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k-Almost consecutive partitions and quantum Jacobi forms

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Abstract

We define the notion of k-almost consecutive partitions, and study associated combinatorial and modular aspects. We establish quantum Jacobi properties of their corresponding two-variable partition generating functions. Further, we provide related asymptotics, formulas, and combinatorial identities, and make connections to Ramanujan's third order mock-theta function $\psi(q)$ and Cohen's $\sigma^*(q)$. We conclude by recording a proof of a related conjecture of Xiong.

Keywords: Partitions, Quantum modular forms, Jacobi forms, q-series

Mathematics Subject Classification: Primary 05A17, 11F37, 11F50; Secondary 11F03, 11F30, 33D70

1 Introduction and statement of results

Partitions into distinct parts, and distinct consecutive parts, have been studied for centuries by the likes of Leonhard Euler, J.J. Sylvester, and others. We present a bridge between these two types of partitions in the form of *k-almost consecutive partitions*, which we define to be partitions with strictly greater than k distinct parts, all of which are consecutive with the possible exception of the k smallest parts. We will interchangeably refer to k-almost consecutive partitions as having *k-almost consecutive parts*. For a fixed $k \in \mathbb{N}_0$, we define

$$p_{ka}(n) := p(n\colon k\text{-almost consecutive parts}),$$

where $p(n) := \#\{\text{partitions of } n\}$ is the ordinary partition function. (In particular, $p_{ka}(0) = 0$.)

Example 1 For $k = 0$ and $n \in \mathbb{N}$, we have that $p_{0a}(n) = p_c(n)$, where the consecutive parts partition function is defined by $p_c(n) := p(n\colon \text{consecutive parts})$. Consequently, 2025, $5+4+3$, and $60+59+58+57+56$ are all examples of 0-almost consecutive partitions.

Example 2 For $k = 1$ and $n \in \mathbb{N}$, we have that $p_{1a}(n) = p_a(n) - 1$ in the notation of [18] whose definition of almost consecutive partitions we extend (to $k > 1$). Further discussion of results from [18] in the context of the results of this paper is given in the sections that follow; in particular, see Remark 3.

Example 3 For $k = 3$, 3-almost consecutive partitions allow a consecutive segment with up to 3 lesser distinct parts following (if the parts of the partition are arranged in non-increasing order as usual). The partitions $90 + 54 + 32 + 2$, $10 + 9 + 8 + 5 + 4 + 2$, and $6 + 5 + 4 + 3 + 2 + 1$ are all examples of 3-almost consecutive partitions.

Example 4 For fixed $k \geq 0$, all distinct parts partitions with $k + 1$ parts are k-almost consecutive.

In what follows, we will more generally study $p_{ka}(m, n) := p_{ka}(n\colon \text{largest part } m)$, and its corresponding two-variable generating function

$$P_{ka}(w; q) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{ka}(m, n) w^m q^n,$$

also noting that $P_{ka}(1; q) = \sum_{n=0}^{\infty} p_{ka}(n) q^n =: P_{ka}(q)$. We also extend several of our results (for integers $k \geq 0$) to the setting of negative integers $k < 0$, as found below and further discussed in Section 2.2.

Our results on k-almost consecutive partitions include connections to the theory of modular forms, combinatorial identities, exact formulas, asymptotic behaviors, and more. To motivate the first of these, we recall the generating function for partitions into distinct parts

$$P_d(q) = \sum_{n=0}^{\infty} p_d(n) q^n = \prod_{n=1}^{\infty} (1 + q^n),$$

where $p_d(n) := p(n\colon \text{distinct parts})$, which may be realized (up to multiplication by $q^{1/24}$) as a modular η-quotient of weight 0 with the usual modular variable $q = e^{2\pi i \tau}$, with $\tau \in \mathbb{H} := \{x + iy \in \mathbb{C}\colon x, y \in \mathbb{R}, y > 0\}$. That is, $q^{\frac{1}{24}} P_d(q) = \eta(2\tau)/\eta(\tau)$, where $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ is Dedekind's weight $1/2$ cusp form, satisfying

$$\eta(\tau + 1) = \zeta_{24} \eta(\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau)$$

for all $\tau \in \mathbb{H}$, with respect to the action of the generators $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. Here and throughout, we let $\zeta_N := e^{2\pi i/N}$ denote the Nth root of unity ($N \in \mathbb{N}$).

Our first set of results realizes k-almost consecutive partition generating functions within the theory of quantum modular forms. Loosely speaking, quantum modular forms exhibit modular-like transformations with respect to the action of a suitable subgroup of $\mathrm{SL}_2(\mathbb{Z})$; however, the domain of a quantum modular form is not the upper half-plane $\mathbb{H}$, but rather the set of rationals $\mathbb{Q}$ or an appropriate subset, and the functions transform with a "well-behaved"error to modularity on $\mathbb{Q}$. The formal definition of a quantum modular form was originally introduced by Zagier in [22] as follows:

Definition 1 A weight $k \in \frac{1}{2}\mathbb{Z}$ quantum modular form is a complex-valued function f on $\mathbb{Q}$, such that for all $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the functions $h_A : \mathbb{Q} \setminus A^{-1}(i\infty) \to \mathbb{C}$ defined by

$$h_A(x) = h_{A,f}(x) := f(x) - \varepsilon^{-1}(A)(\gamma x + \delta)^{-k} f\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

satisfy a "suitable"property of continuity or analyticity in a subset of $\mathbb{R}$.

(Here, $|\varepsilon(A)| = 1$ as in the theory of ordinary modular forms.) We point out that this definition has been slightly modified since its origin to allow for half-integral weights, subgroups of $\mathrm{SL}_2(\mathbb{Z})$, etc. (see [2]). Moreover, Zagier has since defined and developed the notion of a *holomorphic quantum modular form* [23], which is a holomorphic $f : \mathbb{H} \to \mathbb{C}$ such that the functions h_A are holomorphic on a larger domain than $\mathbb{H}$, such as the slit plane $\mathbb{C} \setminus (-\infty, 0]$. (See also [14, 15].)

Bringmann and the first author later defined the notion of a quantum Jacobi form and offered the first example in [1], an example which admits a combinatorial interpretation. Quantum Jacobi forms are a marriage of quantum modular forms with Jacobi forms, the theory of which was largely developed by Eichler and Zagier in the 1980s [7]. Quantum Jacobi forms take values in $\mathbb{C}$, are defined in $\mathbb{Q} \times \mathbb{Q}$ (as opposed to $\mathbb{C} \times \mathbb{H}$ in the case of Jacobi forms) and exhibit Jacobi transformation properties there, up to the addition of smooth error functions in $\mathbb{R} \times \mathbb{R}$. For a precise definition, see Section 2.1.

To state our first main result, we use the following notation for slightly normalized k-almost partition generating functions

$$\mathcal{P}_{ka}(z;\tau) := q^{\frac{1}{8}} w^{\frac{1}{4}} P_{ka}(w^{\frac{1}{2}}; q),$$

where $w = e^{2\pi i z}, q = e^{2\pi i \tau}$ are specialized Jacobi variables. In what follows, we also occasionally abuse notation and write $P_{ka}(z;\tau)$ in place of $P_{ka}(w;q)$ with this same Jacobi change of variables for convenience as needed. We note again that P_{ka} for $k < 0$ is discussed in Section 2.2.

Theorem 1.1 *For odd integers $2k-1$ (with $k \in \mathbb{Z}$), we have that the almost consecutive parts partition generating functions*

$$\mathcal{P}_{(2k-1)a}(z;-\tau)$$

are depth two quantum Jacobi forms of weight $1/2$, index $-1/8$, with respect to $G_k \ltimes (4k\mathbb{Z} \times 2\mathbb{Z})$, (modular Jacobi) multiplier ψ_γ, and with quantum Jacobi sets T_k for $k \in \mathbb{N}$, and S_2 for $k \in -\mathbb{N}_0$.

Moreover, for negative even integers $2k$ (with $k \in -\mathbb{N}$), we have that the almost consecutive parts partition generating functions $P_{(2k)a}(z;\tau)$ are (depth one) quantum Jacobi forms of weight and index 0, trivial (modular Jacobi) multiplier, with respect to $\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$, and with quantum Jacobi sets $\mathbb{Q} \times \mathbb{Q}$.

The subsets S_2 and T_k of $\mathbb{Q}^2$ are defined in (5.1) and (5.2) respectively, the modular subgroups G_k are defined in (5.3), and the multiplier ψ_γ is as defined in [12] (see also Section 2.1). The C^∞ nature of the errors to (Jacobi) modularity of $\mathcal{P}_{(2k-1)a}(z;-\tau)$ is described in more detail in the proof of Theorem 1.1.

Remark 1 Motivated by Theorem 1.1 and work of Bringmann, Kaszian, and Milas [3] which defines the notion of higher depth quantum modular forms, we offer a definition of higher depth quantum Jacobi forms in Definition 3, blending the notion of higher depth quantum modular forms with that of quantum Jacobi forms. Formally, as stated above, our combinatorial generating functions $P_{(2k-1)a}(w;q)$ (modestly normalized) are what we define to be depth two quantum Jacobi forms, as the proof of Theorem 1.1 reveals that they are products of quantum Jacobi forms (see Example 5). We emphasize that it is a question of interest to find other, non-decomposable, examples of higher depth quantum Jacobi and higher depth quantum modular forms, including those arising from pertinent combinatorial generating functions.

Remark 2 We refer the interested reader to [10] for quantum and analytic properties associated to the functions $P_{(2k)a}$ for $k \in \mathbb{N}_0$, owing to the fact that $p_{0a}(n) = d_o(n)$, where $d_o(n) := \#\{\text{odd (positive) divisors of } n\}$ (see (1.1)), along with (1.9) of Proposition 1.8, also using that $p_c(n) = d_o(n)$ (for $n \in \mathbb{N}$) after an identity of Franklin and Sylvester [13].

In addition to understanding any potential modularity of combinatorial generating functions as discussed above, it is also a question of interest to understand the asymptotic growth of their coefficients. In the case of the partition function $p(n)$ for example, Hardy and Ramanujan famously showed the exponential growth $p(n) \sim \frac{1}{4n\sqrt{3}} e^{2\pi\sqrt{n/6}}$ as $n \to \infty$. For the k-almost consecutive partition functions $p_{ka}(n)$ we establish polynomial growth in n for fixed k as follows.

Proposition 1.2 *For $k \in \mathbb{N}$, as $n \to \infty$ we have that*

$$p_{ka}(n) \sim \frac{n^k}{k(k!)^2}.$$

Remark 3 This proposition corrects and generalizes [18, Theorem 17] which corresponds to the case $k = 1$ in our notation (see Example 2).

For small k as a function of n, we also offer formulas for the almost partition functions in Theorem 1.3 below. For $k = 0$, using Franklin and Sylvester's identity between partitions with consecutive parts and distinct odd parts [13] we have that

$$p_{0a}(n) = d_o(n) \tag{1.1}$$

for $n \in \mathbb{N}$. For $k = 1, 2,$ and 3, we establish the following.

Theorem 1.3 *We have that*

$$p_{1a}(n) = n - \left\lceil \tfrac{1}{2}(\sqrt{1+8n} - 1) \right\rceil, \textit{ for } n \in \mathbb{N}_0;$$

$$p_{2a}(n) = \frac{n^2}{8} - \left(\frac{1 + (-1)^n}{8} \right) n - \frac{1 + (-1)^{n+1}}{16} - \sum_{m=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} d_o(n - 2m), \textit{ for } n \in \mathbb{N}_0;$$

$$p_{3a}(n) = \frac{n^3}{108} - \frac{5n^2}{24} + \frac{n\sqrt{8n-31}}{9} + a_{2,n}n + a_{1,n}\sqrt{8n-31} + a_{0,n}, \textit{ for integers } n \geq 7,$$

where $a_{0,n}$, $a_{1,n}$ and $a_{2,n}$ are bounded in absolute value by fixed constants independent of n.

Remark 4 The bounded terms $a_{j,n}$ appearing in the above theorem may be found in the proof of the result in Section 4. For example, $a_{2,n} \in \{\frac{17}{36}, \frac{7}{12}\}$, $a_{1,n} \in (-\frac{2}{9}, -\frac{5}{144})$, and $a_{0,n} \in (-\frac{191}{216}, \frac{25}{54}]$.

We also establish a recursive formula for the k-almost consecutive partition functions, in the spirit of Euler's well-known recursion for the partition function $p(n)$:

$$p(n) = \sum_{j=1}^{\infty} (-1)^{j+1} p(n - j(3j \pm 1)/2).$$

Proposition 1.4 *For all $k \in \mathbb{N}$, we have for $n \in \mathbb{N}$ that*

$$p_{(2k-1)a}(n) = \sum_{j=k+1}^{\infty} (-1)^{j+k+1} p(n - k^2 - j(j-1) \colon \textit{parts in } T_{j,k}),$$

$$p_{(2k)a}(n) = \sum_{j=k+1}^{\infty} (-1)^{j+k+1} p(n - j^2 - k(k+1) : \textit{parts in } E_{j,k}),$$

where the sets $T_{j,k}$ and $E_{j,k}$ are defined in (4.4) and (4.8), respectively.

Playing a key role in establishing the above results is the following proposition, which we state and prove both combinatorially and analytically in Section 3.

Proposition 1.5 *For all integers $k \in \mathbb{Z} \setminus \{0\}$ we have that*

$$P_{ka}(w;q) = \frac{q^k}{1-q^k} \left(\frac{w^{k-1} q^{k(k-1)/2}}{(wq;q)_k} - P_{(k-2)a}(w;q) \right). \tag{1.2}$$

In particular, we have that for all integers $k \geq 2$, and $m, n \in \mathbb{N}_0$,

$$\begin{aligned} &p_{ka}(m,n) - p_{(k-2)a}(m,n) \\ &\quad = p_{ka}(m, n+k) - p(n+k : \textit{exactly } k \textit{ distinct parts, largest part } m+1). \end{aligned} \tag{1.3}$$

The above proposition implies the following recursive relationship for the coefficients p_{ka} in terms of the ordinary partition function.

Corollary 1.6 *For all integers $k \geq 2$, and $n \geq k(k-1)/2$, we have that*

$$p_{ka}(n+k) - p_{ka}(n) = p\left(n - \frac{k(k-1)}{2} : \textit{parts in } \{1, ..., k\}\right) - p_{(k-2)a}(n).$$

Corollary 1.7 *We have for $n \in \mathbb{N}_0$ that*

$$p_{1a}(n+1) - p_{1a}(n) = \begin{cases} 0, & n \textit{ is a triangular number}, \\ 1, & \textit{otherwise}, \end{cases}$$

$$p_{2a}(n+2) - p_{2a}(n) = \left\lfloor \tfrac{n+1}{2} \right\rfloor - d_o(n),$$

$$p_{3a}(n+3) - p_{3a}(n) = \left\lfloor \frac{n^2+6}{12} \right\rfloor - n + \left\lceil \tfrac{1}{2}(\sqrt{1+8n} - 1) \right\rceil.$$

Let

$$S(w;q) := \sum_{m=0}^{\infty} w^m q^{m(m+1)/2}, \tag{1.4}$$

and

$$P_c(w;q) := 1 + \sum_{n=1}^{\infty} \frac{w^n q^{n(n+1)/2}}{1 - wq^n}. \tag{1.5}$$

These q-series may also be realized as combinatorial partition generating functions and possess associated quantum modular properties (loosely speaking) as discussed further in what follows. Moreover, these series are prominently related to our P_{ka} as explained below. Towards this, we next establish the following closed form two-variable generating functions for $P_{ka}(w;q)$ for all $k \in \mathbb{Z}$. Here and throughout, for $n \in \mathbb{N}_0 \cup \{\infty\}$, the q-Pochhammer symbol is defined by

$$(a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$$

and also satisfies [16]

$$(a;q)_{-n} = (aq^{-n};q)_n^{-1}.$$

Proposition 1.8 *For $k \in \mathbb{N}$, we have that*

$$P_{(2k-1)a}(w;q) = \frac{(-1)^k q^{k^2}}{(q;q^2)_k} \left(S(w;q) + \sum_{n=1}^{k} \frac{(-1)^n w^{2n-2} q^{n(n-1)} (q;q^2)_{n-1}}{(wq;q)_{2n-1}} \right), \tag{1.6}$$

$$P_{(-2k+1)a}(w;q) = (q;q^2)_{k-1} \left(S(w;q) - w^{-1} \sum_{n=1}^{k-1} \frac{q^{2n-1} (w^{-1};q)_{2n-1}}{(q;q^2)_n} \right), \tag{1.7}$$

$$P_{(-2k)a}(w;q) = (q^2;q^2)_{k-1} w^{-1} \sum_{n=0}^{k-1} \frac{q^{2n} (w^{-1};q)_{2n}}{(q^2;q^2)_n}. \tag{1.8}$$

Further, for $k \in \mathbb{N}_0$, we have that

$$P_{(2k)a}(w;q) = \frac{(-1)^k q^{k(k+1)}}{(q^2;q^2)_k} \left((P_c(w;q) - 1) + \sum_{n=1}^{k} \frac{w^{2n-1} (-1)^n q^{n^2} (q^2;q^2)_{n-1}}{(wq;q)_{2n}} \right). \tag{1.9}$$

Our last set of main results offers interesting connections between these k-almost consecutive generating functions and the mock theta function ψ, as well as Cohen's σ^*. That is, we recall Cohen's companion to $\sigma(q)$ from Ramanujan's Lost Notebook [6]

$$\sigma^*(q) := 2 \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2}}{(q;q^2)_k}$$

which gives rise to a (weight 0) Maass wave form, as well as Ramanujan's third order mock theta function (see, e.g., [17])

$$\psi(q) := \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q;q^2)_k}$$

which is a weight $1/2$ mock modular form [1,24].

Proposition 1.9 *We have that*

$$\sum_{k=1}^{\infty} P_{(2k-1)a}(w;q) = \frac{1}{2}\sigma^*(q)\cdot S(w;q) + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2}}{(q;q^2)_k} \sum_{n=1}^{k} \frac{(-1)^n w^{2n-2} q^{n(n-1)}(q;q^2)_{n-1}}{(wq;q)_{2n-1}}, \tag{1.10}$$

$$\sum_{k=1}^{\infty} (-1)^k P_{(2k-1)a}(w;q) = \psi(q)\cdot S(w;q) + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q;q^2)_k} \sum_{n=1}^{k} \frac{(-1)^n w^{2n-2} q^{n(n-1)}(q;q^2)_{n-1}}{(wq;q)_{2n-1}}, \tag{1.11}$$

and

$$\sum_{k=0}^{\infty} P_{2ka}(w;q) = q^{-\frac{1}{12}}\eta(2\tau)\cdot (P_c(w;q)-1) + \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}}{(q^2;q^2)_k} \sum_{n=1}^{k} \frac{w^{2n-1}(-1)^n q^{n^2}(q^2;q^2)_{n-1}}{(wq;q)_{2n}}, \tag{1.12}$$

$$\sum_{k=0}^{\infty} (-1)^k P_{2ka}(w;q) = q^{-\frac{1}{12}}\frac{\eta(4\tau)}{\eta(2\tau)}\cdot (P_c(w;q)-1) + \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q^2;q^2)_k} \sum_{n=1}^{k} \frac{w^{2n-1}(-1)^n q^{n^2}(q^2;q^2)_{n-1}}{(wq;q)_{2n}}. \tag{1.13}$$

Exploring potential (nontrivial) mock modular or other consequences of the identities established in Proposition 1.9 is a problem of interest which we pose to the interested reader.

The remainder of the paper is structured as follows. In Section 2, we provide some preliminary results and definitions related to quantum modular forms, k-almost consecutive partitions including generating functions, and q-hypergeometric series. In Section 3 we prove the combinatorial results Proposition 1.5, Corollaries 1.6 and 1.7, and Proposition 1.2. In Section 4, we prove the q-series results Propositions 1.8, 1.9, and 1.4. In Section 5 we prove the quantum Jacobi result Theorem 1.1, and in Section 6 we prove the combinatorial Theorem 1.3. Finally, in Section 7 we provide a proof of a related conjecture of Xiong and record associated combinatorial interpretations.

2 Preliminaries

In this section, we provide some preliminary results and definitions. In particular, in Section 2.1 we discuss quantum modular forms, in Section 2.2 we discuss negative k-almost consecutive partitions, and in Section 2.3 we discuss q-hypergeometric series.

2.1 Quantum modularity

From [1], we have the following definition, extending Zagier's original definition of a quantum modular form (see Section 1) to the Jacobi setting.

Definition 2 A weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form** is a complex-valued function ϕ on $\mathbb{Q} \times \mathbb{Q}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$, the functions $h_\gamma : \mathbb{Q} \times (\mathbb{Q} \setminus \gamma^{-1}(i\infty)) \to \mathbb{C}$ and $g_{(\lambda,\mu)} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ defined by

$$h_\gamma(z;\tau) := \phi(z;\tau) - \varepsilon_1^{-1}(\gamma)(c\tau+d)^{-k} e^{\frac{-2\pi i m c z^2}{c\tau+d}} \phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right),$$

$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \varepsilon_2^{-1}((\lambda,\mu)) e^{2\pi i m(\lambda^2\tau+2\lambda z)} \phi(z+\lambda\tau+\mu;\tau),$$

satisfy a "suitable"property of continuity or analyticity in a subset of $\mathbb{R} \times \mathbb{R}$.

Remarks.

(1) The complex numbers $\varepsilon_1(\gamma)$ and $\varepsilon_2((\lambda,\mu))$ satisfy $|\varepsilon_1(\gamma)| = |\varepsilon_2((\lambda,\mu))| = 1$; in particular, the $\varepsilon_1(\gamma)$ are such as those appearing in the theory of half-integral weight modular forms.

(2) We may modify the definition to allow modular transformations on appropriate subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domain to be a suitable subset of $\mathbb{Q} \times \mathbb{Q}$.

(3) The "suitable"property of continuity or analyticity required is intentionally left somewhat vague in order to mimic Zagier's definition of a quantum modular form [22].

Like the subject of quantum modular forms, the subject of quantum Jacobi forms continues to develop; like quantum modular forms, quantum Jacobi forms arise in the diverse areas of Number Theory, Combinatorics, Topology, and Mathematical Physics. Some known quantum Jacobi forms to date have been established in [1,11,12] for example. One result of interest which we make use of is from [12]. To state it, we normalize $S(w;q)$ by defining

$$\widetilde{\theta}_2(z;\tau) := q^{\frac{1}{8}} w^{\frac{1}{4}} S(w^{\frac{1}{2}};q),$$

where $w = e^{2\pi i z}, q = e^{2\pi i \tau}$. (This is as defined in [12], noting that $S(w;q)$ here equals the function $\theta_2(w;q)$ in [12].) The following result is part of [12, Theorem 4.3]. We refer the reader there for the definitions of the quantum set $S_2 \subset \mathbb{Q} \times \mathbb{Q}$, character ψ_γ, and modular theta functions $g_{a,b}$ appearing.

Theorem *(F-Pratt-Solomon-Tawfeek [12]) The function $\widetilde{\theta}_2(z;-\tau)$ is a quantum Jacobi form on S_2 of weight $1/2$ and index $-1/8$ with respect to $\Gamma_0(4) \ltimes (4\mathbb{Z} \times 2\mathbb{Z})$ and with character ψ_γ. In particular, for $z \in (-\frac{1}{4}, 0)$, $\tau \neq -\frac{1}{4}$, we have that*

$$\widetilde{\theta}_2(z;-\tau) - (4\tau+1)^{-\frac{1}{2}} e\left(\frac{z^2}{2(4\tau+1)}\right) \widetilde{\theta}_2\left(\frac{z}{4\tau+1}; \frac{-\tau}{4\tau+1}\right)$$

$$= -\frac{1}{2}\int_0^\infty \frac{\sum_{\pm} g_{\mp\frac{1}{4},-z}(1+it)}{\sqrt{-i(1+it+4\tau)}}dt, \tag{2.1}$$

and the difference in (2.1) *extends to a* C^∞ *function on* $(\mathbb{R}\setminus(\mathbb{Z}+\{0,\frac{1}{2},\pm\frac{1}{4}\}))\times(\mathbb{R}\setminus\{-\frac{1}{4}\})$.

Another extension of quantum modular forms was defined and studied by Bringmann, Kaszian, and Milas in [3], namely that of higher depth quantum modular forms. Roughly speaking, (for a precise and more general definition see [3]) depth 1 quantum modular forms are ordinary quantum modular forms, while depth 2 quantum modular forms f in the simplest case satisfy

$$f(x)-(\gamma x+\delta)^{-k}f\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)\in \mathcal{Q}_k(\Gamma)\mathcal{O}(R)+\mathcal{O}(R),$$

where $\mathcal{O}(R)$ is the space of real analytic functions defined on $R\subseteq\mathbb{R}$, and $\mathcal{Q}_k(\Gamma)$ is the space of weight k quantum modular forms on $\Gamma\subseteq \mathrm{SL}_2(\mathbb{Z})$. Motivated by this, as well as the proof of Theorem 1.1 which reveals that our combinatorial generating functions $P_{(2k-1)a}(w;q)$ are products of quantum Jacobi forms, we offer a definition of higher depth quantum Jacobi forms in Definition 3 below. Formally, our combinatorial generating functions $P_{(2k-1)a}(w;q)$ (modestly normalized) are depth two quantum Jacobi forms; see Example 5. We emphasize that it is a question of interest to find other, non-decomposable, examples of higher depth quantum Jacobi and higher depth quantum modular forms, including those arising from pertinent combinatorial generating functions (see also Remark 1).

Definition 3 blends the notion of higher depth quantum modular forms with that of quantum Jacobi forms, and naturally extends the definition of higher depth quantum modular forms from [3].

Definition 3 A function $f:\mathbb{Q}\times\mathbb{Q}\to\mathbb{C}$ is called a *quantum Jacobi form of depth* $N\in\mathbb{N}$, *weight* $k\in\frac{1}{2}\mathbb{Z}$, *index* $m\in\frac{1}{2}\mathbb{Z}$, *multipliers* $\varepsilon_1,\varepsilon_2$, *and quantum Jacobi set* $\mathcal{Q}\subseteq\mathbb{Q}\times\mathbb{Q}$ *for* $\Gamma\ltimes G$ if for all $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ and $(\lambda,\mu)\in G$,

$$f(z,\tau)-\varepsilon_1(M)^{-1}(c\tau+d)^{-k}e^{-\frac{2\pi imcz^2}{c\tau+d}}f\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\in \bigoplus_j \mathcal{J}^{N_j}_{\kappa_j,m_j}(\Gamma\ltimes G,\varepsilon_{1,j},\varepsilon_{2,j})\mathcal{O}(R^2),$$

and

$$f(z,\tau)-\varepsilon_2(M)^{-1}e^{2\pi im(\lambda^2\tau+2\lambda z)}f(z+\lambda\tau+\mu;\tau)\in\bigoplus_j \mathcal{J}^{N_j}_{\kappa_j,m_j}(\Gamma\ltimes G,\varepsilon_{1,j},\varepsilon_{2,j})\mathcal{O}(R^2),$$

where j runs through a finite set, $\kappa_j,m_j\in\frac{1}{2}\mathbb{Z}, N_j\in\mathbb{N}$, with $\max_j(N_j)=N-1$, $\varepsilon_{\ell,j}$ are multipliers, $\mathcal{O}(R^2)$ is the space of real-analytic functions on $R^2\subseteq\mathbb{R}^2$ which contain an open subset of $\mathbb{R}^2$, $\mathcal{J}^1_{k,m}(\Gamma\ltimes G,\varepsilon_1,\varepsilon_2)$ denotes the space of quantum Jacobi forms of weight k, index m, group $\Gamma\ltimes G$, and multipliers $\varepsilon_1,\varepsilon_2$, $\mathcal{J}^0_{k,m}(\Gamma\ltimes G,\varepsilon_1,\varepsilon_2):=1$, and $\mathcal{J}^N_{k,m}(\Gamma\ltimes G,\varepsilon_1,\varepsilon_2)$ denotes the space of quantum Jacobi forms of weight k, index m, depth N, group $\Gamma\ltimes G$, and character χ.

Example 5 For $f\in\mathcal{J}^1_{k_1,m_1}(\Gamma_1\ltimes G_1,\varepsilon_{1,f},\varepsilon_{2,f})$ on $\mathcal{Q}_1\subseteq\mathbb{Q}\times\mathbb{Q}$, and $g\in\mathcal{J}^1_{k_2,m_2}(\Gamma_2\ltimes G_2,\varepsilon_{1,g},\varepsilon_{2,g})$ on $\mathcal{Q}_2\subseteq\mathbb{Q}\times\mathbb{Q}$, we have that fg lies in the space $\mathcal{J}^2_{k_1+k_2,m_1+m_2}((\Gamma_1\ltimes G_1)\cap(\Gamma_2\ltimes G_2),\varepsilon_{1,f}\varepsilon_{1,g},\varepsilon_{2,f}\varepsilon_{2,g})$ on $\mathcal{Q}_1\cap\mathcal{Q}_2$.

2.2 *k*-almost consecutive partitions

In this subsection we establish some basic properties of k-almost consecutive partitions, including for $k < 0$. We first establish a closed form generating function.

Lemma 2.1 *For any* $k \in \mathbb{N}_0$,

$$P_{ka}(w;q) = \sum_{\ell=1}^{\infty} \frac{w^{\ell+k} q^{(\ell+k)(\ell+k+1)/2}}{(wq^{\ell};q)_{k+1}}.$$

Proof of Lemma 2.1 Fix $k \in \mathbb{N}_0$. Any k-almost consecutive partition can be written as

$$\pi := \lambda_1 + \lambda_2 + \cdots + \lambda_k + \lambda_{k+1} + \cdots + \lambda_{k+\ell}$$

where $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k+\ell}$, $\ell \geq 1$, and parts $\lambda_{k+1}, \ldots, \lambda_{k+\ell}$ are consecutive (i.e. $\lambda_{k+j+1} - \lambda_{k+j} = 1$ for $1 \leq j \leq \ell - 1$). Let $a_0, a_1, \ldots, a_k \in \mathbb{N}_0$ be defined by $\lambda_1 := 1 + a_k$ and for $n \geq 1$, $\lambda_{n+1} - \lambda_n = 1 + a_{k-n}$. For a fixed ℓ, notice that any k-almost consecutive partition can be uniquely determined by $a_0, a_1, \ldots a_k$. By the above, we have that

$$\begin{aligned}
\pi &= (1 + a_k) + (2 + a_k + a_{k-1}) + \cdots + (k + a_k + a_{k-1} + \cdots + a_1) \\
&\quad + (k + 1 + a_k + a_{k-1} + \cdots + a_0) + \cdots + (k + \ell + a_k + a_{k-1} + \cdots + a_0) \\
&= \sum_{n=1}^{k} \left(n + \sum_{m=1}^{n} a_{k-m+1} \right) + \sum_{n=1}^{\ell} \left(k + n + \sum_{m=0}^{k} a_m \right) \\
&= \sum_{n=1}^{k+\ell} n + \sum_{n=1}^{\ell} \sum_{m=0}^{k} a_m + \sum_{n=1}^{k} \sum_{m=1}^{n} a_{k-m+1} \\
&= \sum_{n=1}^{k+\ell} n + \ell \sum_{n=0}^{k} a_n + \sum_{n=0}^{k} n a_n \\
&= \sum_{n=1}^{k+\ell} n + \sum_{n=0}^{k} (\ell + n) a_n \\
&= \frac{(\ell + k)(\ell + k + 1)}{2} + \sum_{n=0}^{k} (\ell + n) a_n.
\end{aligned}$$

Using the above notation, the largest part of π equals $k + \ell + a_k + a_{k-1} + \cdots + a_0$. Thus, we have shown that

$$\begin{aligned}
P_{ka}(w;q) &:= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{ka}(m,n) w^m q^n \\
&= \sum_{\ell=1}^{\infty} w^{k+\ell} q^{(k+\ell)(k+\ell+1)/2} \prod_{n=0}^{k} \sum_{a_n=0}^{\infty} w^{a_n} q^{(\ell+n)a_n} \\
&= \sum_{\ell=1}^{\infty} w^{k+\ell} q^{(k+\ell)(k+\ell+1)/2} (1 + wq^{\ell} + w^2 q^{2\ell} + \cdots) \cdot \\
&\quad (1 + wq^{1+\ell} + w^2 q^{2+2\ell} + \cdots) \cdots (1 + wq^{k+\ell} + w^2 q^{2k+2\ell} + \cdots) \\
&= \sum_{\ell=1}^{\infty} \frac{w^{k+\ell} q^{(k+\ell)(k+\ell+1)/2}}{(wq^{\ell};q)_{k+1}}.
\end{aligned}$$

□

We now use the result of Lemma 2.1 to define $P_{ka}(w;q)$ for $k<0$. That is, for $k \in -\mathbb{N}$, we define P_{ka} and $p_{ka}(m,n)$ by

$$P_{ka}(w;q) := \sum_{\ell=1}^{\infty} \frac{w^{\ell+k} q^{(\ell+k)(\ell+k+1)/2}}{(wq^{\ell};q)_{k+1}} =: \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{ka}(m,n) w^m q^n. \tag{2.2}$$

We further define $p_{ka}(n)$ for $k \in -\mathbb{N}$ using (2.2) in the obvious way (as the cofficient of q^n after specializing $w=1$). From (2.2) with $k=-1$ we have that

$$P_{-1a}(w;q) = S(w;q), \tag{2.3}$$

also as in (1.7) with $k=1$ from Proposition 1.8. This additionally motivates our examination of the case of negative k. To this end, we establish the following lemma.

Lemma 2.2 *For $k \in -\mathbb{N}$, we have that*

$$P_{ka}(w;q) = \sum_{\ell=0}^{\infty} w^{\ell} q^{\ell(\ell+1)/2} (wq^{\ell+1};q)_{-k-1} + w^{-1} \sum_{\ell=1}^{-k-1} (-1)^{\ell+k+1} (w;q)_{\ell} (w^{-1}q;q)_{-k-\ell-1}. \tag{2.4}$$

Before proving this lemma, some remarks are in order.

Remark 5 We explain a combinatorial interpretation of $p_{ka}(n)$ (respectively $p_{ka}(m,n)$) for integers $k<0$ in terms of partitions from (2.4) of Lemma 2.2.

To this end, we point out that for integers $k<0$, we have that $-k-1 \geq 0$, and for $k<-1$, we have $-k-\ell-1 \geq 0$ for $1 \leq \ell \leq -k-1$. This reveals that (for $k<0$) the two series on the right-hand side of (2.4) may be easily expanded into q-series (with only non-negative powers of q appearing), and with coefficients admitting a combinatorial interpretation in terms of partitions. (The second sum on the right-hand side of (2.4) is empty and thus equal to 0 when $k=-1$.) That is, after series expanding and equating coefficients, the series on the right-hand side of (2.4) in Lemma 2.2 may be used to provide a combinatorial interpretation in terms of partitions of the coefficients $p_{ka}(m,n)$ (of $w^m q^n$) and $p_{ka}(n)$ (of q^n with $w=1$) for $k<0$ (arising from the left-hand side of (2.4)). However, these combinatorial interpretations are apparently not simple.

Because of this and the fact that the results for p_{ka} or P_{ka} for $k<0$ in this paper do not depend on the combinatorial interpretations of $p_{ka}(m,n)$ and p_{ka} for $k<0$, we leave it to the interested reader to provide simpler (than as arising from (2.4)) combinatorial interpretations of $p_{ka}(m,n)$ and $p_{ka}(n)$ for integers $k<0$, still beginning with (2.2) so they remain aligned with their $k \geq 0$ counterparts and so results of this paper for $k<0$ including Theorem 1.1 still hold. Ideally such a simple combinatorial interpretation would involve almost consecutive parts as in our definition of p_{ka} for integers $k \geq 0$ given in Section 1.

Proof of Lemma 2.2 From (2.2) and standard q-series properties [16] we have that

$$P_{ka}(w;q) = \sum_{\ell=1}^{\infty} \frac{w^{\ell+k} q^{(\ell+k)(\ell+k+1)/2}}{(wq^{\ell};q)_{k+1}}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{\infty} w^{\ell+k} q^{(\ell+k)(\ell+k+1)/2} (wq^{\ell+k+1};q)_{-k-1} \\
&= \sum_{\ell=0}^{\infty} w^{\ell} q^{\ell(\ell+1)/2} (wq^{\ell+1};q)_{-k-1} + \sum_{\ell=1}^{-k-1} w^{\ell+k} q^{(\ell+k)(\ell+k+1)/2} (wq^{\ell+k+1};q)_{-k-1} \\
&= \sum_{\ell=0}^{\infty} w^{\ell} q^{\ell(\ell+1)/2} (wq^{\ell+1};q)_{-k-1} + \sum_{\ell=1}^{-k-1} w^{\ell+k} q^{(\ell+k)(\ell+k+1)/2} (w;q)_{\ell} (wq^{-1};q^{-1})_{-k-\ell-1} \\
&= \sum_{\ell=0}^{\infty} w^{\ell} q^{\ell(\ell+1)/2} (wq^{\ell+1};q)_{-k-1} \\
&\quad + \sum_{\ell=1}^{-k-1} w^{\ell+k} q^{(\ell+k)(\ell+k+1)/2} (-wq^{-1})^{-k-\ell-1} q^{-\frac{(-k-\ell-1)(-k-\ell-2)}{2}} (w;q)_{\ell} (w^{-1}q;q)_{-k-\ell-1} \\
&= \sum_{\ell=0}^{\infty} w^{\ell} q^{\ell(\ell+1)/2} (wq^{\ell+1};q)_{-k-1} + w^{-1} \sum_{\ell=1}^{-k-1} (-1)^{\ell+k+1} (w;q)_{\ell} (w^{-1}q;q)_{-k-\ell-1}.
\end{aligned}
$$

□

2.3 q-hypergeometric series

Recall the definition of $S(w;q)$ from (1.4). In Lemma 2.3 we establish a q-hypergeometric expression for this function.

Lemma 2.3 *We have that*

$$
S(w;q) = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n} q^{n(n+1)} (q;q^2)_n}{(wq;q)_{2n+1}}.
$$

Proof To prove this lemma, we first rewrite

$$
\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n} q^{n(n+1)} (q;q^2)_n}{(wq;q)_{2n+1}} = \frac{1}{1-wq} \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n} q^{n(n+1)} (q;q^2)_n}{(wq^3;q^2)_n (wq^2;q^2)_n}.
$$

Next, we define as in [9]

$$
F(a,b,t;q) = \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n} t^n.
$$

Using [9, (12.2)] with $q \mapsto q^2$, $a \mapsto wq^{-1}$, $b \mapsto w$, and $t \mapsto wq$, we obtain

$$
\begin{aligned}
(1-wq)F(wq^{-1},w,wq;q^2) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(wq^2;q^2)_n (wq^3;q^2)_n} (-wq^{-1}wq)^n q^{2n(n+1)/2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n} q^{n(n+1)} (q;q^2)_n}{(wq^3;q^2)_n (wq^2;q^2)_n} \\
&= (1-wq) \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n} q^{n(n+1)} (q;q^2)_n}{(wq;q)_{2n+1}}. \qquad (2.5)
\end{aligned}
$$

Now, using [9, (14.4)] with $b \mapsto w$, we find that $F(wq^{-1},w,wq;q^2) = S(w;q)$. Combining this with (2.5) proves the result. □

Next, recall the definition of $P_c(w;q)$ from (1.5). In Lemma 2.4 we establish a q-hypergeometric expression for this function.

Lemma 2.4 *We have that*

$$P_c(w;q) = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n w^{2n-1} q^{n^2} (q^2;q^2)_{n-1}}{(wq;q)_{2n}}.$$

Proof We begin similarly to the proof of Lemma 2.3 and rewrite

$$\begin{aligned}
-\sum_{n=1}^{\infty} \frac{(-1)^n w^{2n-1} q^{n^2} (q^2;q^2)_{n-1}}{(wq;q)_{2n}} &= -\sum_{n=0}^{\infty} \frac{(-1)^{n+1} w^{2n+1} q^{(n+1)^2} (q^2;q^2)_n}{(wq;q^2)_{n+1}(wq^2;q^2)_{n+1}} \\
&= \frac{wq}{(1-wq)(1-wq^2)} \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n} q^{n^2+2n} (q^2;q^2)_n}{(wq^4;q^2)_n (wq^3;q^2)_n} \\
&= \frac{wq}{1-wq^2} F(w, wq^2, wq; q^2),
\end{aligned}$$

where the last line follows from [9, (12.2)]. Using the definition of $F(a,b,t;q)$ we find that this equals

$$\begin{aligned}
\frac{wq}{1-wq^2} \sum_{n=0}^{\infty} \frac{(wq)^n (wq^2;q^2)_n}{(wq^4;q^2)_n} &= \frac{wq}{1-wq^2} \sum_{n=0}^{\infty} \frac{1-wq^2}{1-wq^{2n+2}} (wq)^n \\
&= wq \sum_{n=0}^{\infty} \left(\frac{(wq)^n (1+w^{1/2}q^{n+1})}{1-wq^{2n+2}} - \frac{w^{(2n+1)/2} q^{2n+1}}{1-wq^{2n+2}} \right) \\
&= wq \sum_{n=0}^{\infty} \frac{(wq)^n}{1-w^{1/2}q^{n+1}} - wq \sum_{n=0}^{\infty} \frac{w^{(2n+1)/2} q^{2n+1}}{1-wq^{2n+2}} \\
&= \frac{wq}{1-w^{1/2}q} F(w^{1/2}, w^{1/2}q, wq; q) - w^{1/2} \sum_{n=0}^{\infty} \frac{(wq^2)^{n+1}}{1-wq^{2n+2}}.
\end{aligned}$$

Next we use [9, (6.3)] applied to $F(w^{1/2}, w^{1/2}q, wq; q)$ to show that this equals

$$\begin{aligned}
&= \frac{wq}{1-wq} \sum_{n=0}^{\infty} \frac{(wq;q)_n}{(wq^2;q)_n} (w^{1/2}q)^n - w^{1/2} \sum_{n=0}^{\infty} \frac{(wq^2)^{n+1}}{1-wq^{2n+2}} \\
&= wq \sum_{n=0}^{\infty} \frac{(w^{1/2}q)^n}{1-wq^{n+1}} - w^{1/2} \sum_{n=0}^{\infty} \frac{(wq^2)^{n+1}}{1-wq^{2n+2}} \\
&= w^{1/2} \left(\sum_{n=1}^{\infty} \frac{(w^{1/2}q)^n}{1-wq^n} - \sum_{n=1}^{\infty} \frac{(w^{1/2}q)^{2n}}{1-wq^{2n}} \right) \\
&= w^{1/2} \sum_{n=0}^{\infty} \frac{(w^{1/2}q)^{2n+1}}{1-wq^{2n+1}} \\
&= \sum_{n=0}^{\infty} \frac{w^{n+1} q^{2n+1}}{1-wq^{2n+1}}.
\end{aligned}$$

We further rewrite

$$\sum_{n=0}^{\infty} \frac{w^{n+1} q^{2n+1}}{1-wq^{2n+1}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} w^{n+1} q^{2n+1} (wq^{2n+1})^k$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1} w^{n+1}q^{2n+1}(wq^{2n+1})^k + \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} w^{n+1}q^{2n+1}(wq^{2n+1})^k \\
&= \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} w^{n+1}q^{2n+1}(wq^{2n+1})^k + \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} w^{n+1}q^{2n+1}(wq^{2n+1})^k \\
&= \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} w^{k+n+1}q^{2kn+k+2n+1} + \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} w^{n+1}q^{2n+1}(wq^{2n+1})^k \\
&= \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} (wq)^{k+1}(wq^{2k+2})^n + \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} w^{n+1}q^{2n+1}(wq^{2n+1})^k \\
&= \sum_{k=0}^{\infty} \frac{(wq)^{k+1}}{1-wq^{2k+2}}(wq^{2k+2})^{k+1} + \sum_{n=0}^{\infty} \frac{w^{n+1}q^{2n+1}}{1-wq^{2n+1}}(wq^{2n+1})^n \\
&= \sum_{k=0}^{\infty} \frac{w^{2k+2}q^{2k^2+5k+3}}{1-wq^{2k+2}} + \sum_{n=0}^{\infty} \frac{w^{2n+1}q^{2n^2+3n+1}}{1-wq^{2n+1}} \\
&= \sum_{k=0}^{\infty} \frac{w^{2k+2}q^{(2k+2)(2k+3)/2}}{1-wq^{2k+2}} + \sum_{n=0}^{\infty} \frac{w^{2n+1}q^{(2n+1)(2n+2)/2}}{1-wq^{2n+1}} \\
&= \sum_{n=0}^{\infty} \frac{w^{n+1}q^{(n+1)(n+2)/2}}{1-wq^{n+1}} \\
&= P_c(w;q) - 1.
\end{aligned}$$

That is, we have shown that

$$P_c(w;q) - 1 = -\sum_{n=1}^{\infty} \frac{(-1)^n w^{2n-1} q^{n^2} (q^2;q^2)_{n-1}}{(wq;q)_{2n}},$$

which establishes the desired result. □

3 Proofs I: Combinatorial results

In this section, we prove some of our combinatorial results, namely Proposition 1.5, followed by Corollaries 1.6 and 1.7, and Proposition 1.2. Proposition 1.5 plays a key role in establishing the latter (and other) results, and thus we prove it first.

3.1 Proof of Proposition 1.5

We present both a q-series proof ($k \neq 0$) and a combinatorial proof ($k \geq 2$) of Proposition 1.5, beginning with the former.

q-series proof ($k \neq 0$): For integers $k \neq 0$, using Lemma 2.1 and (2.2), we have that

$$\begin{aligned}
P_{ka}(w;q) &= \sum_{\ell=1}^{\infty} \frac{w^{\ell+k}q^{(\ell+k)(\ell+k+1)/2}}{(wq^{\ell};q)_{k+1}} \\
&= \sum_{\ell=1}^{\infty} \frac{w^{\ell+k}q^{(\ell+k)(\ell+k+1)/2}(wq^{\ell+k+1};q)_\infty}{(wq^{\ell};q)_\infty} \\
&= \frac{1}{1-q^k}\sum_{\ell=1}^{\infty}\left(\frac{w^{\ell+k}q^{(\ell+k)(\ell+k+1)/2}(wq^{\ell+k+1};q)_\infty(1-wq^{\ell+k})}{(wq^{\ell};q)_\infty}\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \frac{w^{\ell+k}q^{(\ell+k)(\ell+k+1)/2}(wq^{\ell+k+1};q)_\infty q^k(1-wq^\ell)}{(wq^\ell;q)_\infty}\Bigg) \\
&= \frac{1}{1-q^k}\sum_{\ell=1}^{\infty}\Bigg(\frac{w^{\ell+k}q^{(\ell+k)(\ell+k+1)/2}(wq^{\ell+k};q)_\infty}{(wq^\ell;q)_\infty} \\
&\qquad - \frac{w^{\ell+k}q^{(\ell+k)(\ell+k+1)/2+k}(wq^{\ell+k+1};q)_\infty}{(wq^{\ell+1};q)_\infty}\Bigg) \\
&= \frac{1}{1-q^k}\Bigg(\frac{w^{k+1}q^{(k+1)(k+2)/2}(wq^{k+1};q)_\infty}{(wq;q)_\infty} + \sum_{\ell=2}^{\infty}\frac{w^{\ell+k}q^{(\ell+k)(\ell+k+1)/2}(wq^{\ell+k};q)_\infty}{(wq^\ell;q)_\infty} \\
&\qquad - \sum_{\ell=2}^{\infty}\frac{w^{\ell+k-1}q^{(\ell+k)(\ell+k-1)/2+k}(wq^{\ell+k};q)_\infty}{(wq^\ell;q)_\infty}\Bigg) \\
&= \frac{1}{1-q^k}\Bigg(\frac{w^{k+1}q^{(k+1)(k+2)/2}(wq^{k+1};q)_\infty}{(wq;q)_\infty} \\
&\qquad + \sum_{\ell=2}^{\infty}\frac{w^{\ell+k-1}q^{(\ell+k)(\ell+k-1)/2+k}(wq^\ell-1)(wq^{\ell+k};q)_\infty}{(wq^\ell;q)_\infty}\Bigg) \\
&= \frac{1}{1-q^k}\Bigg(\frac{w^{k+1}q^{(k+1)(k+2)/2}(wq^{k+1};q)_\infty}{(wq;q)_\infty} \\
&\qquad - \sum_{\ell=2}^{\infty}\frac{w^{\ell+k-1}q^{(\ell+k)(\ell+k-1)/2+k}(wq^{\ell+k};q)_\infty}{(wq^{\ell+1};q)_\infty}\Bigg) \\
&= \frac{1}{1-q^k}\Bigg(\frac{w^{k+1}q^{(k+1)(k+2)/2}(wq^{k+1};q)_\infty}{(wq;q)_\infty} + \frac{w^{k-1}q^{k(k-1)/2+k}(wq^k;q)_\infty}{(wq;q)_\infty} \\
&\qquad + \frac{w^kq^{k(k+1)/2+k}(wq^{k+1};q)_\infty}{(wq^2;q)_\infty} \\
&\qquad - q^k\sum_{\ell=1}^{\infty}\frac{w^{\ell+k-2}q^{(\ell+k-2)(\ell+k-1)/2}(wq^{\ell+k-1};q)_\infty}{(wq^\ell;q)_\infty}\Bigg) \\
&= \frac{1}{1-q^k}\Bigg(\frac{w^{k-1}q^{k(k-1)/2+k}(wq^{k+1};q)_\infty}{(wq;q)_\infty}\left(w^2q^{k+1}+(1-wq^k)+wq^k(1-wq)\right) \\
&\qquad - q^kP_{(k-2)a}(w;q)\Bigg) \\
&= \frac{q^k}{1-q^k}\Bigg(\frac{w^{k-1}q^{k(k-1)/2}(wq^{k+1};q)_\infty}{(wq;q)_\infty} - P_{(k-2)a}(w;q)\Bigg) \\
&= \frac{q^k}{1-q^k}\Bigg(\frac{w^{k-1}q^{k(k-1)/2}}{(wq;q)_k} - P_{(k-2)a}(w;q)\Bigg).
\end{aligned}
$$

□

Towards establishing our combinatorial proof of (1.3), we let $p_{=kd}(m,n)$ be the number of partitions of n into exactly k distinct parts with a largest part m, and let $P_{=kd}(w;q)$ be its associated generating function. Then we have that

$$P_{=kd}(w;q) := \sum_{m,n\in\mathbb{Z}} p_{=kd}(m,n)q^n w^m = \frac{w^k q^{k(k+1)/2}}{(wq;q)_k}.$$

It is not difficult to see that for $k \geq 2$, (1.2) is equivalent to

$$q^{-k}P_{ka}(w;q) - P_{ka}(w;q) = w^{-1}q^{-k}P_{=kd}(w;q) - P_{(k-2)a}(w;q)$$

or equivalently that

$$p_{ka}(m,n+k) - p_{ka}(m,n) = p_{=kd}(m+1,n+k) - p_{(k-2)a}(m,n) \tag{3.1}$$

which we will next prove combinatorially (in order to establish (1.3)).

Combinatorial proof for $k \geq 2$: Let $A_{=kd}(m,n)$ be the set of partitions of n with exactly k distinct parts and with largest part m. Also, let $A'_{=kd}(m,n)$ and $A''_{=kd}(m,n)$ be the subsets of $A_{=kd}(m,n)$ where the former contains the partitions that do not contain 1 as a part and the latter contains the partitions that contain 1 as a part. We may obtain a bijection from $A_{=kd}(m,n)$ to the partitions of $A'_{=kd}(m+1,n+k)$ by adding 1 to each part. Similarly, we obtain a bijection from $A_{=(k-1)d}(m,n)$ to $A''_{=kd}(m+1,n+k)$ by adding 1 to each part, and then adding 1 as a new part. Hence,

$$p_{=kd}(m,n) + p_{=(k-1)d}(m,n) = p_{=kd}(m+1,n+k). \tag{3.2}$$

Let $p_{kax}(m,n)$ be the number of k-almost consecutive partitions of n that are "exactly"k-almost consecutive in the sense that the kth smallest and $(k+1)$th smallest elements are non-consecutive, and that have largest part m. Also let $A_{kax}(m,n)$ be the corresponding set of such partitions, similarly defining the subsets $A'_{ka}(m,n)$ and $A''_{ka}(m,n)$ of $A_{ka}(m,n)$ (where $A_{ka}(m,n)$ denotes the set of k-almost consecutive partitions of n with largest part m) as the sets of partitions that do not contain 1 as a part and do contain 1 as a part respectively. Mirroring the above statements, we obtain a bijection from $A_{kax}(m,n)$ to $A'_{ka}(m,n+k)$ by adding 1 to the k smallest parts, and we can obtain a bijection from $A_{(k-1)ax}(m,n)$ to $A''_{ka}(m,n+k)$ by adding 1 to the $k-1$ smallest parts, and then adding 1 as a new part. Hence,

$$p_{kax}(m,n) + p_{(k-1)ax}(m,n) = p_{ka}(m,n+k). \tag{3.3}$$

Now, since the only partitions contained in exactly one of $A_{ka}(m,n)$ or $A_{(k-1)a}(m,n)$ are those that are "exactly"k-almost consecutive in the former and those that have exactly k distinct parts in the latter, we have that $p_{ka}(m,n) = p_{kax}(m,n)+p_{(k-1)a}(m,n)-p_{=kd}(m,n)$. We use this equation within itself to obtain

$$p_{ka}(m,n)=p_{kax}(m,n)+p_{(k-1)ax}(m,n)+p_{(k-2)a}(m,n)-p_{=(k-1)d}(m,n)-p_{=kd}(m,n+k). \tag{3.4}$$

Rearranging (3.4), as well as using (3.2) and (3.3), we find

$$p_{ka}(m,n) - p_{(k-2)a}(m,n) = p_{ka}(m,n+k) - p_{=kd}(m+1,n+k)$$

thus establishing (3.1) and hence (1.3). □

3.2 Proofs of Corollaries 1.6 and 1.7

Proof of Corollary 1.6 We sum the combinatorial statement of Proposition 1.5 over all m to obtain

$$p_{ka}(n+k) - p_{ka}(n) = p\,(n+k : \text{exactly } k \text{ distinct parts}) - p_{(k-2)a}(n).$$

Hence it remains to show that

$$p\,(n+k : \text{exactly } k \text{ distinct parts}) = p\Big(n - \frac{k(k-1)}{2} \ : \ \text{parts in } \{1, 2, ..., k\}\Big).$$

We define $p_{=kd}(n)$ to be the number of partitions of n into exactly k distinct parts, and we define $p_{=k}(n)$ to be the number of partitions of n into exactly k parts. We obtain a bijection between the partitions of the set $A_{=kd}(n+k)$ and the partitions of the set $A_{=k}(n+k-\frac{k(k-1)}{2})$ (similarly defined as in the combinatorial proof of Proposition 1.5) by removing the Ferrers diagram's upper-left triangle with side lengths of $n-1$. For $n \geq \frac{k(k-1)}{2}$ we then obtain a bijection between the partitions of $A_{=k}(n+k-\frac{k(k-1)}{2})$ and the partitions of $A\left(n - \frac{k(k-1)}{2} \ : \ \text{parts in } \{1, 2, ..., k\}\right)$ by, considering the Ferrers diagram, removing the leftmost column and then conjugating. Hence we have that $p_{=kd}(n+k)$ is equal to $p\big(n - k(k-1)/2 \ : \ \text{parts in } \{1, 2, ..., k\}\big)$. □

Proof of Corollary 1.7 When $k = 1$ and $w = 1$, Proposition 1.5 and (2.3) or (1.7) give that

$$P_{1a}(1; q) = \frac{q}{1-q}\left(\frac{1}{1-q} - S(1; q)\right).$$

Noting that $p_{1a}(0) = 0$, we have

$$\begin{aligned}&\frac{1}{1-q} - S(1; q)\\ &\quad = (q^{-1}-1)\sum_{n=0}^{\infty} p_{1a}(n)q^n = \sum_{n=0}^{\infty} p_{1a}(n)q^{n-1} - \sum_{n=0}^{\infty} p_{1a}(n)q^n = \sum_{n=0}^{\infty}(p_{1a}(n+1) - p_{1a}(n))q^n.\end{aligned} \tag{3.5}$$

Letting $T := \{n \geq 0 : n = m(m+1)/2 \text{ for some } m \geq 0\}$ denote the set of nonnegative triangular numbers and recalling the definition of $S(1; q)$, also using (3.5) we have that

$$\begin{aligned}\sum_{n=0}^{\infty}(p_{1a}(n+1) - p_{1a}(n))q^n &= \sum_{n=0}^{\infty} q^n - \sum_{n=0}^{\infty} q^{n(n+1)/2}\\ &= \sum_{n \in \mathbb{N}_0 \setminus T} q^n + \sum_{n=0}^{\infty} q^{n(n+1)/2} - \sum_{n=0}^{\infty} q^{n(n+1)/2} = \sum_{n \in \mathbb{N}_0 \setminus T} q^n.\end{aligned}$$

Equating coefficients, we obtain that for every $n \geq 0$,

$$p_{1a}(n+1) - p_{1a}(n) = \begin{cases} 0, & n \text{ is a triangular number,}\\ 1, & \text{otherwise,}\end{cases}$$

as claimed for $k = 1$.

To establish the claimed result for $k = 2$, by Corollary 1.6 with $k = 2$, when $n \geq 1$, we obtain that

$$p_{2a}(n+2) - p_{2a}(n) = p(n-1 : \text{parts in } \{1,2\}) - p_{0a}(n).$$

One can verify that $p(n : \text{parts in } \{1,2\}) = \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{n+2}{2} \rfloor$ for $n \geq 1$. For $n \geq 1$ we also have $p_{0a}(n) = p_c(n) = d_o(n)$ (using an identity of Franklin and Sylvester [13]). Therefore, for $n \geq 1$, we must have

$$p_{2a}(n+2) - p_{2a}(n) = \left\lfloor \frac{n+1}{2} \right\rfloor - d_o(n)$$

as claimed for $k = 2$. One can check that the desired result also holds when $n = 0$.

Similarly, by Corollary 1.6 with $k = 3$, when $n \geq 3$, we find that

$$p_{3a}(n+3) - p_{3a}(n) = p(n-3 : \text{parts in } \{1,2,3\}) - p_{1a}(n).$$

By Theorem 1.3, for $n \geq 1$, we have $p_{1a}(n) = n - \lceil \frac{1}{2}(\sqrt{1+8n} - 1) \rceil$. Furthermore, it can be shown that, when $n \geq 0$,

$$p(n : \text{parts in } \{1,2,3\}) = \left\lfloor \frac{(n+3)^2 + 6}{12} \right\rfloor,$$

and hence it follows that for $n \geq 3$,

$$p_{3a}(n+3) - p_{3a}(n) = \left\lfloor \frac{n^2+6}{12} \right\rfloor - n + \left\lceil \frac{1}{2}(\sqrt{1+8n} - 1) \right\rceil$$

as claimed. One can check that the desired result also holds when $n \in \{0,1,2\}$. □

3.3 Proof of Proposition 1.2

Proof of Proposition 1.2 We deduce the result for $k \in \{1,2,3\}$ from Theorem 1.3 proved in the next section. Suppose the proposed asymptotic holds for $p_{(k-2)a}(n)$, for some $k \geq 3$. A well-known result (e.g., see [8]) states that

$$p(n \mid \text{parts in } \{1,2,\ldots,k\}) \sim \frac{n^{k-1}}{k!(k-1)!}.$$

Then using Corollary 1.6 and our inductive supposition, we observe that

$$p_{ka}(n+k) - p_{ka}(n) \sim \frac{n^{k-1}}{k!(k-1)!},$$

implying that with $n \mapsto kn$,

$$p_{ka}(k(n+1)) - p_{ka}(kn) \sim \frac{(kn)^{k-1}}{k!(k-1)!}.$$

Hence we have

$$p_{ka}(kn) \sim \sum_{m=1}^{n-1} \frac{(km)^{k-1}}{k!(k-1)!} \sim \frac{k^{k-1}}{k!(k-1)!} \cdot \frac{n^k}{k} \sim \frac{k^{k-1}n^k}{(k!)^2},$$

implying that

$$p_{ka}(n) \sim \frac{n^k}{k(k!)^2}.$$

□

4 Proofs II: *q*-series results

In this section, we prove Propositions 1.8, 1.9, and 1.4. The proofs of the latter two propositions use the former.

4.1 Proofs of Proposition 1.8 and 1.9

Proof of Proposition 1.8 We first prove (1.6).

We proceed inductively, beginning with the expression for $P_{(2k-1)a}$. For a base case, using Proposition 1.5 with $k = 1$ and using that $S(w;q) = P_{-1a}(w;q)$ ((2.3) or (1.7)), we have that

$$P_{1a}(w;q) = \frac{q}{1-q}\left(\frac{1}{1-wq} - S(w;q)\right) = \frac{-q}{(q;q^2)_1}\left(S(w;q) - \frac{1}{(wq;q)_1}\right),$$

which agrees with the claimed expression for $k = 1$. Supposing it holds for some $k \in \mathbb{N}$, we have using Proposition 1.5

$$\begin{aligned}
P_{(2k+1)a}(w;q) &= \frac{q^{2k+1}}{1-q^{2k+1}}\left(\frac{w^{2k}q^{2k^2+k}}{(wq;q)_{2k+1}} - P_{(2k-1)a}(w;q)\right)\\
&= \frac{q^{2k+1}}{1-q^{2k+1}}\left(\frac{w^{2k}q^{2k^2+k}}{(wq;q)_{2k+1}}\right.\\
&\qquad\left. + \frac{(-1)^{k+1}q^{k^2}}{(q;q^2)_k}\left(S(w;q) + \sum_{n=1}^{k}\frac{(-1)^n w^{2n-2}q^{n(n-1)}(q;q^2)_{n-1}}{(wq;q)_{2n-1}}\right)\right)\\
&= \frac{(-1)^{k+1}q^{(k+1)^2}}{(q;q^2)_{k+1}}\left(S(w;q) + \sum_{n=1}^{k+1}\frac{(-1)^n w^{2n-2}q^{n(n-1)}(q;q^2)_{n-1}}{(wq;q)_{2n-1}}\right),
\end{aligned}$$

and so the desired expression holds for all positive integers k by induction.

Next, to prove (1.9), we begin from Proposition 1.5 with $k = 2$ and recall that $P_{0a}(w;q) = P_c(w;q) - 1$:

$$\begin{aligned}
P_{2a}(w;q) &= \frac{q^2}{1-q^2}\left(\frac{wq}{(1-wq)(1-wq^2)} - \sum_{n=1}^{\infty}\frac{w^n q^{n(n+1)/2}}{1-wq^n}\right)\\
&= \frac{-q^2}{(q^2;q^2)_1}\left((P_c(w;q)-1) - \frac{wq}{(wq;q)_2}\right),
\end{aligned}$$

and so the claim for $P_{2ka}(w;q)$ holds for $k = 1$. Now, suppose the claim holds for some $k \in \mathbb{N}$. Observe using Proposition 1.5

$$\begin{aligned}
P_{(2k+2)a} &= \frac{q^{2k+2}}{1-q^{2k+2}}\left(\frac{w^{2k+1}q^{(2k+2)(2k+1)/2}}{(wq;q)_{2k+2}} - P_{(2k)a}\right)\\
&= \frac{q^{2k+2}}{1-q^{2k+2}}\left(\frac{w^{2k+1}q^{(2k+2)(2k+1)/2}}{(wq;q)_{2k+2}}\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(-1)^k q^{k(k+1)}}{(q^2;q^2)_k}\left((P_c(w;q)-1)+\sum_{n=1}^{k}\frac{w^{2n-1}(-1)^n q^{n^2}(q^2;q^2)_{n-1}}{(wq;q)_{2n}}\right)\Bigg)\\
&= \frac{q^{(k+2)(k+1)}(-1)^k}{(q^2;q^2)_{k+1}}\left(\frac{w^{2k+1}q^{(k+1)^2}(-1)^k(q^2;q^2)_k}{(wq;q)_{2k+2}}\right.\\
&\qquad\left. -(P_c(w;q)-1)-\sum_{n=1}^{k}\frac{w^{2n-1}(-1)^n q^{n^2}(q^2;q^2)_{n-1}}{(wq;q)_{2n}}\right)\\
&= \frac{q^{(k+2)(k+1)}(-1)^{k+1}}{(q^2;q^2)_{k+1}}\left((P_c(w;q)-1)+\sum_{n=1}^{k+1}\frac{w^{2n-1}(-1)^n q^{n^2}(q^2;q^2)_{n-1}}{(wq;q)_{2n}}\right).
\end{aligned}
$$

This completes the proof of (1.9).

Next we prove (1.7). For $k=1$, we have that

$$
P_{-1a}(w;q)=S(w;q)=(q;q^2)_0\left(S(w;q)-w^{-1}\sum_{n=1}^{0}\frac{q^{2n-1}(w^{-1};q)_{2n-1}}{(q;q^2)_n}\right)
$$

as claimed. Next we suppose the claimed expression for $P_{-2k+1}(w;q)$ holds for some $k\in\mathbb{N}$, and begin by applying Proposition 1.5 with $k\mapsto -2k+1$ so that

$$
\begin{aligned}
P_{(-2k-1)a}(w;q) &= \frac{w^{-2k}q^{(-2k+1)(-2k)/2}}{(wq;q)_{-2k+1}}+\frac{q^{-2k+1}-1}{q^{-2k+1}}\left(P_{(-2k+1)a}(w;q)\right)\\
&= \frac{q^{(2k-1)k}}{w^{2k}(wq;q)_{-2k+1}}+(1-q^{2k-1})\left((q;q^2)_{k-1}\left(S(w;q)-w^{-1}\sum_{n=1}^{k-1}\frac{q^{2n-1}(w^{-1};q)_{2n-1}}{(q;q^2)_n}\right)\right)\\
&= (q;q^2)_k\left(S(w;q)+\frac{q^{(2k-1)k}}{w^{2k}(wq;q)_{-2k+1}(q;q^2)_k}-w^{-1}\sum_{n=1}^{k-1}\frac{q^{2n-1}(w^{-1};q)_{2n-1}}{(q;q^2)_n}\right)\\
&= (q;q^2)_k\left(S(w;q)+\frac{q^{(2k-1)k}(wq^{-2k+2};q)_{2k-1}}{w^{2k}(q;q^2)_k}-w^{-1}\sum_{n=1}^{k-1}\frac{q^{2n-1}(w^{-1};q)_{2n-1}}{(q;q^2)_n}\right)\\
&= (q;q^2)_k\left(S(w;q)+\frac{q^{(2k-1)k}(w;q^{-1})_{2k-1}}{w^{2k}(q;q^2)_k}-w^{-1}\sum_{n=1}^{k-1}\frac{q^{2n-1}(w^{-1};q)_{2n-1}}{(q;q^2)_n}\right)\\
&= (q;q^2)_k\left(S(w;q)-w^{-1}\sum_{n=1}^{k}\frac{q^{2n-1}(w^{-1};q)_{2n-1}}{(q;q^2)_n}\right)
\end{aligned}
$$

as wanted, where we have used the identity

$$
(a;q^{-1})_n=(a^{-1};q)_n(-a)^n q^{-\frac{n(n-1)}{2}}
$$

to obtain the last line above.

The proof of (1.8) follow similarly by induction using Proposition 1.5, and is omitted for brevity's sake. □

Proof of Proposition 1.9 The first identity in the proposition follows by summing (1.6) over k, noting convergence. The second identity follows similarly by multiplying (1.6) by $(-1)^k$ and summing over k. The third and fourth identities in the proposition follow similarly using (1.9), along with the well-known identity (see e.g., [16])

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2} z^k}{(q;q)_k} = (-zq;q)_\infty.$$

□

4.2 Proof of Proposition 1.4

Proof By Proposition 1.8, we have

$$P_{(2k-1)a}(w;q) = \frac{(-1)^k q^{k^2}}{(q;q^2)_k} \left(S(w;q) + \sum_{j=1}^{k} \frac{(-1)^j w^{2j-2} q^{j(j-1)} (q;q^2)_{j-1}}{(wq;q)_{2j-1}} \right), \tag{4.1}$$

and in Lemma 2.3, we showed that

$$S(w;q) = \sum_{j=0}^{\infty} \frac{(-1)^j w^{2j} q^{j(j+1)} (q;q^2)_j}{(wq;q)_{2j+1}}. \tag{4.2}$$

Using (4.1) and (4.2), we find

$$P_{(2k-1)a}(w;q) = \frac{(-1)^{k+1} q^{k^2}}{(q;q^2)_k} \sum_{j=k+1}^{\infty} \frac{(-1)^j w^{2j-2} q^{j(j-1)} (q;q^2)_{j-1}}{(wq;q)_{2j-1}},$$

and hence

$$\begin{aligned} P_{(2k-1)a}(1;q) &= \frac{(-1)^{k+1} q^{k^2}}{(q;q^2)_k} \sum_{j=k+1}^{\infty} \frac{(-1)^j q^{j(j-1)} (q;q^2)_{j-1}}{(q;q)_{2j-1}} \\ &= \sum_{j=k+1}^{\infty} \frac{(-1)^{k+j+1} q^{k^2+j(j-1)}}{(1-q^{2j-1})(q;q^2)_k (q^2;q^2)_{j-1}}. \end{aligned} \tag{4.3}$$

Recall that the generating function for partitions with parts in T, $P_T(q)$, is given by $P_T(q) = \prod_{t\in T}(1-q^t)^{-1}$. Then, letting

$$T_{j,k} = \{1,3,5,...,2k-1\} \cup \{2,4,6,...,2j-2\} \cup \{2j-1\}, \tag{4.4}$$

with the observation that $j > k$ implies that $2j-1 \notin \{1,3,5,...,2k-1\}$, and using (4.3), we find that

$$P_{(2k-1)a}(1;q) = \sum_{j=k+1}^{\infty} (-1)^{k+j+1} q^{k^2+j(j-1)} P_{T_{j,k}}(q).$$

Thus,

$$\sum_{n=0}^{\infty} p_{(2k-1)a}(n) q^n = \sum_{j=k+1}^{\infty} (-1)^{k+j+1} q^{k^2+j(j-1)} \sum_{n=0}^{\infty} p(n : \text{parts in } T_{j,k}) q^n$$

$$= \sum_{j=k+1}^{\infty} \sum_{n=0}^{\infty} (-1)^{k+j+1} p(n : \text{parts in } T_{j,k}) q^{n+k^2+j(j-1)}$$
$$= \sum_{j=k+1}^{\infty} \sum_{n=k^2+j(j-1)}^{\infty} (-1)^{k+j+1} p(n-k^2-j(j-1) : \text{parts in } T_{j,k}) q^n.$$

Now, if $n < k^2+j(j-1)$, then $n-k^2-j(j-1) < 0$, so that $p(n-k^2-j(j-1)) = 0$, hence

$$\sum_{n=0}^{\infty} p_{(2k-1)a}(n) q^n = \sum_{j=k+1}^{\infty} \sum_{n=0}^{\infty} (-1)^{k+j+1} p(n-k^2-j(j-1) : \text{parts in } T_{j,k}) q^n,$$

and hence

$$p_{(2k-1)a}(n) = \sum_{j=k+1}^{\infty} (-1)^{k+j+1} p(n-k^2-j(j-1) : \text{parts in } T_{j,k}).$$

For the other case, recall that by Proposition 1.8, we have

$$P_{2ka}(w;q) = \frac{(-1)^k q^{k(k+1)}}{(q^2;q^2)_k} \left((P_c(w;q)-1) + \sum_{j=1}^{k} \frac{w^{2j-1}(-1)^j q^{j^2} (q^2;q^2)_{j-1}}{(wq;q)_{2j}} \right), \tag{4.5}$$

and in Lemma 2.4, we showed that

$$P_c(w;q) - 1 = -\sum_{j=1}^{\infty} \frac{(-1)^j w^{2j-1} q^{j^2} (q^2;q^2)_{j-1}}{(wq;q)_{2j}}. \tag{4.6}$$

Using (4.5) and (4.6), we find that

$$P_{2ka}(w;q) = \frac{(-1)^{k+1} q^{k(k+1)}}{(q^2;q^2)_k} \sum_{j=k+1}^{\infty} \frac{(-1)^j w^{2j-1} q^{j^2} (q^2;q^2)_{j-1}}{(wq;q)_{2j}},$$

and hence

$$P_{2ka}(1;q) = \frac{(-1)^{k+1} q^{k(k+1)}}{(q^2;q^2)_k} \sum_{j=k+1}^{\infty} \frac{(-1)^j q^{j^2} (q^2;q^2)_{j-1}}{(q;q)_{2j}}$$
$$= \sum_{j=k+1}^{\infty} \frac{(-1)^{j+k+1} q^{k(k+1)+j^2}}{(1-q^{2j})(q^2;q^2)_k (q;q^2)_j}. \tag{4.7}$$

Then, letting

$$E_{j,k} = \{2,4,6,\ldots,2k\} \cup \{1,3,5,\ldots,2j-1\} \cup \{2j\}, \tag{4.8}$$

with the observation that $j > k$ implies that $2j \notin \{2,4,6,\ldots,2k\}$, and using (4.7), we have

$$P_{2ka}(1;q) = \sum_{j=k+1}^{\infty} (-1)^{k+j+1} q^{k(k+1)+j^2} P_{E_{j,k}}(q).$$

Thus,

$$\sum_{n=0}^{\infty} p_{2ka}(n)q^n = \sum_{j=k+1}^{\infty} (-1)^{k+j+1} q^{k(k+1)+j^2} \sum_{n=0}^{\infty} p(n : \text{parts in } E_{j,k})q^n$$
$$= \sum_{j=k+1}^{\infty} \sum_{n=0}^{\infty} (-1)^{k+j+1} p(n : \text{parts in } E_{j,k})q^{n+k(k+1)+j^2}$$
$$= \sum_{j=k+1}^{\infty} \sum_{n=k(k+1)+j^2}^{\infty} (-1)^{k+j+1} p(n-k(k+1)-j^2 : \text{parts in } E_{j,k})q^n.$$

Now, if $n < k(k+1)+j^2$, then $n-k(k+1)-j^2 < 0$, so that $p(n-k(k+1)-j^2) = 0$, hence,

$$\sum_{n=0}^{\infty} p_{2ka}(n)q^n = \sum_{j=k+1}^{\infty} \sum_{n=0}^{\infty} (-1)^{k+j+1} p(n-k(k+1)-j^2 : \text{parts in } E_{j,k})q^n,$$

and thus

$$p_{2ka}(n) = \sum_{j=k+1}^{\infty} (-1)^{k+j+1} p(n-k(k+1)-j^2 : \text{parts in } E_{j,k}).$$

□

5 Proofs III: Quantum Jacobi results

In this section, we prove Theorem 1.1. We first prove the theorem for $\mathcal{P}_{(2k-1)a}$ with $k \in \mathbb{N}$. To do so, we begin by applying (1.6) from Proposition 1.8. By [12, Theorem 4.3] (see also Section 2.1) we have that $\widetilde{\theta}_2(z;-\tau)$ (where $\widetilde{\theta}_2(z;\tau) = q^{\frac{1}{8}} w^{\frac{1}{4}} S(w^{\frac{1}{2}};q)$ with $w = e^{2\pi i z}$, $q = e^{2\pi i\tau}$) is a quantum Jacobi form on

$$S_2 := \left\{\left(\tfrac{a}{b}, \tfrac{r}{s}\right) \in \mathbb{Q}^2 : \tfrac{a}{b}, \tfrac{r}{s} \text{ are reduced, } b \mid s, s \text{ is even, and } as/b \text{ is even}\right\}, \tag{5.1}$$

with weight $1/2$, index $-1/8$ and character ψ_γ, with respect to $\Gamma_0(4) \ltimes (4\mathbb{Z} \times 2\mathbb{Z})$. Moreover, it is shown in [12, Theorem 4.3] that the errors to (Jacobi) modularity of $\widetilde{\theta}_2(z;-\tau)$ on S_2 extend to C^∞ functions on $(\mathbb{R} \setminus (\mathbb{Z} + \{0, \frac{1}{2}, \pm\frac{1}{4}\})) \times (\mathbb{R} \setminus \{-\frac{1}{4}\})$. Here, for $k \in \mathbb{N}$ we define the subsets $T_k \subseteq S_2$ by

$$T_k := \left\{\left(\tfrac{a}{b}, \tfrac{r}{s}\right) \in S_2 : 4bk \mid s\right\}, \tag{5.2}$$

as well as the subgroups $G_k \subseteq \Gamma_0(4)$ by

$$G_k := \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 4k & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle. \tag{5.3}$$

To show that T_k is closed under the Jacobi action of G_k, it suffices to show the added divisibility condition defining T_k is preserved (as closure of S_2 under the Jacobi action of $\Gamma_0(4)$ follows from [12]). Indeed, we have for $\left(\frac{a}{b}, \frac{r}{s}\right) \in T_k$ that

$$\left(\begin{smallmatrix} 1 & 0 \\ 4k & 1 \end{smallmatrix}\right) \cdot \left(\tfrac{a}{b}, \tfrac{r}{s}\right) = \left(\tfrac{A}{B}, \tfrac{R}{S}\right)$$

in reduced form, where

$$A = \pm ab'/g,\ B = S/g,\ B' = g,\ R = \pm r,\ S = |4kr + s|,$$

and

$$b' := s/b = 4ku \text{ (for some } u \in \mathbb{Z}),\ \ g := \gcd(S, ab').$$

We seek to show that $4Bk \mid S$. Note that $g' := g/(4k)$ is an integer, because $4k \mid s$ and hence S, and $4k \mid b'$. Moreover, we have that $g' \mid S$. Thus, $4Bk = S/g'$ is an integer divisor of S as wanted.

We proceed similarly for $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Again considering the reduced form of

$$\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\tfrac{a}{b}, \tfrac{r}{s}\right) = \left(\tfrac{A}{B}, \tfrac{R}{S}\right)$$

for $\left(\frac{a}{b}, \frac{r}{s}\right) \in T_k$ we have

$$A = \pm a,\ B = b,\ R = \pm(r + s),\ S = s,$$

since it is clear that R and S are coprime. Hence the divisibility condition of $4Bk \mid S$ is met. The case of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is clear.

Next, we pick some $(\lambda, \mu) \in (4k\mathbb{Z} \times 2\mathbb{Z})$. To check whether membership of T_k is preserved under the corresponding elliptic transformation, we have

$$\left(\frac{a}{b} + \frac{\lambda r}{s} + \mu, \frac{r}{s}\right) = \left(\frac{as + \lambda rb + \mu bs}{bs}, \frac{r}{s}\right) = \left(\frac{A}{B}, \frac{R}{S}\right)$$

with, defining $g := gcd(as + \lambda rb + \mu bs, bs)$,

$$A = (as + \lambda rb + \mu bs)/g,\ B = bs/g,\ R = r,\ S = s.$$

To show $4Bk \mid S$, or $4bsk \mid sg$, it is equivalent to show $4bk \mid g$, which is clear as $4bk$ divides bs and each term of of $as + \lambda rb + \mu bs$. Hence we have that T_k is closed under the necessary Jacobi transformations.

Let

$$F_k(w; q) := \frac{(-1)^k q^{k^2}}{(q; q^2)_k} \sum_{n=1}^{k} \frac{(-1)^n w^{2n-2} q^{n(n-1)} (q; q^2)_{n-1}}{(wq; q)_{2n-1}}$$

as appearing in (1.6). Given what has been established above, it now suffices to show that $F_k(w^{\frac{1}{2}}; q^{-1})$ and $(q^{-1}; q^{-2})_k^{-1}$ (when viewed as a function of (z, τ) as above) is defined on T_k. It is not difficult to show that the latter function is defined on S_2. Regarding the former, it suffices to show that for $\left(\frac{a}{b}, \frac{r}{s}\right) \in T_k$, we have that $\zeta_{2b}^{a} \zeta_s^{-jr} \neq 1$ for all $1 \leq j \leq 2k - 1$; equivalently, that $ab'/2 - jr \not\equiv 0 \pmod{s}$ for all $1 \leq j \leq 2k - 1$. With notation as in [12] we let r' be an integer such that $rr' \equiv -1 \pmod{s}$. The above is equivalent to establishing that for all j, $1 \leq j \leq 2k - 1$, we have $j \not\equiv -ab'r'/2 \pmod{s}$; equivalently, $j \not\equiv -2kuar' \pmod{4kub}$. If such a congruence did hold, it would imply that $2k \mid j$, which is impossible since $1 \leq j \leq 2k - 1$.

The proof (in the case of $(2k-1)$ with $k \in \mathbb{N}$) now follows from the argument above, as well as Example 5 following our Definition 3, and the fact that rational functions in (w, q) with $w = e^{2\pi i z}, q = e^{2\pi i \tau}$ are holomorphic on their domains in $\mathbb{R} \times \mathbb{R}$; one can check directly they satisfy the required Jacobi properties.

The result for $(2k-1)$ with $k \in -\mathbb{N}_0$ follows similarly to the argument above in the case $k \in \mathbb{N}$, using (1.7) from Proposition 1.8, Theorem 4.3 from [12] (as in Section 2.1) and Definition 3. We also remark that Jacobi elliptic transformation properties may be directly verified. The result for $2k$ with $k \in -\mathbb{N}$ follows in a similar manner.

6 Proofs IV: Proof of Theorem 1.3

In this section, we prove Theorem 1.3, beginning with proofs of the results for p_{1a} and p_{2a}, followed by a (longer) proof of the result for p_{3a}, in Sections 6.1, 6.2, and 6.3, respectively.

6.1 Proof of Theorem 1.3 in the case of $p_{1a}(n)$.

From Corollary 1.7, we have for all $n \geq 0$ that

$$p_{1a}(n+1) - p_{1a}(n) = \begin{cases} 0, & n \text{ is a triangular number,} \\ 1, & \text{otherwise.} \end{cases}$$

From this, combined with the observation that $p_{1a}(0) = 0$, it follows that $p_{1a}(n)$ is the difference between n and the number of nonnegative triangular numbers (including 0) strictly less than n. In particular, $p_{1a}(n) = n - m$, where $m \geq 0$ is the smallest nonnegative integer such that $m(m+1)/2 \geq n$. We remark that $m(m+1)/2 \geq n$ if and only if $m^2 + m - 2n \geq 0$. Furthermore, the roots of the polynomial $f(x) = x^2 + x - 2n$ are given by $x = \frac{1}{2}(\pm\sqrt{8n+1} - 1)$, so since $f(x) < 0$ when $x \in (\frac{1}{2}(-\sqrt{8n+1} - 1), \frac{1}{2}(\sqrt{8n+1} - 1))$, and $\frac{1}{2}(-\sqrt{8n+1} - 1) < 0$, $\frac{1}{2}(\sqrt{8n+1} - 1) \geq 0$, it follows that the smallest nonnegative integer such that $m^2 + m - 2n \geq 0$ is given by $m = \lceil \frac{1}{2}(\sqrt{8n+1} - 1) \rceil$. Hence, $p_{1a}(n) = n - \lceil \frac{1}{2}(\sqrt{8n+1} - 1) \rceil$.

6.2 Proof of Theorem 1.3 in the case of $p_{2a}(n)$.

The result for $p_{2a}(n)$ is easily checked for $n = 0$. For integers $n \geq 1$, we separately induct on odd and even n, beginning with the odd case. To this end, we write $n = 2k + 1$ for some $k \in \mathbb{N}_0$ and induct on k. If $k = 0$, then $p_{2a}(2k+1) = p_{2a}(1) = 0$ as claimed in the statement of the theorem. Now suppose the result holds for some odd $n = 2k+1$, $k \in \mathbb{N}_0$, i.e.,

$$p_{2a}(2k+1) = \frac{k(k+1)}{2} - \sum_{m=1}^{k} d_o(2k+1-2m),$$

(which agrees with the expression in the theorem after some minor simplifications). From Corollary 1.7, we have for each $n \in \mathbb{N}$ that

$$p_{2a}(n+2) - p_{2a}(n) = \left\lfloor \frac{n+1}{2} \right\rfloor - d_o(n),$$

hence

$$p_{2a}(2k+3) = p_{2a}(2k+1) + \left\lfloor \frac{2k+2}{2} \right\rfloor - d_o(2k+1)$$

$$= \frac{k(k+1)}{2} + k + 1 - d_o(2k+1) - \sum_{m=1}^{k} d_o(2k+1-2m)$$
$$= \frac{(k+1)(k+2)}{2} - \sum_{m=1}^{k+1} d_o(2k+3-2m)$$

as wanted.

The proof of the result in the case of even $n = 2k, k \in \mathbb{N}$, follows similarly. For $k = 1$, the result is easily verified. Supposing the result holds for some $k \in \mathbb{N}$, i.e,

$$p_{2a}(2k) = \frac{k(k-1)}{2} - \sum_{m=1}^{k-1} d_o(2k-2m)$$

(which agrees with the statement in the theorem after some minor simplifications) then from this and Corollary 1.7,

$$p_{2a}(2k+2) = p_{2a}(2k) + \left\lfloor \frac{2k+1}{2} \right\rfloor - d_o(2k)$$
$$= \frac{k(k-1)}{2} + k - d_o(2k) - \sum_{m=1}^{k-1} d_o(2k-2m)$$
$$= \frac{(k+1)k}{2} - \sum_{m=1}^{k} d_o(2k+2-2m)$$

as wanted, completing the proof of the result for p_{2a}.

6.3 Proof of Theorem 1.3 in the case of p_{3a}(n).

The proof of the result for $p_{3a}(n)$ is comparatively lengthy (considering the above proofs of the analogous results for p_{ka} for $k \in \{1, 2\}$) and requires several lemmas.

Lemma 6.1 *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, and $k \in \mathbb{N}$, we have that*

$$p_{(2k-1)a}(n : \textit{largest part } m) = (-1)^k p(n - k^2 - \tfrac{m(m+1)}{2} : \textit{parts in } U_k)$$
$$+ \sum_{\ell=0}^{k-1} (-1)^{k+\ell+1} \sum_{j=0}^{\infty} p(n - j - k^2 - \ell(\ell+1) : m - 2\ell \textit{ parts in } T_\ell) p(j : \textit{parts in } S_{k,\ell}),$$

where $S_{k,\ell} := \{1 + 2\ell, 3 + 2\ell, ..., 2k - 1\}$, $T_\ell := \{1, 2, ..., 2\ell + 1\}$, and $U_k := \{1, 3, ..., 2k - 1\}$.

Proof of Lemma 4.1 Fix $k \in \mathbb{N}$. By Proposition 1.8, we have

$$P_{(2k-1)a}(w; q) = \frac{(-1)^k q^{k^2}}{(q; q^2)_k} \left(S(w; q) + \sum_{\ell=1}^{k} \frac{(-1)^\ell w^{2\ell-2} q^{\ell(\ell-1)} (q; q^2)_{\ell-1}}{(wq; q)_{2\ell-1}} \right)$$
$$= \frac{(-1)^k q^{k^2} S(w; q)}{(q; q^2)_k} + \sum_{\ell=0}^{k-1} \frac{(-1)^{k+\ell+1} w^{2\ell} q^{k^2+\ell(\ell+1)}}{(wq; q)_{2\ell+1} (q^{2\ell+1}; q^2)_{k-\ell}}.$$

Recall that the two-variable generating function for partitions of n with m parts in the set T, $P_T(w; q)$, is given by $P_T(w; q) = \prod_{t \in T} (1 - wq^t)^{-1}$. Referring to the sets defined in the statement of the lemma, note that since $0 \le \ell \le k-1$, we have $S_{k,\ell} \neq \emptyset$, $T_\ell \neq \emptyset$, and $U_k \neq \emptyset$. Hence,

$$\begin{aligned}
P_{(2k-1)a}(w;q) &= (-1)^k q^{k^2}\left(\sum_{\ell=0}^{\infty} w^\ell q^{\ell(\ell+1)/2}\right) P_{U_k}(q) \\
&\qquad + \sum_{\ell=0}^{k-1} (-1)^{k+\ell+1} w^{2\ell} q^{k^2+\ell(\ell+1)} P_{T_\ell}(w;q) P_{S_{k,\ell}}(q) \\
&= (-1)^k \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p(n : \text{parts in } U_k) w^m q^{n+k^2+m(m+1)/2}\right) \\
&\quad + \sum_{\ell=0}^{k-1} (-1)^{k+\ell+1} \left(\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} p(n : m \text{ parts in } T_\ell)\right. \\
&\qquad\qquad \left. \times\, p(j : \text{parts in } S_{k,\ell}) w^{m+2\ell} q^{n+j+k^2+\ell(\ell+1)}\right) \\
&= (-1)^k \left(\sum_{m=0}^{\infty} \sum_{n=k^2+\frac{m(m+1)}{2}}^{\infty} p\left(n-k^2-\tfrac{m(m+1)}{2} : \text{parts in } U_k\right) w^m q^n\right) \\
&\quad + \sum_{\ell=0}^{k-1} (-1)^{k+\ell+1} \left(\sum_{m=2\ell}^{\infty}\sum_{j=0}^{\infty} \sum_{n=j+k^2+\ell(\ell+1)}^{\infty}\right. \\
&\qquad \left. p(n-j-k^2-\ell(\ell+1) : m-2\ell \text{ parts in } T_\ell) p(j : \text{parts in } S_{k,\ell}) w^m q^n\right).
\end{aligned}$$

Observe that $n < k^2+m(m+1)/2$ implies that $p(n-k^2-m(m+1)/2 : \text{parts in } S_{k,\ell}) = 0$, and $m < 2\ell$ or $n < j+k^2+\ell(\ell+1)$ implies that $p(n-j-k^2-\ell(\ell+1) : m-2\ell \text{ parts in } T_\ell) = 0$. Hence,

$$\begin{aligned}
P_{(2k-1)a}(w;q) &= (-1)^k \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p\left(n-k^2-\tfrac{m(m+1)}{2} : \text{parts in } U_k\right) w^m q^n \\
&\quad + \sum_{\ell=0}^{k-1} (-1)^{k+\ell+1} \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \\
&\qquad p(n-j-k^2-\ell(\ell+1) : m-2\ell \text{ parts in } T_\ell) p(j : \text{parts in } S_{k,\ell}) w^m q^n.
\end{aligned}$$

Equating series coefficients yields the desired result that for all $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&p_{(2k-1)a}(n : \text{largest part } m) = (-1)^k p\left(n-k^2-\tfrac{m(m+1)}{2} : \text{parts in } U_k\right) \\
&\quad + \sum_{\ell=0}^{k-1} (-1)^{k+\ell+1} \sum_{j=0}^{\infty} p(n-j-k^2-\ell(\ell+1) : m-2\ell \text{ parts in } T_\ell) p(j : \text{parts in } S_{k,\ell}).
\end{aligned}$$

□

Let

$$\begin{aligned}
\xi_1(n) &:= \sum_{m=0}^{\infty} p\left(n-4-\tfrac{m(m+1)}{2} : \text{parts in } \{1,3\}\right), \\
\xi_2(n) &:= \sum_{m=0}^{n-4} p(n-m-4 : \text{parts in } \{1,3\}),
\end{aligned}$$

$$\xi_3(n) := \sum_{m=0}^{\infty} p(n-3m-6 : \text{parts in } \{1,2,3\}).$$

We establish additional lemmas below, which compute ξ_1, ξ_2, and ξ_3. In the statement and proof of Lemma 6.2, we abuse notation and write $a \equiv b \pmod{\frac{1}{3}}$ with $0 \le b < \frac{1}{3}$ (and $a, b \in \mathbb{R}$) to define b as a kind of least residue $\pmod{\frac{1}{3}}$ for a, that is, b is defined to be such that $0 \le b < \frac{1}{3}$, and such that there exists an integer $m \in \mathbb{Z}$ for which $a - b = \frac{1}{3}m$.

Lemma 6.2 *Let* $n \ge 7$, $n \equiv \beta \pmod 3$, $\frac{\sqrt{8n-31}-1}{6} \equiv c \pmod 1$, *and* $\frac{\sqrt{8n-31}-1}{6} \equiv c' \pmod{\frac{1}{3}}$, *where* $0 \le \beta < 3$, $0 \le c < 1$, *and* $0 \le c' < \frac{1}{3}$. *Then we have that*

$$\xi_1(n) = \frac{n\sqrt{8n-31}}{9} + \left(f(c') + \frac{\beta(3-\beta)}{12} - \frac{5}{18}\right)\sqrt{8n-31} + g(c') - h(\beta, c) + \frac{5c}{3} - \frac{c\beta(3-\beta)}{2},$$

where

$$h(x,c) := \begin{cases} \frac{15x^2-33x+14}{36}, & 0 \le c < \frac{1}{3}, \\ \frac{-3x^2-3x+26}{36}, & \frac{1}{3} \le c < \frac{2}{3}, \\ \frac{25}{18} - \frac{5x(3-x)}{12}, & \frac{2}{3} \le c < 1, \end{cases}$$

$$f(x) := \frac{-27x^2+9x+2}{36}, \quad \text{and} \quad g(x) := \frac{18x^3-9x^2-35x+6}{12}.$$

Proof Observe that we can split $\xi_1(n)$ into the three sums,

$$\begin{aligned} \xi_1(n) = &\sum_{m=0}^{\infty} p\left(n-4-\frac{3m(3m+1)}{2} : \text{parts in } \{1,3\}\right) \\ &+ \sum_{m=0}^{\infty} p\left(n-5-\frac{3m(3m+3)}{2} : \text{parts in } \{1,3\}\right) \\ &+ \sum_{m=0}^{\infty} p\left(n-7-\frac{3m(3m+5)}{2} : \text{parts in } \{1,3\}\right). \end{aligned} \tag{6.1}$$

For $m \equiv \nu \pmod 3$ and $m \ge 0$, we note the partition identity

$$p(m : \text{parts in } \{1,3\}) = \sum_{j=0}^{m} p(j : \text{parts in } \{3\}) = \frac{m+3-\nu}{3}. \tag{6.2}$$

For $m \in \mathbb{Z}$, $n-4-\frac{3m(3m+1)}{2} \equiv (n+2) \pmod 3$, $n-5-\frac{3m(3m+3)}{2} \equiv (n+1) \pmod 3$, and $n-7-\frac{3m(3m+5)}{2} \equiv (n+2) \pmod 3$. So, let $n+1 \equiv \nu_1 \pmod 3$, $n+2 \equiv \nu_2 \pmod 3$, with $0 \le \nu_1, \nu_2 < 3$. Let $k_1 = \lfloor \frac{\sqrt{8n-31}-1}{6} \rfloor$, $k_2 = \lfloor \frac{\sqrt{8n-31}-3}{6} \rfloor$, and $k_3 = \lfloor \frac{\sqrt{8n-31}-5}{6} \rfloor$. Then (6.1) with (6.2) gives that

$$\xi_1(n) = \sum_{m=0}^{k_1} \frac{n-1-\frac{3m(3m+1)}{2}-\nu_2}{3} + \sum_{m=0}^{k_2} \frac{n-2-\frac{3m(3m+3)}{2}-\nu_1}{3}$$

$$
\begin{aligned}
&+\sum_{m=0}^{k_3}\frac{n-4-\frac{3m(3m+5)}{2}-\nu_2}{3} \qquad (6.3)\\
&=(k_1+1)\left(\frac{n-1-\nu_2}{3}\right)+(k_2+1)\left(\frac{n-2-\nu_1}{3}\right)+(k_3+1)\left(\frac{n-4-\nu_2}{3}\right)\\
&\quad-\sum_{m=0}^{k_1}\left(\frac{3m^2}{2}+\frac{m}{2}\right)-\sum_{m=0}^{k_2}\left(\frac{3m^2}{2}+\frac{3m}{2}\right)-\sum_{m=0}^{k_3}\left(\frac{3m^2}{2}+\frac{5m}{2}\right). \qquad (6.4)
\end{aligned}
$$

After some algebra, (6.3) becomes

$$
\begin{aligned}
\xi_1(n)&=n+\frac{n}{3}(k_1+k_2+k_3)-\frac{1}{2}(k_1^3+k_2^3+k_3^3)-k_1^2-\frac{3}{2}k_2^2-2k_3^2 \qquad (6.5)\\
&-\frac{(5+2\nu_2)}{6}k_1-\frac{(5+\nu_1)}{3}k_2-\frac{(17+2\nu_2)}{6}k_3-\frac{(7+\nu_1+2\nu_2)}{3}.
\end{aligned}
$$

Let $c=\frac{\sqrt{8n-31}-1}{6}-k_1$. Then $k_1=\frac{\sqrt{8n-31}-1}{6}-c$, $k_2\equiv\frac{\sqrt{8n-31}-3}{6}-(c-\frac{1}{3}) \pmod 1$, and $k_3\equiv\frac{\sqrt{8n-31}-5}{6}-(c-\frac{2}{3}) \pmod 1$. As such, we can simplify $\xi_3(n)$ in three separate cases.

First, suppose $0\le c<\frac{1}{3}$. Then $k_2=\frac{\sqrt{8n-31}-3}{6}-c-\frac{2}{3}$ and $k_3=\frac{\sqrt{8n-31}-5}{6}-c-\frac{1}{3}$. Rather, we can write $k_2=\frac{\sqrt{8n-31}-1}{6}-c-1=k_1-1$ and $k_3=\frac{\sqrt{8n-31}-1}{6}-c-1=k_1-1$. Plugging this in to (6.5) and simplifying gives

$$
\begin{aligned}
\xi_1(n)&=\frac{n}{3}+nk_1-\frac{3}{2}k_1^3-\frac{3}{2}k_1^2-\frac{(4+2\nu_2+\nu_1)}{3}k_1-\frac{1+\nu_2}{3}\\
&=\frac{n\sqrt{8n-31}}{9}+\frac{(-27c^2+9c+2-2(\nu_1+2\nu_2))\sqrt{8n-31}}{36}\\
&+\frac{3c^3}{2}-\frac{3c^2}{4}-\frac{35c}{12}+\frac{1}{2}+\frac{c(\nu_1+2\nu_2)}{3}+\frac{(\nu_1-4\nu_2)}{18}. \qquad (6.6)
\end{aligned}
$$

Similarly, if $\frac{1}{3}\le c<\frac{2}{3}$, then we have $k_2=\frac{\sqrt{8n-31}-3}{6}-c+\frac{1}{3}=\frac{\sqrt{8n-31}-1}{6}-c=k_1$ and $k_3=\frac{\sqrt{8n-31}-5}{6}-c-\frac{1}{3}=\frac{\sqrt{8n-31}-1}{6}-c-1=k_1-1$, so simplifying (6.5) gives

$$
\begin{aligned}
\xi_1(n)&=\frac{2n}{3}+nk_1-\frac{3}{2}k_1^3-3k_1^2-\frac{(17+4\nu_2+2\nu_1)}{6}k_1-\frac{(3+\nu_1+\nu_2)}{3}\\
&=\frac{n\sqrt{8n-31}}{9}+\frac{(-27c^2+27c-4-2(\nu_1+2\nu_2))\sqrt{8n-31}}{36}\\
&+\frac{3c^3}{2}-\frac{9c^2}{4}-\frac{23c}{12}+\frac{4}{3}+\frac{c(\nu_1+2\nu_2)}{3}-\frac{(5\nu_1+4\nu_2)}{18}. \qquad (6.7)
\end{aligned}
$$

Lastly, if $\frac{2}{3}\le c<1$, then we have $k_2=\frac{\sqrt{8n-31}-3}{6}-c+\frac{1}{3}=\frac{\sqrt{8n-31}-1}{6}-c=k_1$ and $k_3=\frac{\sqrt{8n-31}-5}{6}-c+\frac{2}{3}=\frac{\sqrt{8n-31}-1}{6}-c=k_1$, so simplifying (6.5) gives

$$
\begin{aligned}
\xi_1(n)&=n+nk_1-\frac{3}{2}k_1^3-\frac{9}{2}k_1^2-\frac{(32+4\nu_2+2\nu_1)}{6}k_1-\frac{(7+\nu_1+2\nu_2)}{3}\\
&=\frac{n\sqrt{8n-31}}{9}+\frac{(-27c^2+45c-16-2(\nu_1+2\nu_2))\sqrt{8n-31}}{36}\\
&+\frac{3c^3}{2}-\frac{15c^2}{4}+\frac{c}{12}+\frac{5}{3}+\frac{c(\nu_1+2\nu_2)}{3}-\frac{5(\nu_1+2\nu_2)}{18}. \qquad (6.8)
\end{aligned}
$$

Notice that, in (6.6), (6.7), and (6.8), for the terms before the square root,

$$-27c^2+9c+2=-27(c+\tfrac{1}{3})^2+27(c+\tfrac{1}{3})-4=-27(c+\tfrac{2}{3})^2+45(c+\tfrac{2}{3})-16,$$

and for the constant terms,

$$\begin{aligned}18c^3-9c^2-35c+6&=18(c+\tfrac{1}{3})^3-27(c+\tfrac{1}{3})^2-23(c+\tfrac{1}{3})+16\\&=18(c+\tfrac{2}{3})^3-45(c+\tfrac{2}{3})^2+(c+\tfrac{2}{3})+20.\end{aligned}$$

So, if we let

$$f(x):=\frac{-27x^2+9x+2}{36},\quad g(x):=\frac{18x^3-9x^2-35x+6}{12},$$
$$h(n):=\begin{cases}\frac{4\nu_2-\nu_1}{18}, & 0\le c<\frac{1}{3},\\ \frac{5\nu_1+4\nu_2}{18}, & \frac{1}{3}\le c<\frac{2}{3},\\ \frac{5(\nu_1+2\nu_2)}{18}, & \frac{2}{3}\le c<1,\end{cases}$$

for $c\equiv c' \pmod{\frac{1}{3}}$ with $0\le c'<\frac{1}{3}$, we can rewrite (6.6), (6.7), and (6.8) as

$$\xi_1(n)=\frac{n\sqrt{8n-31}}{9}+(f(c')-\frac{(\nu_1+2\nu_2)}{18})\sqrt{8n-31}+g(c')-h(n)+\frac{c(\nu_1+2\nu_2)}{3}.$$

One can verify that, if $n\equiv\beta \pmod 3$, with $0\le\beta<3$, we have

$$\begin{aligned}\xi_1(n)=&\frac{n\sqrt{8n-31}}{9}+(f(c')+\frac{\beta(3-\beta)}{12}-\frac{5}{18})\sqrt{8n-31}+g(c')-h(\beta,c)\\&+\frac{5c}{3}-\frac{c\beta(3-\beta)}{2},\end{aligned}$$

where we abuse notation and redefine $h(*)=h(*,*)$ as

$$h(x,c):=\begin{cases}\frac{15x^2-33x+14}{36}, & 0\le c<\frac{1}{3},\\ \frac{-3x^2-3x+26}{36}, & \frac{1}{3}\le c<\frac{2}{3},\\ \frac{25}{18}-\frac{5x(3-x)}{12}, & \frac{2}{3}\le c<1.\end{cases}$$

□

Lemma 6.3 *Let $n\ge 4$, and $0\le\beta<3$ with $n\equiv\beta \pmod 3$. Then*

$$\xi_2(n)=\frac{n^2}{6}-\frac{n}{2}+\frac{\beta(3-\beta)}{6}.$$

Proof First, observe that

$$\xi_2(n)=\sum_{m=0}^{n-4}p(n-m-4:\text{parts in }\{1,3\})=\sum_{m=0}^{n-4}p(m:\text{parts in }\{1,3\}).$$

We also have the partition identity (see also (6.2))

$$p(m:\text{parts in }\{1,3\})=\sum_{j=0}^{m}p(j:\text{parts in }\{3\}).$$

Thus, we may rewrite

$$\xi_2(n) = \sum_{m=0}^{n-4} \sum_{j=0}^{m} p(j : \text{parts in } \{3\}) = \sum_{m=0}^{n-4} (n-3-m) p(m : \text{parts in } \{3\}).$$

This becomes

$$\xi_2(n) = \sum_{m=0}^{\lfloor \frac{n-4}{3} \rfloor} (n-3-3m) = n \left\lfloor \frac{n-1}{3} \right\rfloor - 3 \left\lfloor \frac{n-1}{3} \right\rfloor - \frac{3 \lfloor \frac{n-4}{3} \rfloor \lfloor \frac{n-1}{3} \rfloor}{2}.$$

If we let $n - 1 \equiv \nu \pmod 3$ where $0 \leq \nu < 3$, then

$$\begin{aligned} \xi_2(n) &= n \left(\frac{n-1-\nu}{3} \right) - 3 \left(\frac{n-1-\nu}{3} \right) - \frac{3 \left(\frac{n-4-\nu}{3} \right) \left(\frac{n-1-\nu}{3} \right)}{2} \\ &= \frac{n^2}{6} - \frac{n}{2} - \frac{(\nu+1)(\nu-2)}{6}. \end{aligned}$$

Thus,

$$\begin{aligned} \xi_2(n) &= \begin{cases} \frac{n^2}{6} - \frac{n}{2}, & n \equiv 0 \pmod 3 \\ \frac{n^2}{6} - \frac{n}{2} + \frac{1}{3}, & n \equiv 1 \pmod 3 \\ \frac{n^2}{6} - \frac{n}{2} + \frac{1}{3}, & n \equiv 2 \pmod 3 \end{cases} \\ &= \frac{n^2}{6} - \frac{n}{2} + \frac{\beta(3-\beta)}{6} \end{aligned}$$

where $n \equiv \beta \pmod 3$, with $0 \leq \beta < 3$. □

Lemma 6.4 *Let $n \geq 6$, and $0 \leq \beta < 3$ with $n \equiv \beta \pmod 3$. Then*

$$\xi_3(n) = \frac{n^3}{108} - \frac{n^2}{24} + \left(\frac{1}{12} - \frac{\beta(3-\beta)}{18} \right) n + r(\beta) + \frac{(-1)^n}{16},$$

where

$$r(\beta) := \begin{cases} \frac{-1}{16}, & \beta = 0, \\ \frac{53}{432}, & \beta = 1, \\ \frac{37}{432}, & \beta = 2. \end{cases}$$

Proof We first note the following partition identity and its inductive successor

$$\begin{aligned} p(m : \text{parts in } \{1,2,3\}) &= p(m-3 : \text{parts in } \{1,2,3\}) + p(m : \text{parts in } \{1,2\}) \\ &= \sum_{j=0}^{\infty} p(m-3j : \text{parts in } \{1,2\}). \end{aligned}$$

Then we have

$$\xi_3(n) = \sum_{m=0}^{\infty} p(n-3m-6 : \text{parts in } \{1,2,3\})$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} p(n - 3(j+m) - 6 : \text{parts in } \{1,2\})$$

$$= \sum_{m=0}^{\infty} (m+1) p(n - 3m - 6 : \text{parts in } \{1,2\})$$

$$= \sum_{m=0}^{\infty} (2m+1) p(n - 6m - 6 : \text{parts in } \{1,2\})$$

$$+ \sum_{m=0}^{\infty} (2m+2) p(n - 6m - 9 : \text{parts in } \{1,2\}). \tag{6.9}$$

Using a partition identity and its inductive successor, with $m \equiv \nu \pmod 6$, $0 \le \nu < 6$, we have that

$$p(m : \text{parts in } \{1,2\}) = p(m-6 : \text{parts in } \{1,2\}) + 3 = p(\nu : \text{parts in } \{1,2\}) + \sum_{j=1}^{\frac{m-\nu}{6}} 3$$
$$= p(\nu : \text{parts in } \{1,2\}) + \frac{m-\nu}{2}. \tag{6.10}$$

Observe that $n - 6j - 6 \equiv n \pmod 6$ and $n - 6j - 9 \equiv n - 3 \pmod 6$. If $n \equiv \alpha \pmod 6$ and $n - 3 \equiv \mu \pmod 6$, with $0 \le \alpha, \mu < 6$, then using (6.9) and (6.10), we have

$$\xi_3(n) = \sum_{m=0}^{\frac{n-6-\alpha}{6}} (2m+1) p(n - 6m - 6 : \text{parts in } \{1,2\})$$
$$+ \sum_{m=0}^{\frac{n-9-\mu}{6}} (2m+2) p(m - 6m - 9 : \text{parts in } \{1,2\})$$
$$= \sum_{m=0}^{\frac{n-6-\alpha}{6}} (2m+1) \left(p(\alpha : \text{parts in } \{1,2\}) + \frac{n - 6m - 6 - \alpha}{2} \right)$$
$$+ \sum_{m=0}^{\frac{n-9-\mu}{6}} (2m+2) \left(p(\mu : \text{parts in } \{1,2\}) + \frac{n - 6m - 9 - \mu}{2} \right)$$
$$= \left(p(\alpha : \text{parts in } \{1,2\}) + \frac{n-\alpha}{2} - 3 \right) \left(\frac{n-\alpha}{6} \right)^2 - \sum_{m=0}^{\frac{n-6-\alpha}{6}} (6m^2 + 3m)$$
$$+ \left(p(\mu : \text{parts in } \{1,2\}) + \frac{n-\mu-9}{2} \right) \left(\frac{n-\mu-3}{6} \right) \left(\frac{n-\mu+3}{6} \right)$$
$$- \sum_{m=0}^{\frac{n-9-\mu}{6}} (6m^2 + 6m). \tag{6.11}$$

Using algebra, identity (6.11) becomes

$$\xi_3(n) = \frac{n^3}{108} - (\alpha + \mu + 6 - 2p(\alpha : \text{parts in } \{1,2\}) - 2p(\mu : \text{parts in } \{1,2\})) \frac{n^2}{72}$$
$$+ (3 + 6\alpha + \alpha^2 + 6\mu + \mu^2 - 4\alpha p(\alpha : \text{parts in } \{1,2\}) - 4\mu p(\mu : \text{parts in } \{1,2\})) \frac{n}{72}$$

$$+\frac{81-18\alpha-9\alpha^2-\alpha^3+9\mu-9\mu^2-\mu^3}{216}$$
$$+\frac{6\alpha^2 p(\alpha : \text{parts in } \{1,2\}) + 6\mu^2 p(\mu : \text{parts in } \{1,2\}) - 54 p(\mu : \text{parts in } \{1,2\})}{216}.$$

Since α determines μ, one can verify that

$$\xi_3(n) = \begin{cases} \frac{n^3}{108} - \frac{n^2}{24} + \frac{n}{12}, & n \equiv 0 \pmod 6, \\ \frac{n^3}{108} - \frac{n^2}{24} - \frac{n}{36} + \frac{13}{216}, & n \equiv 1 \pmod 6, \\ \frac{n^3}{108} - \frac{n^2}{24} - \frac{n}{36} + \frac{4}{27}, & n \equiv 2 \pmod 6, \\ \frac{n^3}{108} - \frac{n^2}{24} + \frac{n}{12} - \frac{1}{8}, & n \equiv 3 \pmod 6, \\ \frac{n^3}{108} - \frac{n^2}{24} - \frac{n}{36} + \frac{5}{27}, & n \equiv 4 \pmod 6, \\ \frac{n^3}{108} - \frac{n^2}{24} - \frac{n}{36} + \frac{5}{216}, & n \equiv 5 \pmod 6, \end{cases}$$
$$= \begin{cases} \frac{n^3}{108} - \frac{n^2}{24} + \frac{n}{12} - \frac{1}{16} + \frac{(-1)^n}{16}, & n \equiv 0 \pmod 3, \\ \frac{n^3}{108} - \frac{n^2}{24} - \frac{n}{36} + \frac{53}{432} + \frac{(-1)^n}{16}, & n \equiv 1 \pmod 3, \\ \frac{n^3}{108} - \frac{n^2}{24} - \frac{n}{36} + \frac{37}{432} + \frac{(-1)^n}{16}, & n \equiv 2 \pmod 3, \end{cases}$$
$$= \frac{n^3}{108} - \frac{n^2}{24} + \left(\frac{1}{12} - \frac{\beta(3-\beta)}{18}\right)n + r(\beta) + \frac{(-1)^n}{16},$$

where $n \equiv \beta \pmod 3$, with $0 \le \beta < 3$, and

$$r(\beta) := \begin{cases} \frac{-1}{16}, & \beta = 0, \\ \frac{53}{432}, & \beta = 1, \\ \frac{37}{432}, & \beta = 2. \end{cases}$$

□

With the above lemmas, we now prove the claimed exact formula for $p_{3a}(n)$ in Theorem 1.3.

Proof of formula for $p_{3a}(n)$ in Theorem 1.3 Using Lemma 6.1 with $k = 2$, we have

$$\begin{aligned} p_{3a}(n : \text{largest part } m) &= p\left(n - 4 - \tfrac{m(m+1)}{2} : \text{parts in } \{1,3\}\right) \\ &\quad - \sum_{j=0}^{\infty} p(n-j-4 : m \text{ parts in } \{1\}) p(j : \text{parts in } \{1,3\}) \\ &\quad + \sum_{j=0}^{\infty} p(n-j-6 : m-2 \text{ parts in } \{1,2,3\}) p(j : \text{parts in } \{3\}). \end{aligned}$$

Notice that $p(j : \text{parts in } \{3\}) = \begin{cases} 1, & 3 \mid j, \\ 0, & \text{else,} \end{cases}$ so we can write

$$\begin{aligned} &\sum_{j=0}^{\infty} p(n-j-6 : m-2 \text{ parts in } \{1,2,3\}) p(j : \text{parts in } \{3\}) \\ &\quad = \sum_{j=0}^{\infty} p(n-3j-6 : m-2 \text{ parts in } \{1,2,3\}). \end{aligned}$$

Summing over m, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} p_{3a}(n : \text{largest part } m) &= \sum_{m=0}^{\infty} p(n-4-\tfrac{m(m+1)}{2} : \text{parts in } \{1,3\}) \\ &\quad - \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} p(n-j-4 : m \text{ parts in } \{1\}) p(j : \text{parts in } \{1,3\}) \\ &\quad + \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} p(n-3j-6 : m-2 \text{ parts in } \{1,2,3\}).\end{aligned} \tag{6.12}$$

Now,

$$p(n-j-4 : m \text{ parts in } \{1\}) = \begin{cases} 1, & j = n-4-m, \\ 0, & \text{else,} \end{cases}$$

and there exists some $j \geq 0$ such that $j = n-4-m$ if and only if $m \leq n-4$. Thus,

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} p(n-j-4 : m \text{ parts in } \{1\}) p(j : \text{parts in } \{1,3\}) &= \sum_{m=0}^{n-4} p(n-4-m : \text{parts in } \{1,3\}) \\ &= \xi_2(n).\end{aligned} \tag{6.13}$$

Similarly, we also have:

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} p(n-3j-6 : m-2 \text{ parts in } \{1,2,3\}) &= \sum_{j=0}^{\infty} p(n-3j-6 : \text{parts in } \{1,2,3\}) \\ &= \xi_3(n).\end{aligned} \tag{6.14}$$

Recall that

$$\sum_{m=0}^{\infty} p(n-4-\tfrac{m(m+1)}{2} : \text{parts in } \{1,3\}) = \xi_1(n). \tag{6.15}$$

Using (6.12) with (6.13), (6.14), and (6.15), we have

$$p_{3a}(n) = \xi_1(n) - \xi_2(n) + \xi_3(n).$$

We combine Lemmas 6.2, 6.3, and 6.4 to obtain for all $n \geq 7$, where $n \equiv \beta \pmod 3$, $\frac{\sqrt{8n-31}-1}{6} \equiv c \pmod 1$, and $\frac{\sqrt{8n-31}-1}{6} \equiv c' \pmod{\frac{1}{3}}$, with $0 \leq \beta < 3$, $0 \leq c < 1$, and $0 \leq c' < \frac{1}{3}$,

$$\begin{aligned}&p_{3a}(n) \\ &= \frac{n^3}{108} - \frac{5n^2}{24} + \frac{n\sqrt{8n-31}}{9} + \left(\frac{7}{12} - \frac{\beta(3-\beta)}{18}\right) n + \left(f(c') + \frac{\beta(3-\beta)}{12} - \frac{5}{18}\right)\sqrt{8n-31} \\ &\quad + g(c') - h(\beta, c) + r(\beta) + \frac{(-1)^n}{16} - \frac{\beta(3-\beta)(3c+1)}{6} + \frac{5c}{3},\end{aligned}$$

where $f(x)$, $g(x)$, $h(x, c)$, and $r(x)$ are defined as in Lemmas 6.2, 6.3, and 6.4.

If we let $a_{0,n} = g(c') - h(\beta, c) + r(\beta) + \frac{(-1)^n}{16} - \frac{\beta(3-\beta)(3c+1)}{6} + \frac{5c}{3}$, $a_{1,n} = f(c') + \frac{\beta(3-\beta)}{12} - \frac{5}{18}$, and $a_{2,n} = \frac{7}{12} - \frac{\beta(3-\beta)}{18}$, where each are bounded in absolute value by a fixed constant independent of n, we obtain the desired result that, when $n \geq 7$,

$$p_{3a}(n) = \frac{n^3}{108} - \frac{5n^2}{24} + \frac{n\sqrt{8n-31}}{9} + a_{2,n}n + a_{1,n}\sqrt{8n-31} + a_{0,n}.$$

In particular, $a_{0,n} = g(c') + \frac{(-1)^n}{16} + q(c, \beta)$, where

$$q(c,\beta) := \begin{cases} -\frac{65}{144} + \frac{5c}{3}, & \beta = 0,\ c \in [0, \frac{1}{3}), \\ -\frac{43}{432} + \frac{2c}{3}, & \beta = 1,\ c \in [0, \frac{1}{3}), \\ -\frac{203}{432} + \frac{2c}{3}, & \beta = 2,\ c \in [0, \frac{1}{3}), \\ -\frac{113}{144} + \frac{5c}{3}, & \beta = 0,\ c \in [\frac{1}{3}, \frac{2}{3}), \\ -\frac{331}{432} + \frac{2c}{3}, & \beta = 1,\ c \in [\frac{1}{3}, \frac{2}{3}), \\ -\frac{203}{432} + \frac{2c}{3}, & \beta = 2,\ c \in [\frac{1}{3}, \frac{2}{3}), \\ -\frac{209}{144} + \frac{5c}{3}, & \beta = 0,\ c \in [\frac{2}{3}, 1), \\ -\frac{331}{432} + \frac{2c}{3}, & \beta = 1,\ c \in [\frac{2}{3}, 1), \\ -\frac{347}{432} + \frac{2c}{3}, & \beta = 2,\ c \in [\frac{2}{3}, 1). \end{cases}$$

After a calculation, it follows that $a_{0,n} \in (-\frac{191}{216}, \frac{25}{54}]$. Similarly, we find that $a_{1,n} \in (-\frac{2}{9}, -\frac{5}{144})$, and $a_{2,n} \in \{\frac{17}{36}, \frac{7}{12}\}$. □

7 On a conjecture of Xiong

A function related to some studied in this paper was investigated in [21], namely

$$h(q) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}}{1-q^n} =: \sum_{n=1}^{\infty} h(n)q^n,$$

e.g., compare to our $P_{0a}(w;q)$ (which also equals $P_c(w;q) - 1$). We state Conjecture 3.1 from [21].

Conjecture *(Conjecture 3.1, [21]) The function $h(n)$ can attain each positive integer infinitely many times. In particular, we have* $\limsup h(n) = \infty$.

We provide a proof of the latter half of [21, Conjecture 3.1], which to our knowledge has not been recorded previously, and also offer two equivalent combinatorial interpretations, one related to partitions with consecutive parts, namely to

$$p_{c,o}(n) := p(n\colon \text{odd number of consecutive parts}),$$
$$p_{c,e}(n) := p(n\colon \text{even number of consecutive parts}),$$

and the other to a divisor function

$$d_m(n) := \sum_{\substack{d \mid n \\ d \in [\sqrt{n/2}, \sqrt{2n})}} 1$$

which counts the number of "middle divisors" of n.

Proposition 7.1 *We have that*

$$\limsup_{n\to\infty} h(n) = \infty.$$

Equivalently, we have that

$$\limsup_{n\to\infty} (p_{c,o}(n) - p_{c,e}(n)) = \infty;$$

or

$$\limsup_{n\to\infty} d_m(n) = \infty.$$

Proof By an argument similar to the one provided above establishing that P_c is a generating function for partitions into consecutive parts, we conclude that $h(n) = p_{c,o}(n) - p_{c,e}(n)$. Further, it is shown in [5] (see also [4, 19]) that $h(n) = d_m(n)$. Thus, we have established the equivalances stated in the proposition; hence, it suffices to show that $\limsup_{n\to\infty} d_m(n) = \infty$. A proof of this fact is established in [20]. □

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