \geq 0$ and sequences of integers $M = \{m_j\}_{j=0}^{\infty}$ with each $m_j \geq 3$, define

$$A_{n,M} = A_n := \{M\text{-sequence non-squashing partitions of } k : k = 0, \ldots, n\},$$
$$B_{n,M} = B_n := \{M^+\text{-sequence box stacking partitions from } n \text{ boxes}\}.$$

We will show that for all n, $|A_n| = |B_n|$ by giving injective maps between the two sets. First, let $\varphi : A_n \to B_n$ be defined by $\varphi(\{p_1, \ldots, p_k\}) = \{p_1, \ldots, p_{k-1}, p_1 + \cdots + p_k\}$. The map φ indeed maps to B_n because

(i) $p_1 < p_2 < \cdots < p_{k-1} < p_1 + \cdots + p_k$, since $m_j \geq 3$ for all $j \geq 0$,
(ii) $p_1 + \cdots + p_k \leq n$, since $\{p_1, \ldots, p_k\} \in A_n$,
(iii) $(m_{k-j} - 1)(p_1 + \cdots + p_{j-1}) \leq p_j$ for $2 \leq j \leq k-1$, by the non-squashing condition (3) in Definition 1.4,
(iv) $m_0(p_1 + \cdots + p_{k-1}) \leq p_1 + \cdots + p_k$ since $(m_0 - 1)(p_1 + \cdots + p_{k-1}) \leq p_k$, by the non-squashing condition (3) in Definition 1.4.

It is easy to see by its definition that the map φ is injective. Now let $\tau : B_n \to A_n$ be defined by $\tau(\{p_1, \ldots, p_k\}) = \{p_1, \ldots, p_{k-1}, p_k - (p_1 + \cdots + p_{k-1})\}$. The map τ indeed maps to A_n because

(i) $0 \leq p_1 + \cdots + p_{k-1} + p_k - (p_1 + \cdots + p_{k-1}) = p_k \leq n$,
(ii) $p_1 \leq \cdots \leq p_{k-1} \leq p_k - (p_1 + \cdots + p_{k-1})$, since each $m_j \geq 3$,
(iii) $(m_{k-j} - 1)(p_1 + \cdots + p_{j-1}) \leq p_j$ for $2 \leq j \leq k-1$, by the box stacking condition (2) in Definition 1.15,
(iv) $(m_0 - 1)(p_1 + \cdots + p_{k-1}) \leq p_k$ since $m_0(p_1 + \cdots + p_{k-1}) \leq p_1 + \cdots + p_k$, by the box stacking condition (2) in Definition 1.15.

It is easy to see that τ is injective by its definition. Since we have established two injective maps (φ and τ) between A_n and B_n, we have that $|A_n| = |B_n|$. By Proposition 3.4, we have that for all integers $n \geq 0$ and $c \geq 2$, $|A_n| = \alpha_{M^c}(n)$, hence the desired result follows. □

5.3. Sequence box stacking asymptotic and congruence properties

Using Theorem 1.16, we obtain asymptotic results similar to Theorem 1.12. We also give a congruence conjecture.

Corollary 5.3. *Let $M = \{m_0 + 1, m_1, m_2, \dots\}$ be a sequence of integers with $m_j \geq 2$ for each j. Let $n > 0$ be an integer, and let N be the integer such that $m_0m_1 \cdots m_{N-2} \leq n < m_2m_1 \cdots m_{N-1}$ if $n \geq m_0$, and 1 otherwise. Then we have that*

$$\frac{n^N}{N! \prod_{k=0}^{N-2} m_k^{N-1-k}} \leq a_M(n) \leq \frac{(2n)^N}{\prod_{k=0}^{N-2} m_k^{N-1-k}}.$$

Proof. In Theorem 1.12, let $n \to cn$ and $M \to M^c = \{c, m_0, m_1, m_2, \dots\}$ for some integer $c \geq 2$. Then we have that

$$\frac{(cn)^N}{N!c^N \prod_{k=0}^{N-2} m_k^{N-1-k}} \leq \alpha_{M^c}(cn) \leq \frac{(2cn)^N}{c^N \prod_{k=0}^{N-2} m_k^{N-1-k}},$$

where N is the number such that if $n \geq m_0$, $cm_0m_1 \dots m_{N-2} \leq cn < cm_0m_1 \dots m_{N-1}$, or equivalently, $m_0m_1 \cdots m_{N-2} \leq n < m_2m_1 \cdots m_{N-1}$, and $N = 1$ otherwise. Canceling the c^j in both the numerator and denominator of the two sides of the inequality and using the identity from Theorem 1.16 gives the result. □

Conjecture 5.4. *Assume the notation as above. Let $M = \{m_0 + 1, m_1, m_2, \dots\}$ be any sequence of integers with $m_j \geq 2$ for each $j \in \mathbb{N}_0$, and let $\widetilde{M} := \{m_0, m_1, m_2, \dots\}$. Define $\widetilde{\sigma} = \widetilde{\sigma}_{r,M} := \epsilon_0 m_0 + \sigma_{r,\widetilde{M}}(\boldsymbol{\epsilon})$, where $\epsilon_j \in \{0, 1\}$, for each $j \in \mathbb{N}_0$. Then, for all integers $r, n \geq 1$, we have that*

$$a_M(nm_0m_1 \cdots m_r - \widetilde{\sigma} - 1) \equiv 0 \bmod \mu_0\mu_1\mu_2 \cdots \mu_r.$$

Remark. Conjecture 5.4 follows from Conjecture 1.10. To see this, in Conjecture 1.10, let $r \to r + 1$ and $M \to M^c = \{c, m_0, m_1, m_2, \dots\}$ for some integer $c \geq 2$. Then,

$$\alpha_{M^c}(cnm_0m_1 \cdots m_r - c\widetilde{\sigma} - c) \equiv 0 \bmod \mu_0\mu_1\mu_2 \cdots \mu_r.$$

The result now follows from Theorem 1.16.

6. Asymptotics for colored *m*-ary partitions

In this section, we prove another asymptotic result (Theorem 1.18) on colored m-ary partitions as described in Section 1.2.5 that in a special case also recovers the m-ary asymptotic of Mahler (1.3) (see also our Corollary 1.13) by different (elementary) means. In particular, we generalize methods of Knuth [15], who proved an asymptotic for the binary partition function. In Section 6.1, we provide some lemmas that we use to prove Theorem 1.18 (in addition to the tools from Section 2.3), which we do in Section 6.2.

6.1. Preliminaries

The following is a lemma that we will use to prove our main asymptotic theorem.

Lemma 6.1. *For all integers $r \geq 0$ $i \geq 1$ and $n \geq 1$, let*

$$s_i^r := \sum_{j=1}^{i} j^r \quad \text{and} \quad t_i^r = t_i^r(n) := \sum_{j=1}^{i} (n - j + 1)^r.$$

Then, for any fixed integer $m \geq 2$, as $n \to \infty$, we have

$$s_n^r m^n + s_{n-1}^r m^{n-1} + \cdots + s_1^r m = O(n^{r+1}m^n) \tag{6.1}$$

$$t_1^r m^{n-1} + t_2^r m^{n-2} + \cdots + t_{n-1}^r m + t_n^r = O(n^r m^n). \tag{6.2}$$

Proof. We first prove (6.1). By Bernoulli's Formula (Lemma 2.7),

$$s_i^r = \frac{1}{r+1} \sum_{k=0}^{r} \binom{r+1}{k} B_k i^{r+1-k}.$$

Thus,

$$s_n^r m^n + s_{n-1}^r m^{n-1} + \cdots + s_1^r m = O(n^{r+1})(m^n + m^{n-1} + \cdots + m) = O(n^{r+1}m^n),$$

as desired.

To prove (6.2), first note that $t_1^r m^n = O(n^r m^n)$, $t_n^r = O(n^{r+1})$, and

$$t_2^r m^{n-2} + \cdots + t_{n-1}^r m \le (n-1)^r (m^{n-1} + \cdots + m) = O(n^r m^{n-1}).$$

Therefore,

$$t_1^r m^{n-1} + t_2^r m^{n-2} + \cdots + t_{n-1}^r m + t_n^r = O(n^r m^n) + O(n^{r+1}) + O(n^r m^{n-1}) = O(n^r m^n). \quad \square$$

6.2. Proof of Theorem 1.18

We are now ready to prove Theorem 1.18.

Proof of Theorem 1.18. First, by the definition of P-c-colored m-ary partitions, we obtain the following generating function:

$$F(q) = \sum_{n=0}^{\infty} \beta_m^{P,c}(n) q^n = \frac{1}{(1-q)^c \prod_{k=1}^{\infty} (1-q^{m^k})^{P(k)}}.$$

Now, by taking the logarithm and expanding the power series, we find that

$$\ln F(q) = -c \ln(1-q) - \sum_{k=1}^{\infty} P(k) \ln\left(1 - q^{m^k}\right) = c \sum_{r=1}^{\infty} \frac{q^r}{r} + \sum_{k=1}^{\infty} P(k) \sum_{r=1}^{\infty} \frac{q^{m^k r}}{r},$$

$$\frac{F'(q)}{F(q)} = c \sum_{r=1}^{\infty} q^{r-1} + \sum_{k=1}^{\infty} P(k) \sum_{r=1}^{\infty} m^k q^{m^k r - 1} = \sum_{k=1}^{\infty} \theta_k q^{k-1},$$

where if p is the largest nonnegative integer such that $m^p | k$, then

$$\theta_k = \begin{cases} c + mP(1) + m^2 P(2) + \cdots + m^p P(p) & \text{if } p > 0 \\ c & \text{if } p = 0. \end{cases}$$

Next,

$$\frac{F'(q)}{F(q)} = (1-q) \sum_{k=1}^{\infty} \psi_k q^{k-1},$$

where

$$\psi_k := \theta_1 + \theta_2 + \cdots + \theta_k.$$

Then, let $k = \sum_{i=0}^{r} b_i m^i$, where $0 \le b_i \le m-1$ for $0 \le i \le r$, that is, let $b_r b_{r-1} \ldots b_0$ be the base m representation of k. In this way, $b_0, b_1, \ldots, b_r$ are uniquely determined. Now we have that

$$\begin{aligned} \psi_k &= ck + mP(1) \cdot |\{n \le k : m|n\}| + m^2 P(2) \cdot |\{n \le k : m^2|n\}| + \cdots + m^r P(r) \cdot |\{n \le k : m^r|n\}| \\ &= ck + mP(1) \left\lfloor \frac{k}{m} \right\rfloor + m^2 P(2) \left\lfloor \frac{k}{m^2} \right\rfloor + \cdots + m^r P(r) \left\lfloor \frac{k}{m^r} \right\rfloor \\ &= ck + mP(1) \sum_{i=1}^{r} b_i m^{i-1} + m^2 P(2) \sum_{i=2}^{r} b_i m^{i-2} + \cdots + m^r P(r) \sum_{i=r}^{r} b_i m^{i-r} \\ &= ck + \sum_{i=1}^{r} Q(i) b_i m^i, \end{aligned}$$

where $Q(n) = P(1) + P(2) + \cdots + P(n)$ for positive integers n. Now, we find that an upper bound for ψ_k is

$$\psi_k \le ck + Q(r)k = O(k) + \frac{a}{\ell - 1} k (\log_m k)^{\ell - 1} + O(k (\log_m k)^{\ell - 2}),$$

by applying Bernoulli's Formula (Lemma 2.7) and the fact that $r \le \log_m(k) \le r + 1$. Let $R(n) = P(r) + P(r-1) + \cdots + P(r - n + 1)$. Then, using Lemma 6.1, Bernoulli's Formula, and the fact that $r \ge \log_m k - 1$, we find that a lower bound for

ψ_k is

$$\begin{aligned}\psi_k &\geq kQ(r) - (R(1)m^{r-1} + R(2)m^{r-2} + \cdots + R(r))(m-1)\\ &= \frac{a}{\ell-1}k(\log_m k)^{\ell-1} + O(k(\log_m k)^{\ell-2}) + O(r^{\ell-2}m^r) + O(r^{\ell-1})\\ &\geq \frac{a}{\ell-1}k(\log_m k)^{\ell-1} + O(k(\log_m k)^{\ell-2}) + O(k(\log_m k)^{\ell-2}) + O((\log_m k)^{\ell-1})\\ &= \frac{a}{\ell-1}k(\log_m k)^{\ell-1} + O(k(\log_m k)^{\ell-2}).\end{aligned}$$

Therefore, we may conclude that

$$\psi_k = \frac{a}{\ell-1}k(\log_m k)^{\ell-1} + O(k(\log_m k)^{\ell-2}).$$

Now let $G(q) := \exp\left(\frac{a\ln^{\ell}(1-q)}{\ell(\ell-1)\ln^{\ell-1}m}\right)$. Then,

$$\begin{aligned}\ln G(q) &= \frac{a\ln^{\ell}(1-q)}{\ell(\ell-1)\ln^{\ell-1}m},\\ \frac{G'(q)}{G(q)} &= \frac{-a\ln^{\ell-1}(1-q)}{(\ell-1)(\ln^{\ell-1}m)(1-q)} = (1-q)\sum_{k=1}^{\infty}\chi_k q^{k-1},\end{aligned}$$

where

$$\chi_k = \frac{ak}{(\ell-1)\ln^{\ell-1}m}P_{\ell-1}(H_k^{(1)}-1, H_k^{(2)}-1, \ldots, H_k^{(\ell-1)}-1),$$

where $H_k^{(s)} = \sum_{j=1}^{k} j^{-s}$, $P_t(x_1, x_2, \ldots, x_t) = (-1)^t Y_t(-0!x_1, -1!x_2, \ldots, -(t-1)!x^t)$, and Y_t is the Bell polynomial as found in Zave [22]. Since $H_n^{(1)} = \ln n + O(1)$ and $H_n^{(s)} \leq H_n^{(2)} \leq \frac{\pi^2}{6}$ for all $s \geq 2$, we have

$$\chi_k = \frac{a}{\ell-1}k(\log_m k)^{\ell-1} + O(k(\log_m k)^{\ell-2}).$$

Hence, $\psi_k = \chi_k + O(k(\log_m k)^{\ell-2})$. Now for any given $\epsilon > 0$, there exists N such that for all $n > N$, $(1-\epsilon)\chi_n < \psi_n < (1+\epsilon)\chi_n$. Define

$$\begin{aligned}A(q) &= \exp\int_0^q (1-t)a(t)dt,\\ B(q) &= \exp\int_0^q (1-t)b(t)dt.\end{aligned}$$

First, we let $A(q) = F(q)$, (i.e. $a(q) = \sum_{k=1}^{\infty}\psi_k q^{k-1}$) and

$$b(q) = \sum_{k=1}^{N}\psi_k q^{k-1} + \sum_{k=N+1}^{\infty}(1+\epsilon)\chi_k q^{k-1}.$$

Note that $a(q) \leq b(q)$. Thus, by Lemma 2.12, $A(q) \leq B(q)$. Then, by Theorem 2.9 and Corollary 2.11, $\beta_m^{P,c}(n) \leq C_n$, where

$$\ln C_n \sim \frac{(1+\epsilon)(a)(\ln^{\ell} n)}{\ell(\ell-1)\ln^{\ell-1}m} + O((\ln^{\ell-1} n)(\ln\ln n)).$$

Now we apply Lemma 2.12 again with $A(q) = F(q)$ and

$$b(q) = \sum_{k=1}^{N}\psi_k q^{k-1} + \sum_{k=N+1}^{\infty}(1-\epsilon)\chi_k q^{k-1}.$$

In this case, $a(q)$ is the same as above, and now $a(q) \geq b(q)$. By Theorem 2.9 and Corollary 2.11, $\beta_m^{P,c}(n) \geq C_n'$, where

$$\ln C_n' \sim \frac{(1-\epsilon)(a)(\ln^{\ell} n)}{\ell(\ell-1)\ln^{\ell-1}m} + O((\ln^{\ell-1} n)(\ln\ln n)).$$

Thus,

$$\left|\frac{\ln \beta_m^{P,c}(n)}{\ln^\ell n} - \frac{a}{\ell(\ell-1)\ln^{\ell-1} m}\right|$$

is arbitrarily small for large enough n. The general asymptotic result stated in Theorem 1.18 now follows.

By setting $\ell = 2$, $c = 1$, and $P(x) = 1$, we can recover Mahler's asymptotic for m-ary partitions (1.3) by this different method (see also our Corollary 1.13). Similarly, by setting $\ell = 3$, $c = 1$, and $P(x) = x$, we recover the asymptotic given for the n-color m-ary partition function $\beta_m^{x,1}(n)$ as stated in Theorem 1.18. □

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