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Ramanujan's radial limits

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Abstract. Ramanujan's famous deathbed letter to G. H. Hardy concerns the asymptotic properties of modular forms and his so-called *mock theta functions*. For his mock theta function $f(q)$, he asserts, as q approaches an even order $2k$ root of unity, that we have

$$f(q) - (-1)^k(1-q)(1-q^3)(1-q^5)\cdots\bigl(1-2q+2q^4-\cdots\bigr) = O(1).$$

We give two proofs of this claim by offering exact formulas for these limiting values. One formula is a specialization of a general result which relates Dyson's rank mock theta function and the Andrews-Garvan crank modular form. The second formula is ad hoc, and it relies on the elementary manipulation of q-series which are themselves mock modular forms. Both proofs show that the $O(1)$ constants are not mysterious; they are values of special q-series which are finite sums at even order roots of unity.

1. Introduction and Statement of Results

Ramanujan's enigmatic last letter to Hardy [**7**] gave tantalizing hints of his theory of *mock theta functions*. By work of Zwegers [**31, 32**], it is now known that Ramanujan's examples are essentially holomorphic parts of weight $1/2$ *harmonic weak Maass forms* (see [**13**] for their definition). This realization has resulted in many applications in combinatorics, number theory, physics, and representation theory (for example, see [**25, 29**]).

Here we revisit Ramanujan's original claims from this letter [**7**]. The letter begins by summarizing the asymptotic properties, near roots of unity, of *Eulerian series* which are modular forms. He then asks whether other Eulerian series with similar asymptotics are necessarily the sum of a modular form and a function which is $O(1)$ at all roots of unity. He writes:

"The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function."

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Ramanujan offers a specific example for the q-hypergeometric function

$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots, \tag{1.1}$$

which he claims (but does not prove) is a mock theta function according to his imprecise definition. This function is convergent for $|q| < 1$ and those roots of unity q with odd order. For even order roots of unity, it has exponential singularities. For example, as $q \to -1$, we have

$$f(-0.994) \sim -1.08 \cdot 10^{31}, \quad f(-0.996) \sim -1.02 \cdot 10^{46}, \quad f(-0.998) \sim -6.41 \cdot 10^{90}.$$

To cancel the exponential singularity at $q = -1$, Ramanujan found the function $b(q)$, which is modular[1] up to multiplication by $q^{-\frac{1}{24}}$, defined by

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots\left(1 - 2q + 2q^4 - \cdots\right). \tag{1.2}$$

The exponential behavior illustrated above is canceled in the numerics below.

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\dots$	$3.969\dots$	$3.976\dots$	$3.984\dots$	$3.992\dots$

It appears that $\lim_{q\to -1}(f(q) + b(q)) = 4$. More generally, as q approaches an even order $2k$ root of unity radially within the unit disk, Ramanujan claimed that

$$f(q) - (-1)^k b(q) = O(1). \tag{1.3}$$

Ramanujan's point is that $b(q)$ is a "near miss", a modular form which almost cuts out the exponential singularities of $f(q)$. He asserts that $b(q)$ cuts out the exponential singularities of $f(q)$ for half of the even order roots of unity, while $-b(q)$ cuts out the exponential singularities for the remaining even order roots of unity. Of course, if $f(q)$ is a mock theta function according to his definition, then there are no modular forms which exactly cut out its exponential singularitites.

Remark. Claim (1.3) is intimately related to the problem of determining the asymptotics of the coefficients of $f(q)$. Andrews [**1**] and Dragonette [**16**] obtained asymptotics for these coefficients, and Bringmann and the second author [**11**] later obtained an exact formula for these coefficients.

In a recent paper [**20**], the authors proved (1.3) by obtaining a simple closed formula for the implied $O(1)$ constants.

Theorem 1.1 (Theorem 1.1 of [**20**]). *If ζ is a primitive even order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(f(q) - (-1)^k b(q)\right) = -4 \cdot \sum_{n=0}^{k-1} (1+\zeta)^2(1+\zeta^2)^2 \cdots (1+\zeta^n)^2 \zeta^{n+1}.$$

Example. Since empty products equal 1, Theorem 1.1 confirms that

$$\lim_{q\to -1}\left(f(q) + b(q)\right) = 4.$$

[1] Here $q^{-\frac{1}{24}} b(q)$ is modular with respect to z where $q := e^{2\pi i z}$.

EXAMPLE. For $k = 2$, Theorem 1.1 gives $\lim_{q\to i}(f(q) - b(q)) = 4i$. The table below nicely illustrates this fact:

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$\sim 1.9 \cdot 10^6 - 4.6 \cdot 10^6 i$	$\sim 1.6 \cdot 10^8 - 3.9 \cdot 10^8 i$	$\sim 1.0 \cdot 10^{12} - 2.5 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.0577 + 3.855i$	$\sim 0.0443 + 3.889i$	$\sim 0.0303 + 3.924i$

REMARK. Zudilin [**30**] has recently obtained an elementary proof of Theorem 1.1.

Theorem 1.1 is a special case of a more general theorem, one which surprisingly relates two well-known q-series: Dyson's rank function $R(w;q)$ and the Andrews-Garvan crank function $C(w;q)$. These series play a prominent role in the study of integer partition congruences (for example, see [**4, 6, 12, 17, 24**]).

To define these series, throughout we let $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$ and for $n \in \mathbb{Z}$

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Dyson's rank function is given by

$$R(w;q) = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} N(m,n)w^m q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n \cdot (w^{-1}q;q)_n}. \tag{1.4}$$

Here $N(m,n)$ is the number of partitions of n with rank m, where the rank of a partition is defined to be its largest part minus the number of its parts. If $w \neq 1$ is a root of unity, then it is known that $R(w;q)$ is (up to a power of q) a mock theta function (i.e. the holomorphic part of a weight 1/2 harmonic Maass form) (for example, see [**12**] or [**29**]). The Andrews-Garvan crank function is defined by

$$C(w;q) = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} M(m,n)w^m q^n := \frac{(q;q)_\infty}{(wq;q)_\infty \cdot (w^{-1}q;q)_\infty}. \tag{1.5}$$

Here $M(m,n)$ is the number of partitions of n with crank m [**4**]. For roots of unity w, $C(w;q)$ is (up to a power of q) a modular form. We also require the series $U(w;q)$ [**2, 5, 14, 26**], which arises in the study of unimodal sequences. This q-hypergeometric series is defined by

$$U(w;q) = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} u(m,n)(-w)^m q^n := \sum_{n=0}^{\infty}(wq;q)_n \cdot (w^{-1}q;q)_n q^{n+1}. \tag{1.6}$$

Here $u(m,n)$ is the number of strongly unimodal sequences of size n with rank m [**14**].

Theorem 1.1 is a special case of the following theorem which relates these three q-series. Throughout, we let $\zeta_n := e^{2\pi i/n}$.

THEOREM 1.2. (Theorem 1.2 of [**20**]) *Let $1 \le a < b$ and $1 \le h < k$ be integers with $\gcd(a,b) = \gcd(h,k) = 1$ and $b \mid k$. If h' is an integer satisfying $hh' \equiv -1 \pmod{k}$, then, as q approaches ζ_k^h radially within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\left(R(\zeta_b^a;q) - \zeta_{b^2}^{-a^2h'k}C(\zeta_b^a;q)\right) = -(1-\zeta_b^a)(1-\zeta_b^{-a}) \cdot U(\zeta_b^a;\zeta_k^h).$$

Four remarks.
1) There is an integer $c(a,b,h,k)$ such that the limit in Theorem 1.2 reduces to the finite sum

$$-(1-\zeta_b^a)(1-\zeta_b^{-a})\sum_{n=0}^{c(a,b,h,k)}(\zeta_b^a\zeta_k^h;\zeta_k^h)_n\cdot(\zeta_b^{-a}\zeta_k^h;\zeta_k^h)_n\cdot\zeta_k^{h(n+1)}.$$

2) Theorem 1.1 is the $a=1$ and $b=2$ case of Theorem 1.2 because $R(-1;q)=f(q)$, combined with the elementary fact that $C(-1;q)=b(q)$.

3) A variant of Theorem 1.2 holds when $b\nmid k$. This is obtained by modifying the proof to guarantee that the two resulting asymptotic expressions match.

4) Garvan [**21**] was the first to compare the asymptotics of the rank and crank generating functions. His observations were made in the context of the moments for the rank and crank statistic. A precise form of these results has been obtained by Bringmann and Mahlburg and the third author [**9, 10**].

Theorem 1.2, which in turn implies Theorem 1.1, relies on an identity of Choi [**15**] and Ramanujan (see Entry 3.4.7 in [**3**]). This identity reduces the proof of Theorem 1.2 to the claim, upon appropriate specialization of variables, that a certain mixed mock modular form is asymptotic to a suitable multiple of the modular crank function.

Here we offer a second proof of (1.3). This proof is ad hoc, and gives a different formula for the $O(1)$ constants in (1.3).

Theorem 1.3. *If ζ is a primitive even order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have*

$$\lim_{q\to\zeta}\left(f(q)-(-1)^k b(q)\right)$$
$$=\begin{cases} 4\displaystyle\sum_{n=0}^{\frac{k}{2}-1}(-1)^n\zeta^{n+1}(1+\zeta^2)(1+\zeta^4)\cdots(1+\zeta^{2n}) & \text{if } k\equiv 0 \pmod 2,\\ 2+2\displaystyle\sum_{n=0}^{\frac{k-1}{2}}(-1)^{n+1}\zeta^{2n+1}(1+\zeta)(1+\zeta^3)\cdots(1+\zeta^{2n-1}) & \text{if } k\equiv 1 \pmod 2.\end{cases}$$

Remark. We leave it as a challenge to find an elementary proof that the constants appearing in Theorem 1.3 match those appearing in Theorem 1.1.

In Section 2 we sketch[2] the proof of Theorem 1.2, and in Section 3 we give the proof of Theorem 1.3. In the last section we conclude with a discussion of the observations which played a role in the discovery of Theorem 1.3.

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[2]The complete proof is contained in [**20**].

2. Sketch of the proof of Theorem 1.2

The proof of Theorem 1.2 requires the theory of modular units and mock modular forms. In particular, we employ the modular properties of Dedekind's η-function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

and certain Klein forms $\mathfrak{t}_{(r,s)}(z) = \mathfrak{t}_{(r,s)}^{(N)}(z)$ [**23**]. We also require the Appell-Lerch function which played an important role in the work of Zwegers [**32**] on Ramanujan's mock theta functions. For $q = e^{2\pi i z}$, $z \in \mathbb{H}$, and $u, v \in \mathbb{C} \setminus (\mathbb{Z}z + \mathbb{Z})$, this function is defined by

$$\mu(u, v; z) := \frac{e^{\pi i u}}{\vartheta(v; z)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}.$$

Here the Jacobi theta function is defined by

$$\begin{aligned}\vartheta(v; z) &:= i \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{2\pi i v (n+\frac{1}{2})} \\ &= -iq^{\frac{1}{8}} e^{-\pi i v} \prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i v}q^{n-1})(1-e^{-2\pi i v}q^n).\end{aligned} \tag{2.1}$$

The μ-function satisfies the following beautiful bilateral series identity.

Theorem 2.1 (see p.67 of [**3**]). *Let $q = e^{2\pi i z}$, where $z \in \mathbb{H}$. For suitable complex numbers $\alpha = e^{2\pi i u}$ and $\beta = e^{2\pi i v}$, we have*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\alpha\beta)^n q^{n^2}}{(\alpha q; q)_n (\beta q; q)_n} &+ \sum_{n=1}^{\infty} q^n (\alpha^{-1}; q)_n (\beta^{-1}; q)_n \\ &= iq^{\frac{1}{8}}(1-\alpha)(\beta\alpha^{-1})^{\frac{1}{2}} \left(q\alpha^{-1}; q\right)_\infty \left(\beta^{-1}; q\right)_\infty \mu(u, v; z).\end{aligned}$$

Remark. Theorem 2.1 is also obtained in a beautiful paper by Choi [**15**].

To make use of this identity, we study the function $A(u, v; z) := \vartheta(v; z)\mu(u, v; z)$ which was previously studied by Zwegers [**33**] and the first author and Bringmann [**8**]. We employ the "completed" function $\widehat{A}(u, v; z)$, defined by Zwegers as

$$\widehat{A}(u, v; z) := A(u, v; z) + \frac{i}{2}\vartheta(v; z) R(u - v; z), \tag{2.2}$$

where

$$\begin{aligned}R(v; z) := \sum_{n\in\mathbb{Z}} &\left\{ \operatorname{sgn}\left(n + \frac{1}{2}\right) - E\left(\left(n + \frac{1}{2} + \frac{\operatorname{Im}(v)}{\operatorname{Im}(z)}\right)\sqrt{2 \cdot \operatorname{Im(z)}}\right)\right\} \\ &\times (-1)^n q^{-\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{-2\pi i v\left(n+\frac{1}{2}\right)},\end{aligned}$$

and for $w \in \mathbb{C}$ we have

$$E(w) := 2\int_0^w e^{-\pi u^2} du.$$

Using the transformation properties [**32**] of the functions μ and ϑ, we have, for integers m, n, r, s and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, that

(2.3)

$$\widehat{A}\,(u+mz+n,v+rz+s;z)=(-1)^{m+n}e^{2\pi iu(m-r)}e^{-2\pi ivm}q^{\frac{m^2}{2}-mr}\widehat{A}(u,v;z),$$

$$\widehat{A}\left(\frac{u}{cz+d},\frac{v}{cz+d};\gamma z\right)=(cz+d)e^{\pi ic\frac{(-u^2+2uv)}{(cz+d)}}\widehat{A}(u,v;z). \tag{2.4}$$

Sketch of the Proof of Theorem 1.2. The proof makes use of the modular transformation properties described above. We consider Choi's identity with $\alpha=\zeta_b^{-a}$ and $\beta=\zeta_b^a$, (hence $u=-\frac{a}{b}$, $v=\frac{a}{b}$), and q replaced by $e^{\frac{2\pi i}{k}(h+iz)}$, and we define

$$m(a,b;u):=ie^{\frac{\pi iu}{4}}(1-\zeta_b^{-a})\zeta_b^a(\zeta_b^ae^{2\pi iu};e^{2\pi iu})_\infty(\zeta_b^{-a};e^{2\pi iu})_\infty. \tag{2.5}$$

To prove Theorem 1.2, noting that the function $U(\zeta_b^a;\zeta_k^h)$ is a finite convergent sum when $b|k$, it suffices to prove that upon appropriate specialization of variables, the mixed mock modular form $m\cdot\mu$ is asymptotic to a suitable multiple of the modular crank generating function C.

To be precise, let $b|k, \gcd(a,b)=1, \gcd(h,k)=1$, where a,b,h,k are positive integers. As $z\to 0^+$, we must prove that

$$m\left(a,b;\frac{1}{k}\,(h+iz)\right)\mu\left(-\frac{a}{b},\frac{a}{b};\frac{1}{k}\,(h+iz)\right)\sim\zeta_{b^2}^{-a^2h'k}C\left(\zeta_b^a;\frac{1}{k}(h+iz)\right). \tag{2.6}$$

Above and in what follows, we let $z\in\mathbb{R}^+$, and let $z\to 0^+$. This corresponds to the radial limit $q=e^{\frac{2\pi i}{k}(h+iz)}\to\zeta_k^h$ from within the unit disk.

The claim (2.6) is obtained by comparing separate asymptotic results for the crank function and the mixed mock modular form. To describe this, we let

$$q:=e^{\frac{2\pi i}{k}(h+iz)},\quad q_1:=e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}. \tag{2.7}$$

For the mixed mock modular $m\cdot\mu$, we obtain the following asymptotics. Let $b|k, \gcd(a,b)=1, \gcd(h,k)=1$, where a,b,h,k are positive integers, and let b' and h' be positive integers such that $bb'=k$ and $hh'\equiv -1 \pmod{k}$. For $z\in\mathbb{R}^+$, as $z\to 0^+$, we established in Theorem 3.2 of [**20**] that there is an $\alpha>1/24$ for which

$$\begin{aligned}&m\left(a,b;\frac{1}{k}\,(h+iz)\right)\mu\left(-\frac{a}{b},\frac{a}{b};\frac{1}{k}\,(h+iz)\right)\\&\qquad=\left(\frac{i}{z}\right)^{\frac{1}{2}}(\psi(\gamma))^{-1}q^{\frac{1}{24}}q_1^{-\frac{1}{24}}(-1)^{ab'}\zeta_{2b}^{ah'-a}\zeta_{2b^2}^{-3a^2kh'}\frac{\zeta_b^a-1}{1-\zeta_b^{ah'}}(1+O(q_1^\alpha)).\end{aligned}$$

Here, $\gamma=\gamma(h,k)\in\mathrm{SL}_2(\mathbb{Z})$, and $\psi(\gamma)$ is a 24th root of unity, both of which are defined in [**20**].

Under the same hypotheses, we show in Proposition 3.3 of [**20**] that the modularity of Dedekind's eta-function and the Klein forms implies, for the crank function, that

$$\begin{aligned}&C\left(\zeta_b^a;\frac{1}{k}(h+iz)\right)\\&\qquad=\left(\frac{i}{z}\right)^{\frac{1}{2}}(\psi(\gamma))^{-1}q^{\frac{1}{24}}q_1^{-\frac{1}{24}}(-1)^{ab'}\zeta_{2b}^{ah'-a}\zeta_{2b^2}^{-a^2kh'}\frac{\zeta_b^a-1}{1-\zeta_b^{ah'}}(1+O(q_1^\beta)),\end{aligned} \tag{2.8}$$

for some $\beta > 1/24$. Combining these asymptotics gives (2.6), which in turn implies Theorem 1.2.

$\square$

It is important to explain the automorphic reasons which underlie the asymptotic relationship of Theorem 1.2. The rank generating function $R(w;q)$ and the crank generating function $C(w;q)$ are Jacobi forms with the same weight and multiplier. This coincidence explains why their radial asymptotics are closely related. However, they are not of the same index, and this difference accounts for the $(-1)^k$ in Theorem 1.1, and the $\zeta_{b^2}^{-a^2h'k}$ in Theorem 1.2. Of course, these facts alone do not directly lead to Theorem 1.2. To make this step requires the bilateral series of Theorem 2.1, namely,

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(wq)_n(w^{-1}q)_n} = R(w;q) + (1-w)(1-w^{-1})\sum_{n=0}^{\infty} q^{n+1}(wq)_n(w^{-1}q)_n.$$

It turns out that this series is related to a mixed-mock Jacobi form which has asymptotics resembling that of crank generating function. Although such coincidences are mysterious, it is not uncommon that such a bilateral series possesses better modular properties than either half of the series (See [**22**] for more dealing with bilateral series and mock theta functions).

3. Proof of Theorem 1.3

Here we prove Theorem 1.3, a second formula for the $O(1)$ constants in (1.3).

Proof of Theorem 1.3. In his deathbed letter, Ramanujan defined four "third order" mock theta functions. Three of them are $f(q)$, which we saw above, and

$$\phi(q) := 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}$$

$$\psi(q) := \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}.$$

In his final letter, Ramanujan stated the following relation between these three functions

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = b(q), \tag{3.1}$$

where $b(q)$ is defined in (1.2). These relations were proved by Watson [**28**]. Later, Fine [**19**] gave elegant proofs which rely on his detailed study of the basic hypergeometric series

$$F(a,b;t) = F(a,b;t,q) := \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n} t^n. \tag{3.2}$$

From (3.1), we have

$$f(q) - b(q) = -4\psi(-q). \tag{3.3}$$

Note that if $q \to \zeta$ and ζ is a $2k$ root of unity with k even, then both functions on the left hand side of (3.3) must have singularities, while the function on the right hand side of (3.3) is a constant. Likewise, if $q \to \zeta$ and ζ is a $2k$ root of unity with k odd, then we have

$$f(q) + b(q) = 2\phi(-q). \tag{3.4}$$

The function on the right hand side of (3.4) tends to a finite number, while the functions on the left hand side of (3.4) each have singularities.

Therefore, to establish the theorem it is enough to show that

$$\psi(q) = \sum_{n=0}^{\infty} q^{n+1}(-q^2;q^2)_n \quad \text{and} \quad \phi(q) = 1 + \sum_{n=0}^{\infty} q^{2n+1}(-1)^n(q;q^2)_n. \tag{3.5}$$

Both of these identities follow as special cases of Fine [**19**] (7.31), which is

$$F(b/t, 0; t) = \frac{1}{1-t}\sum_{n=0}^{\infty} \frac{(-b)^n q^{\frac{n^2+n}{2}}}{(tq;q)_n}.$$

To prove the first claim in (3.5), we take $b = -q^{\frac{1}{2}}$ and $t = q^{\frac{1}{2}}$ to obtain

$$\frac{1}{1-q^{\frac{1}{2}}}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}+n}}{(q^{\frac{3}{2}};q)_n} = \sum_{n=0}^{\infty} q^{\frac{n}{2}}(-q;q)_n.$$

Sending $q \to q^2$ and multiplying by q gives the first claim. To prove the second claim in (3.5), take $b = -q^{\frac{1}{2}}$ and $t = -q$ to obtain

$$\frac{1}{1+q}\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}+n}}{(-q^2;q)_n} = \sum_{n=0}^{\infty}(-q)^n(q^{\frac{1}{2}};q)_n.$$

Sending $q \to q^2$ and multiplying by q, then adding 1, gives the second claim of (3.5). $\square$

4. Discussion related to Theorem 1.3

The remainder of the paper will explain the discovery of Theorem 1.3. Recall that

$$\lim_{t\to 0^+} \left(f(-e^{-t}) + b(-e^{-t})\right) = -4U(-1;-1) = 4.$$

However, the calculations of [**20**] yield the stronger asymptotic relation

$$f(-e^{-t}) + b(-e^{-t}) = -4U(-1;-e^{-t}) \quad \text{as } t \to 0^+.$$

Computation gives

$$-U(-1;-e^{-t}) = 1 - t + 7\frac{t^2}{2!} - 127\frac{t^3}{3!} + 4315\frac{t^4}{4!} - 235831\frac{t^5}{5!} + 1811467\frac{t^6}{6!} - \cdots \tag{4.1}$$

The coefficients of this t-series can be given in *closed form*.

Theorem 4.1. *We have*

$$f(-e^{-t}) + b(-e^{-t}) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(-t)^n \quad \textit{as } t \to 0^+,$$

where the a_n are given by

$$a_n = (-1)^n 2 \sum_{a+2b+c=n} \frac{n!}{a!(2b)!c!} \left(\frac{3}{2}\right)^a \left(\frac{5}{2}\right)^{2b} E_{2a+2b}$$
$$+ 2(-1)^n \sum_{a+2b=n} \frac{n!}{a!(2b)!} \left(\frac{3}{2}\right)^a \left(\frac{1}{2}\right)^{2b} E_{2a+2b},$$

where E_n are the Euler numbers, and summation is taken over $a, b, c \in \mathbb{N}_0$.

This closed form is not enlightening. For instance, it is not clear from this formula that the a_n are all positive.

The significance of this result lies in (4.1). A search in the Online Encyclopedia of Integer Sequences [**27**] shows that these numbers are also the values of $G(e^{-t})$ as $t \to 0^+$ where

$$G(q) := 1 + \sum_{n=1}^{\infty} (-1)^n (q; q^2)_n. \tag{4.2}$$

This is a function which, as defined, exists only for q a root of unity.

The asymptotic relationship between $U(-1; -e^{-t})$ and $G(e^{-t})$ led us to guess that there might be a relation at other roots of unity. We computed the following table with $\zeta_k = e^{2\pi i/k}$.

k	$-U(-1; -\zeta_k)$	$G(\zeta_k)$
1	$1.00000 + 0.00000i$	$1.00000 + 0.00000i$
3	$-0.50000 + 0.86603i$	$-0.50000 + 0.86603i$
5	$2.11803 - 0.36327i$	$2.11803 - 0.36327i$
7	$-1.06853 + 0.78183i$	$-1.06853 + 0.78183i$
9	$2.85844 - 0.11878i$	$2.85844 - 0.11878i$
11	$-1.54408 + 0.32013i$	$-1.54408 + 0.32013i$
13	$3.36485 + 0.42938i$	$3.36485 + 0.42938i$
15	$-1.83087 - 0.37987i$	$-1.83087 - 0.37987i$
17	$3.60849 + 1.17128i$	$3.60849 + 1.17128i$
19	$-1.86847 - 1.21821i$	$-1.86847 - 1.21821i$
21	$3.56061 + 2.00857i$	$3.56061 + 2.00857i$
23	$-1.62459 - 2.09747i$	$-1.62459 - 2.09747i$
25	$3.21160 + 2.84281i$	$3.21160 + 2.84281i$
27	$-1.09530 - 2.92075i$	$-1.09530 - 2.92075i$
29	$2.57645 + 3.57814i$	$2.57645 + 3.57814i$

From this table and additional computation it is clear that for odd roots of unity ζ we have

$$-U(-1; -\zeta) = G(\zeta).$$

Remark. We would like an elementary proof of this claim.

Next, we wondered if it was possible to extend the function G to converge in the domain $|q|<1$. The obvious thing to try is to write

$$\begin{aligned}G(q) =&1+\sum_{n=0}^{\infty}(-1)^{n+1}(q;q^2)_{n+1}\\ =&1+\sum_{n=0}^{\infty}(-1)^{n+1}(q;q^2)_n-\sum_{n=0}^{\infty}(-1)^{n+1}q^{2n+1}(q;q^2)_n\\ =&-\sum_{n=1}^{\infty}(-1)^{n}(q;q^2)_n+\sum_{n=0}^{\infty}(-1)^{n}q^{2n+1}(q;q^2)_n.\end{aligned}$$

This implies that

$$\begin{aligned}2G(q) &= 1+\sum_{n=0}^{\infty}(-1)^{n}q^{2n+1}(q;q^2)_n\\ &= 1+q-q^3+q^4+q^5-q^6-q^7+2q^9-2q^{11}\cdots-q^{14}-2q^{15}+q^{16}+3q^{17}+\cdots\end{aligned}$$

A search in the Online Encyclopedia of Integer Sequences [**27**] shows that this q-series matches that of the mock theta function $\phi(q)$. A literature search for relations between $f(q)$ and $\phi(q)$ turned up the result $f(q)+b(q)=2\phi(-q)$, which led to Theorem 1.3.

4.1. Proof of Theorem 4.1. Here we prove Theorem 4.1.

Proof of Theorem 4.1. From [**22**] page 18 with $q=e^{-t}$ we have

$$\begin{aligned}f(-q)+\sqrt{\frac{\pi}{t}}\exp\left(\frac{\pi^2}{24t}-\frac{t}{24}\right)f(-e^{-\frac{\pi^2}{t}})\\ =\sqrt{\frac{24t}{\pi}}e^{-\frac{t}{24}}\int_0^{\infty}e^{-\frac{3}{2}tx^2}\frac{\cosh\left(\frac{5}{2}tx\right)+\cosh\left(\frac{1}{2}tx\right)}{\cosh\left(3tx\right)}dx.\end{aligned}\tag{4.3}$$

Notice that $f(-e^{-\frac{\pi^2}{t}})=1+O\left(e^{-\frac{\pi^2}{t}}\right)$ as $t\to 0^+$. Next, using the fact that $b(e^{2\pi i\tau})=e^{\pi i\tau/12}\eta^3(\tau)/\eta^2(2\tau)$, together with the modular transformation law for the Dedekind η-function, a straightforward calculation shows that

$$b(-e^{-t})=\sqrt{\frac{\pi}{t}}\exp\left(\frac{\pi^2}{24t}-\frac{t}{24}\right)(1+O(e^{-\frac{\pi^2}{t}}))$$

as $t\to 0^+$. Thus, using these facts, to prove Theorem 4.1 it suffices to analyze the asymptotic as $t\to 0^+$ of the right hand side of (4.3). A simple change of variables gives

$$\begin{aligned}\sqrt{\frac{24t}{\pi}}\int_0^{\infty}e^{-\frac{3}{2}tx^2}\frac{\cosh\left(\frac{5}{2}tx\right)+\cosh\left(\frac{1}{2}tx\right)}{\cosh\left(3tx\right)}dx\\ =\sqrt{\frac{2\pi}{3t}}\int_{\mathbb{R}}e^{-\frac{\pi^2w^2}{6t}}\frac{\cosh\left(\frac{5}{6}\pi w\right)+\cosh\left(\frac{1}{6}\pi w\right)}{\cosh\left(\pi w\right)}dw.\end{aligned}$$

Recall that the Mordell integral as defined by Zwegers [**32**] for $z\in\mathbb{C},\tau\in\mathbb{H}$, is given by

$$h(z;\tau):=\int_{\mathbb{R}}\frac{e^{\pi i\tau x^2-2\pi zx}}{\cosh\left(\pi x\right)}dx=\int_{\mathbb{R}}e^{\pi i\tau w^2}\frac{\cosh\left(2\pi zw\right)}{\cosh\left(\pi w\right)}dw.\tag{4.4}$$

By Proposition 1.2 (5) of [**32**] we have

$$h(z;\tau)=\frac{1}{\sqrt{-i\tau}}e^{\frac{\pi i z^2}{\tau}}\int_{\mathbb{R}} e^{-\frac{\pi i}{\tau}w^2}\frac{\cosh\left(2\pi w\frac{z}{\tau}\right)}{\cosh\left(\pi w\right)}dw.$$

Thus, for rational $0<A<1$ we have

$$\int_{\mathbb{R}} e^{-\frac{\pi^2 w^2}{6t}}\frac{\cosh(A\pi w)}{\cosh(\pi w)}dw=\sqrt{\frac{6t}{\pi}}e^{\frac{3}{2}A^2 t}\int_{\mathbb{R}} e^{-6tw^2}\frac{\cos(6Atw)}{\cosh(\pi w)}dw. \tag{4.5}$$

Letting $A=\frac{5}{6}$, and then $A=\frac{1}{6}$ in (4.5), and adding, we find that

$$\sqrt{\frac{24t}{\pi}}e^{-\frac{t}{24}}\int_0^\infty e^{-\frac{3}{2}tx^2}\frac{\cosh\left(\frac{5}{2}tx\right)+\cosh\left(\frac{1}{2}tx\right)}{\cosh\left(3tx\right)}dx=2\int_{-\infty}^{\infty}\frac{g(w;t)}{\cosh(\pi w)}dw,$$

where

$$\begin{aligned} g(w;t) :=& e^{-6tw^2}\left(e^t\cos(5tw)+\cos(tw)\right) \\ =& 2+(1-12w^2)t+\left(\frac{1}{2}-19w^2+36w^4\right)t^2+O(t^3). \end{aligned} \tag{4.6}$$

We make use of the well known identity (see [**18**] for example)

$$\int_{\mathbb{R}}\frac{w^{2n}}{\cosh(\pi w)}dw=\frac{(-1)^n E_{2n}}{2^{2n}},$$

where E_n are the Euler numbers. The Taylor expansion of $g(w;t)$ and these constants give the asymptotic as stated in Theorem 4.1. $\square$

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