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Quantum Jacobi forms and sums of tails identities

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Abstract

Our results are on the interconnected topics of quantum Jacobi sums of tails identities, quantum Jacobi and mock Jacobi properties of two-variable q-hypergeometric series and partial theta functions, and two-variable q-hypergeometric generating functions for certain L-values by way of related asymptotic expansions. More specifically, we establish five two-variable quantum Jacobi sums of tails identities. As corollaries, we recover known one-variable quantum sum of tails identities due to Zagier, Andrews-Jiménez-Urroz-Ono, and more. Further, justifying the "quantum Jacobi" description of our two-variable sums of tails identities, we establish the quantum Jacobi and mock Jacobi properties of a number of two-variable q-hypergeometric series and partial Jacobi theta functions which appear in our two-variable sums of tails identities, inspired by related results of Zagier and Rolen-Schneider in the one-variable quantum modular setting. Finally, by establishing related asymptotic expansions, we realize generating functions for certain L-values in terms of two-variable q-hypergeometric series and Jacobi partial theta functions, inspired by earlier work in this direction by Andrews–Jiménez-Urroz–Ono for the Riemann ζ-function and Dirichlet and Hecke L-functions.

1 Introduction and statement of results

Early sums of tails identities include the following, found in Ramanujan's "Lost" Notebook [24, p14]:

$$\sum_{n=0}^{\infty}((-q;q)_\infty-(-q;q)_n)=(-q;q)_\infty\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right)+\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n},$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q;q^2)_\infty}-\frac{1}{(q;q^2)_{n+1}}\right)=\frac{1}{(q;q^2)_\infty}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\right)+\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n},$$

where here and throughout, the q-Pochhammer symbol is defined for $n\in\mathbb{N}_0\cup\infty$ by $(a;q)_n:=\prod_{j=0}^{n-1}(1-aq^j)$. These are referred to as "sums of tails" identities, as their left-hand sides are sums of differences between infinite products and their nth partial products. The q-hypergeometric series

$$R(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n}$$

on the right-hand sides may be viewed as a combinatorial generating function, namely, for the difference between the number of partitions into distinct parts with even rank minus those with odd rank. In 1986, Andrews proved the above sums of tails identities, roughly 10 years after unearthing the "Lost" Notebook [1]. Shortly thereafter, the coefficients of the function $R(q)$ were also shown to be defined by a Hecke L-function

$$L(\chi, s) := \sum_{\mathfrak{a} \subseteq \mathbb{Z}[\sqrt{6}]} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where χ is an order 2 character of conductor $4(3+\sqrt{6})$ on ideals in $\mathbb{Z}[\sqrt{6}]$ [2,12].

Partially motivated by finding values of certain other L-functions, in 2001, Zagier in [27], and Andrews, Jiménez-Urroz, and Ono in [4], established other sums of tails identities, including this one from [27]

$$\sum_{n=0}^{\infty} \left((q;q)_\infty - (q;q)_n\right) = (q;q)_\infty \left(-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\right) + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) n q^{\frac{n^2-1}{24}}, \tag{1.1}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol, and these two from [4]

$$\sum_{n=0}^{\infty} \left(\frac{(q;q)_\infty}{(-q;q)_\infty} - \frac{(q;q)_n}{(-q;q)_n}\right) = 2\frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} + 4\sum_{n=1}^{\infty} (-1)^n n q^{n^2}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} - \frac{(q^2;q^2)_n}{(q;q^2)_{n+1}}\right) = -\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1-q^n} + \sum_{n=1}^{\infty} n q^{\frac{n^2+n}{2}}. \tag{1.3}$$

In a different direction, later, in 2010, Zagier introduced the notion of a quantum modular form in [28]. Loosely speaking, a quantum modular form is a $\mathbb{C}$-valued function defined in $\mathbb{Q}$ (as opposed to the upper half-plane $\mathbb{H}$ in the case of modular forms) and which exhibits modular-like transformation properties there, up to the addition of smooth error functions in $\mathbb{R}$ (see [8,28] for further details, and for more on the developing theory of quantum modular forms). In [28] Zagier also offered the first handful of examples of quantum modular forms, one of which is the function

$$\phi(x) := e^{\pi i x/12} \sum_{n=0}^{\infty} (e^{2\pi i x}; e^{2\pi i x})_n.$$

In particular, Zagier shows that for $x \in \mathbb{Q} \setminus \{0\}$, the function ϕ satisfies

$$\phi(x) - \operatorname{sgn}(x)\zeta_8 |x|^{\frac{3}{2}} \phi(-1/x) = g(x),$$

where g is analytic in $\mathbb{R} \setminus \{0\}$. Here and throughout, we let $\zeta_m := e^{2\pi i/m}$. A key ingredient used by Zagier in establishing the quantum modularity of $\phi(x)$ is his sum of tails identity (1.1). More recently in [25], Rolen and Schneider show that three other q-hypergeometric sums (similar to $\phi(x)$) are also quantum modular forms, and they use the sums of tails identities of Andrews, Jiménez-Urroz, and Ono in (1.2) and (1.3) to do so.

More recently in 2016, Bringmann and the first author defined the notion of a quantum Jacobi form and offered the first example in [7], an example which has combinatorial meaning (like $R(q)$ above). Quantum Jacobi forms are a marriage of quantum modular forms (as originally defined by Zagier as mentioned above) with Jacobi forms, the theory of which was largely developed by Eichler and Zagier in the 1980s [13]. These functions take values in $\mathbb{C}$, are defined in $\mathbb{Q} \times \mathbb{Q}$ (as opposed to $\mathbb{C} \times \mathbb{H}$ in the case of Jacobi forms) and which exhibit Jacobi transformation properties there, up to the addition of smooth error functions in $\mathbb{R} \times \mathbb{R}$. Precisely, we have the following definition from [7].

Definition 1.1 A weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form** is a complex-valued function ϕ on $\mathbb{Q} \times \mathbb{Q}$ such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$, the functions $h_\gamma : \mathbb{Q} \times (\mathbb{Q} \setminus \gamma^{-1}(i\infty)) \to \mathbb{C}$ and $g_{(\lambda,\mu)} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ defined by

$$h_\gamma(z;\tau) := \phi(z;\tau) - \varepsilon_1^{-1}(\gamma)(c\tau + d)^{-k} e^{\frac{-2\pi i m c z^2}{c\tau+d}} \phi\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right),$$

$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \varepsilon_2^{-1}((\lambda,\mu)) e^{2\pi i m(\lambda^2\tau + 2\lambda z)} \phi(z + \lambda\tau + \mu; \tau),$$

satisfy a "suitable" property of continuity or analyticity in a subset of $\mathbb{R} \times \mathbb{R}$.

Remarks (1) The complex numbers $\varepsilon_1(\gamma)$ and $\varepsilon_2((\lambda,\mu))$ satisfy $|\varepsilon_1(\gamma)| = |\varepsilon_2((\lambda,\mu))| = 1$; in particular, the $\varepsilon_1(\gamma)$ are such as those appearing in the theory of half-integral weight modular forms.

(2) We may modify the definition to allow modular transformations on appropriate subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domain to be a suitable subset of $\mathbb{Q} \times \mathbb{Q}$.

(3) The "suitable" property of continuity or analyticity required is intentionally left somewhat vague in order to mimic Zagier's definition of a quantum modular form [28].

Like the subject of quantum modular forms, the subject of quantum Jacobi forms continues to develop; the known examples of quantum Jacobi forms to date have been established in [5–7, 11, 15]. Moreover, like quantum modular forms, quantum Jacobi forms arise in the diverse areas of Number Theory, Combinatorics, Topology, and Mathematical Physics.

Here, our results are multifold and are motivated by the interconnected topics which have been discussed above. Our sums of tails results are given in Sect. 1.1; our related quantum Jacobi results are given in Sect. 1.2, and our related results on L-values are given in 1.3.

1.1 Jacobi sums of tails identities

First, we establish two-variable sums of tails identities in Proposition 1.2; from Proposition 1.2, we obtain as corollaries Zagier's sum of tails identity (1.1), and Andrews, Jiménez-Urroz, and Ono's identities (1.2) and (1.3), and one more. Before stating these results, we define the combinatorial q-hypergeometric series $P_n(w;q)$, $n \in \mathbb{N}_0$, which makes an appearance in our results:

$$P_n(w;q) := \sum_{m=0}^{\infty} (-w^2)^m q^{\frac{m(m+1)}{2}} \begin{bmatrix} m+n \\ m \end{bmatrix}.$$

Here and throughout, the q-binomial coefficients are defined by

$$\begin{bmatrix} m \\ n \end{bmatrix} := \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}.$$

We later explain in the proof of Lemma 2.10 that

$$P_n(w;q) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\alpha_n(k,m)(-w^2)^m q^k,$$

where $\alpha_n(k,m) = p(k \mid m \text{ distinct parts, rank} \leq n)$ counts the number of integer partitions of n into k distinct parts with rank at most m.

We also define the following two-variable partial Jacobi theta functions $\theta_j(w;q)$, $1 \leq j \leq 3$, by

$$\theta_1(w;q) := 1 + 2\sum_{n=1}^{\infty}(-w^2)^n q^{n^2}, \tag{1.4}$$

$$\theta_2(w;q) := \sum_{n=0}^{\infty} w^n q^{\frac{n^2+n}{2}}, \tag{1.5}$$

$$\theta_3(w;q) := \sum_{n=1}^{\infty}\left(\frac{12}{n}\right) w^{\frac{n-1}{2}} q^{\frac{n^2-1}{24}}, \tag{1.6}$$

and the two-variable divisor-type function

$$D(w;q) := (w^2q;q)_\infty \sum_{n=1}^{\infty}\sum_{d|n} w^{2d} q^n.$$

Proposition 1.2 *We have the following sums of tails identities:*

$$\text{(i.)}\quad (1-w)\sum_{n=0}^{\infty}\left(\frac{(wq;q)_\infty}{(-wq;q)_\infty} - \frac{(wq;q)_n}{(-wq;q)_n}\right) w^n = \frac{(wq;q)_\infty}{(-wq;q)_\infty} - \theta_1(w;q),$$

$$\text{(ii.)}\quad (1-w)\sum_{n=0}^{\infty}\left(\frac{(wq^2;q^2)_\infty}{(wq;q^2)_\infty} - \frac{(wq^2;q^2)_n}{(wq;q^2)_{n+1}}\right) w^n = \frac{(wq^2;q^2)_\infty}{(wq;q^2)_\infty} - \theta_2(w;q),$$

$$\text{(iii.)}\quad (1-w)\sum_{n=0}^{\infty}\left((wq;q)_\infty - (wq;q)_n\right) w^n = (wq;q)_\infty - \theta_3(w;q),$$

$$\text{(iv.)}\quad (1-w)\sum_{n=0}^{\infty}\left((w^2q;q)_\infty - P_n(w;q)\right) w^n = (w^2q;q)_\infty - \theta_3(w;q),$$

$$\text{(v.)}\quad \sum_{n=0}^{\infty}\left((w^2q;q)_\infty - P_n(w;q)\right) - D(w;q) = \sum_{n=0}^{\infty}\left((w^2q;q)_\infty - (w^2q;q)_n\right).$$

Remark We may view the identities (*iv.*) and (*v.*) in Proposition 1.2 involving the combinatorial q-hypergeometric series $P_n(w;q)$ as sums of tails identities due to the fact that

$$\lim_{n\to\infty} P_n(w;q) = (w^2q;q)_\infty,$$

which we explain in Lemma 2.11.

As corollaries to results in Proposition 1.2, we deduce Zagier's (1.1), and (1.2) and (1.3) by Andrews, Jiménez-Urroz, and Ono (see Corollary 1.3 (*i*) − −(*iii*)), as well as a similar identity (see Corollary 1.3 (*iv.*)) due to Patkowski [22]. To state this corollary, we define the partial theta functions $\theta_1(q)$, $\theta_2(q)$, $\theta_3(q)$ by

$$\theta_1(q) := 4\sum_{n=1}^{\infty}(-1)^n n q^{n^2},$$

$$\theta_2(q) := \sum_{n=1}^{\infty} n q^{\frac{n^2+n}{2}},$$

$$\theta_3(q) := \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{12}{n}\right) n q^{\frac{n^2-1}{24}}.$$

Corollary 1.3 *We have that*

$$\begin{aligned}
&\text{(i)} \sum_{n=0}^{\infty} \frac{(q;q)_\infty}{(-q;q)_\infty} - \frac{(q;q)_n}{(-q;q)_n} &&= 2\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}} + \theta_1(q),\\
&\text{(ii)} \sum_{n=0}^{\infty}\left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} - \frac{(q^2;q^2)_n}{(q;q^2)_{n+1}}\right) &&= -\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1-q^n} + \theta_2(q),\\
&\text{(iii)} \sum_{n=0}^{\infty}((q;q)_\infty - (q;q)_n) &&= (q;q)_\infty\left(-\frac{1}{2} + \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right) + \theta_3(q),\\
&\text{(iv)} \sum_{n=0}^{\infty}((q;q)_\infty - P_n(1;q)) &&= (q;q)_\infty\left(-\frac{1}{2} + 2\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right) + \theta_3(q).
\end{aligned}$$

1.2 Quantum Jacobi forms from sums of tails identities

Next, motivated by the quantum modularity of certain (one-variable) q-hypergeometric series appearing in sums of tails identities (see (1.1)-(1.3)) as established by Zagier and Rolen-Schneider, here we show how the two-variable q-hypergeometric series

$$H_1(w;q) := \sum_{n=0}^{\infty}\frac{(wq;q)_n}{(-wq;q)_n}w^n, \quad H_2(w;q) := \sum_{n=0}^{\infty}\frac{(wq^2;q^2)_n}{(wq;q^2)_{n+1}}w^n, \tag{1.7}$$

$$H_{3,1}(w;q) := \sum_{n=0}^{\infty}(wq;q)_n w^n, \quad H_{3,2}(w;q) := \sum_{n=0}^{\infty}P_n(w;q)w^n \tag{1.8}$$

and the partial Jacobi theta functions $\theta_1(w;q)$, $\theta_2(w;q)$, $\theta_3(w;q)$, appearing in our sums of tails identities from Proposition 1.2, are quantum Jacobi forms (up to suitable normalization and changes of variables). In fact, we show a stronger property, that these functions are additionally simultaneously "dual" to mock Jacobi forms in $\mathbb{C}\times\mathbb{H}^-$, where $\mathbb{H}^-$ is the lower half complex plane. By mock Jacobi form, we mean a function which is the holomorphic part of a nonholomorphic Jacobi form (see [8,29]). For more details on the aforementioned duality, see Sects. 4.1–4.3, where we elaborate on and make more precise Theorem 1.4 below.

Theorem 1.4 *Up to suitable normalization and changes of variables, the two variable q-hypergeometric series*

$$H_1(w;q), H_2(w;q), H_{3,1}(w;q), H_{3,2}(w;q),$$

and the partial Jacobi theta functions

$$\theta_1(w;q),\ \theta_2(w;q),\ \theta_3(w;q),$$

are quantum Jacobi forms, when viewed as functions defined in (a subset of) $\mathbb{Q}\times\mathbb{Q}$. Moreover, they are dual to mock Jacobi forms in $\mathbb{C}\times\mathbb{H}^-$.

Remarks (1) We make Theorem 1.4 precise in Theorem 4.1, Theorem 4.3, and Theorem 4.6 in what follows.

(2) Up to suitable normalizations and changes of variables, the functions $\theta_1(q)$, $\theta_2(q)$, $\theta_3(q)$, and $H_1(1;q)$, $H_2(1;q)$, $H_{3;1}(1;q)$, and $H_{3,2}(1;q)$ are quantum modular forms when viewed as functions in $\mathbb{Q}$. For all of these functions except $H_{3,2}(1;q)$, this follows from results in [25] and [28]; in the case of $H_{3,2}(1;q)$, the quantum modularity follows from (*iv.*) of Corollary 1.3 and work in [28]. Similarly, for one-variable quantum modular properties of these functions for other specializations of variables (e.g., for fixed $w = \zeta_b^a$) see [10, 17, 18, 20].

1.3 *q*-Hypergeometric generating functions for *L*-values

Our next set of results establish asymptotic expansions for the aforementioned q-hypergeometric series and partial Jacobi theta functions towards roots of unity; as is shown in Theorem 1.5 below, these asymptotic expansions feature certain L-values as coefficients. We refer the interested reader to [4] for more results along these lines with respect to the Riemann ζ-function and Dirichlet and Hecke L-functions.

Theorem 1.5 also offers two diverse ways to evaluate the relevant q-hypergeometric and partial Jacobi theta functions at pairs of roots of unity. It is a question of interest to find a direct proof of the equivalence of two such given expressions, avoiding the methods of proof used here, and instead using, for example, elements from the theory of q-hypergeometric series, partial theta functions, or other elementary or direct methods.

Here and throughout, $B_r(x)$ denotes the rth Bernoulli polynomial. Throughout, we say that a rational number r/s is *reduced* if $\gcd(r,s) = 1$, $r \in \mathbb{Z}$ and $s \in \mathbb{N}$.

Theorem 1.5 *The following are true.*

(1) *Let*

$$\left(\tfrac{a}{b}, \tfrac{h}{k}\right) \in S_1 := \left\{ \left(\tfrac{a}{b}, \tfrac{h}{k}\right) \in \mathbb{Q}^2 \mid \tfrac{a}{b}, \tfrac{h}{k} \text{ are reduced, } b \mid k, \text{ and } k \text{ is odd} \right\}.$$

We have the following asymptotic expansion as $t \to 0^+$:

$$(1-\zeta_b^a) H_1(\zeta_b^a; \zeta_k^h e^{-t}) = \theta_1(\zeta_b^a; \zeta_k^h e^{-t}) \sim 1 + 2\sum_{r=0}^{\infty} L(-2r, c_1)\frac{(-t)^r}{r!},$$

where the L-values are given by

$$L(-r, c_1) = -\frac{(2k)^r}{r+1}\sum_{n=1}^{2k} c_1(n) B_{r+1}\left(\frac{n}{2k}\right), \quad (r = 0, 1, 2, \dots)$$

with

$$c_1(n) = (-1)^n \zeta_b^{2an} \zeta_k^{hn^2}.$$

Further, these function can be evaluated in the following two different ways:

$$\begin{aligned}(1-\zeta_b^a) H_1(\zeta_b^a; \zeta_k^h) = \theta_1(\zeta_b^a; \zeta_k^h) &= 1 - \frac{1}{k}\sum_{n=1}^{2k} n(-1)^n \zeta_b^{2an}\zeta_k^{hn^2}, \\ &= (1-\zeta_b^a)\sum_{n=0}^{\mathcal{N}_1 - 1} \frac{(\zeta_b^a \zeta_k^h; \zeta_k^h)_n}{(-\zeta_b^a \zeta_k^h; \zeta_k^h)_n}\zeta_b^{an},\end{aligned}$$

where $\mathcal{N}_1 = \mathcal{N}_1(a,b,h,k)$ *is the smallest non-negative integer congruent to* $ah'k/b \pmod{k}$*, where* h' *is any integer satisfying* $hh' \equiv -1 \pmod{k}$*.*

(2) *Let*

$$\left(\tfrac{a}{b},\tfrac{h}{k}\right) \in S_2 := \left\{\left(\tfrac{a}{b},\tfrac{h}{k}\right) \in \mathbb{Q}^2 \mid \tfrac{a}{b},\tfrac{h}{k} \text{ are reduced, } b \mid k, k \text{ is even, and } ak/b \text{ is even}\right\}.$$

We have the following asymptotic expansion as $t \to 0^+$*:*

$$(1-\zeta_b^a)e^{-t/8}H_2(\zeta_b^a;\zeta_k^h e^{-t}) = e^{-t/8}\theta_2(\zeta_b^a;\zeta_k^h e^{-t}) \sim \sum_{r=0}^{\infty} L(-2r,c_2)\frac{(-t)^r}{8^r\cdot r!},$$

where the L-values are given by

$$L(-r,c_2) = -\frac{(8k)^r}{r+1}\sum_{n=1}^{8k} c_2(n)B_{r+1}\left(\frac{n}{8k}\right), \quad (r=0,1,2,\dots)$$

with

$$c_2(n) = \begin{cases} \zeta_{2b}^{a(n-1)}\zeta_{8k}^{h(n^2-1)}, & n \text{ odd},\\ 0, & \text{else}.\end{cases}$$

Further, these function can be evaluated in the following two different ways:

$$\begin{aligned}(1-\zeta_b^a)H_2(\zeta_b^a;\zeta_k^h) = \theta_2(\zeta_b^a;\zeta_k^h) &= -\frac{1}{8k}\sum_{\substack{n=1\\ n\,\text{odd}}}^{8k} n\zeta_{2b}^{a(n-1)}\zeta_{8k}^{h(n^2-1)},\\ &= (1-\zeta_b^a)\sum_{n=0}^{\mathcal{N}_2-1}\frac{(\zeta_b^a\zeta_k^{2h};\zeta_k^{2h})_n}{(\zeta_b^a\zeta_k^h;\zeta_k^{2h})_{n+1}}\zeta_b^{an},\end{aligned}$$

where $\mathcal{N}_2 = \mathcal{N}_2(a,b,h,k)$ *is the smallest non-negative integer congruent to* $ah'k/(2b) \pmod{k/2}$*, where* h' *is any integer satisfying* $hh' \equiv -1 \pmod{k}$*.*

(3) Let

$$\left(\tfrac{a}{b},\tfrac{h}{k}\right) \in S_3 := \left\{\left(\tfrac{a}{b},\tfrac{h}{k}\right) \in \mathbb{Q}^2 \mid \tfrac{a}{b},\tfrac{h}{k} \text{ are reduced, } b \mid k\right\}.$$

We have the following asymptotic expansion as $t \to 0^+$*:*

$$(1-\zeta_b^a)e^{-t/24}H_{3,j}(\zeta_b^a;\zeta_k^h e^{-t}) = e^{-t/24}\theta_3(\zeta_b^a;\zeta_k^h e^{-t}) \sim \sum_{r=0}^{\infty} L(-2r,c_3)\frac{(-t)^r}{24^r r!},$$

for $j \in \{1,2\}$*, where the L-values are given by*

$$L(-r,c_3) = -\frac{(12k)^r}{r+1}\sum_{n=1}^{12k} c_3(n)B_{r+1}\left(\frac{n}{12k}\right), \quad (r=0,1,2,\dots)$$

with

$$c_3(n) = c_3(a,b,h,k;n) := \left(\frac{12}{n}\right)\zeta_{24k}^{h(n^2-1)}\zeta_{2b}^{a(n-1)}.$$

Further, these functions can be evaluated in the following two different ways:

$$\begin{aligned}(1-\zeta_b^a)H_{3,j}(\zeta_b^a;\zeta_k^h) = \theta_3(\zeta_b^a;\zeta_k^h) &= -\frac{1}{12k}\sum_{n=1}^{12k} nc_3(n),\\ &= (1-\zeta_b^a)\sum_{n=0}^{\mathcal{N}_3-1}(\zeta_b^a\zeta_k^h;\zeta_k^h)_n\zeta_b^{an},\end{aligned}$$

for $j \in \{1, 2\}$, where $\mathcal{N}_3 = \mathcal{N}_3(a, b, h, k)$ is the smallest non-negative integer congruent to $ah'k/b \pmod{k}$, where h' is any integer satisfying $hh' \equiv -1 \pmod{k}$.

2 Preliminaries

We devote this section to establishing or recalling some preliminary results, before proving our main results in the sections that follow this one. In addition to the references in the subsections below, we refer the reader to [8] for additional background.

2.1 Modular and Jacobi forms

In this section we recall the definitions and some properties of the functions

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \; \vartheta(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{\pi i n^2\tau + 2\pi i n(z+\frac{1}{2})}, \tag{2.1}$$

where $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$. These functions satisfy the modular and Jacobi transformations in Lemma 2.1 and Lemma 2.2, respectively [23].

Lemma 2.1 *For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,*

$$\eta(\gamma\tau) = \varepsilon(\gamma)(C\tau + D)^{\frac{1}{2}}\eta(\tau),$$

where for $C > 0$,

$$\varepsilon(\gamma) = \begin{cases} \frac{1}{\sqrt{i}}\left(\frac{D}{C}\right) i^{(1-C)/2} e^{\pi i (BD(1-C^2)+C(A+D))/12} & \text{if } C \text{ is odd}, \\ \frac{1}{\sqrt{i}}\left(\frac{C}{D}\right) e^{\pi i D/4} e^{\pi i (AC(1-D^2)+D(B-C))/12} & \text{if } D \text{ is odd}. \end{cases} \tag{2.2}$$

Lemma 2.2 *For $\lambda, \mu \in \mathbb{Z}$, $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$,*

(i) $\vartheta(z + \lambda\tau + \mu; \tau) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z}\vartheta(z;\tau)$,

(ii) $\vartheta\left(\frac{z}{C\tau + D}; \gamma\tau\right) = \varepsilon^3(\gamma)(C\tau + D)^{\frac{1}{2}} e^{\frac{\pi i C z^2}{C\tau + D}}\vartheta(z;\tau)$,

(iii) $\vartheta(z;\tau) = -iq^{\frac{1}{8}} w^{-\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n)(1-wq^{n-1})(1-w^{-1}q^n)$.

Using η and ϑ we define (as in [15]) the functions

$$N(\tau) := \frac{\eta(\tau)}{\eta^2(2\tau)}, \quad \text{and} \quad T(\tau) := \vartheta(-\tau + \tfrac{1}{2}; 4\tau).$$

It is not difficult to show that

$$N(\tau)T(\tau) = -q^{-\frac{1}{8}}, \tag{2.3}$$

a fact which we will use later. We will also use the following lemma on the modular transformation properties of N and T.

Lemma 2.3 *Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with C even, and define $\widetilde{\gamma} := \left(\begin{smallmatrix} A & 2B \\ C/2 & D \end{smallmatrix}\right)$. We have that*

$$N(\gamma\tau) = (c\tau + d)^{-\frac{1}{2}}\varepsilon(\gamma)\varepsilon^{-2}(\widetilde{\gamma})N(\tau). \tag{2.4}$$

Moreover, we have that

$$T\left(\frac{\tau}{4\tau+1}\right)=(4\tau+1)^{\frac{1}{2}}e^{\frac{\pi i\tau^2}{4\tau+1}}T(\tau). \tag{2.5}$$

Proof The proof of (2.4) follows immediately from Lemma 2.1. Similarly, (2.5) follows from 2.2 and that $\vartheta(z;\tau)$ is an odd function of z. □

A specific function defined using η, N, and ϑ, that we will encounter in what follows is

$$J(z;\tau):=-i\frac{N(\tau)\eta^3(\tau)}{\vartheta(z/2;\tau)}.$$

It is not difficult to show using Lemma 2.1 and Lemma 2.2 that J is a Jacobi form, and in particular exhibits a certain transformation property with respect to the variables $(z,\tau)\in\mathbb{C}\times\mathbb{H}$ (see [8,13] for further details on Jacobi forms).

Lemma 2.4 *The function $J(z;\tau)$ is a Jacobi form of weight $1/2$, index $-1/8$, on $\Gamma_0(2)\ltimes(4\mathbb{Z}\times 2\mathbb{Z})$, with character ψ_γ.*

2.2 The level 2 Appell function and related functions

The level 2 Appell function A_2 is defined for $z_1,z_2\in\mathbb{C}$, $\tau\in\mathbb{H}$ by

$$A_2(z_1,z_2;\tau):=\xi_1\sum_{n\in\mathbb{Z}}\frac{\xi_2^n q^{n(n+1)}}{1-\xi_1 q^n}, \tag{2.6}$$

where $\xi_j=e(z_j)$, $j\in\{1,2\}$. (Note. Here and throughout, we use the notation $e(u):=e^{2\pi iu}$.) This function was studied as one of a more general family of Appell functions by Zwegers (see [8]). The completion of A_2, denoted by $\widehat{A}_2$, is defined by

$$\widehat{A}_2(z_1,z_2;\tau):= A_2(z_1,z_2;\tau)+\frac{i}{2}\sum_{j=0}^{1}e^{2\pi ijz_1}\vartheta\left(z_2+j\tau+\frac{1}{2};2\tau\right)R\left(2z_1-z_2-j\tau-\frac{1}{2};2\tau\right), \tag{2.7}$$

where the nonholomorphic function R is defined by

$$R(z;\tau):=\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}\left\{\operatorname{sgn}(\nu)-E\left((\nu+\lambda)\sqrt{2y}\right)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i\nu^2\tau-2\pi i\nu z}, \tag{2.8}$$

with $y:=\operatorname{Im}(\tau)$, $\lambda:=\frac{\operatorname{Im}(z)}{\operatorname{Im}(\tau)}$ and

$$E(z):=2\int_0^z e^{-\pi u^2}du.$$

The completed function exhibits Jacobi-like transformation properties:

Lemma 2.5 *For $n_1,n_2,m_1,m_2\in\mathbb{Z}$, $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$, the function $\widehat{A}_2$ satisfies the following transformation properties:*

(i) $\widehat{A}_2(-z_1,-z_2;\tau)=-\widehat{A}_2(z_1,z_2;\tau)$,
(ii) $\widehat{A}_2(z_1+n_1\tau+m_1,z_2+n_2\tau+m_2;\tau)=\xi_1^{2n_1-n_2}\xi_2^{-n_1}q^{n_1^2-n_1n_2}\widehat{A}_2(z_1,z_2;\tau)$,
(iii) $\widehat{A}_2\left(\frac{z_1}{c\tau+d},\frac{z_2}{c\tau+d};\gamma\tau\right)=(c\tau+d)e^{\frac{\pi ic}{c\tau+d}(-2z_1^2+2z_1z_2)}\widehat{A}_2(z_1,z_2;\tau)$.

From [29, Proposition 1.9, Proposition 1.10], we also have the following transformation properties of R.

Lemma 2.6 *With hypotheses as above, R satisfies the following transformation properties:*

(i) $R(z;\tau+1)=e^{-\frac{\pi i}{4}}R(z;\tau)$,
(ii) $\frac{1}{\sqrt{-i\tau}}e^{\frac{\pi i z^2}{\tau}}R\left(\frac{z}{\tau};-\frac{1}{\tau}\right)+R(z;\tau)=h(z;\tau)$,
(iii) $R(z;\tau)=R(-z;\tau)$,
(iv) $R(z;\tau)+e^{-2\pi i z-\pi i\tau}R(z+\tau;\tau)=2e^{-\pi i z-\pi i\tau/4}$,
(v) $R(z+1;\tau)=-R(z;\tau)$.

The Mordell integral h given in Lemma 2.6 (ii) is defined for $z\in\mathbb{C}$, $\tau\in\mathbb{H}$ by

$$h(z;\tau):=\int_{\mathbb{R}}\frac{e^{\pi i\tau u^2-2\pi zu}}{\cosh(\pi u)}du. \tag{2.9}$$

Under suitable hypotheses, h can be expressed in terms of the weight 3/2 theta functions $g_{A,B}$, defined for $A,B\in\mathbb{R}$ and $\tau\in\mathbb{H}$ by

$$g_{A,B}(\tau):=\sum_{\nu\in A+\mathbb{Z}}\nu e^{\pi i\nu^2\tau+2\pi i\nu B}. \tag{2.10}$$

Due to Zwegers [29], we have

Lemma 2.7 *For* $A,B\in(-\frac{1}{2},\frac{1}{2})$,

$$\int_0^{i\infty}\frac{g_{A+\frac{1}{2},B+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}dz=-e^{-\pi iA^2\tau+2\pi iA\left(B+\frac{1}{2}\right)}h(A\tau-B;\tau).$$

The functions $g_{A,B}$ transform as follows [26,29].

Lemma 2.8 *With hypotheses as above, the functions* $g_{A,B}$ *satisfy:*

(i) $g_{A+1,B}(\tau)=g_{A,B}(\tau)$,
(ii) $g_{A,B+1}(\tau)=e^{2\pi iA}g_{A,B}(\tau)$,
(iii) $g_{A,B}(\tau+1)=e^{-\pi iA(A+1)}g_{A,A+B+\frac{1}{2}}(\tau)$,
(iv) $g_{A,B}\left(-\frac{1}{\tau}\right)=ie^{2\pi iAB}(-i\tau)^{\frac{3}{2}}g_{B,-A}(\tau)$,
(v) $g_{-A,-B}(\tau)=-g_{A,B}(\tau)$.

Using A_2 and its completion, we define

$$B_{1,2}(z;\tau):=e^{\frac{\pi iz}{2}}A_2\left(\frac{-z}{2},-\tau;2\tau\right)$$

and

$$\widehat{B}_{1,2}(z;\tau):=e^{\frac{\pi iz}{2}}\widehat{A}_2\left(\frac{-z}{2},-\tau;2\tau\right).$$

Our notation for these functions are chosen to parallel the notation given for the functions $B_{\alpha,\beta}$ and $\widehat{B}_{\alpha,\beta}$ in [15]. In [15], it is required that $4\mid\beta$, so the functions $B_{1,2}$ and $\widehat{B}_{1,2}$ defined above do not appear in [15], though they are closely related to the families $B_{\alpha,\beta}$ and $\widehat{B}_{\alpha,\beta}$ appearing there.

Using Proposition 2.5 we establish the following proposition.

Proposition 2.9 *The function*

$$\widehat{B}_{1,2}(z;\tau):=e\left(\frac{z}{4}\right)\widehat{A}_2\left(-\frac{z}{2},-\tau;2\tau\right)$$

is a non-holomorphic Jacobi form of weight 1, *index* $-1/8$, *on* $\Gamma_0(2)\ltimes(4\mathbb{Z}\times2\mathbb{Z})$.

Proof of Proposition 2.9 Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(2)$. Then using the Jacobi modular transformation properties of $\widehat{A}_2$ in Lemma 2.5, and the fact that C is even, we have that

$$\widehat{B}_{1,2}\left(\frac{z}{C\tau+D};\frac{A\tau+B}{C\tau+D}\right)$$
$$= (C\tau+D)e\left(\frac{z}{4(C\tau+D)}\right)e\left(\frac{C\left(-\frac{z^2}{2}+z(A\tau+B)\right)}{4(C\tau+D)}\right)\widehat{A}_2\left(-\frac{z}{2},-(A\tau+B);2\tau\right) \tag{2.11}$$
$$= (C\tau+D)e\left(\frac{z}{4(C\tau+D)}\right)e\left(\frac{C\left(-\frac{z^2}{2}+z(A\tau+B)\right)}{4(C\tau+D)}\right) \tag{2.12}$$
$$\times\, e\left(\frac{z(1-A)}{4}\right)\widehat{A}_2\left(-\frac{z}{2},-\tau;2\tau\right). \tag{2.13}$$

To move from (2.11) to (2.12), we have also used that the number A is odd, and the Jacobi elliptic transformation properties of $\widehat{A}_2$ in Lemma 2.5. Simplifying, and using that $AD - BC = 1$, we see that (2.12) equals

$$(C\tau+D)e\left(\frac{-Cz^2}{8(C\tau+D)}\right)\widehat{B}_{1,2}(z;\tau).$$

This establishes the claimed Jacobi modular transformation property of $\widehat{B}_{1,2}(z;\tau)$.

To claimed Jacobi elliptic transformation properties of $\widehat{B}_{1,2}(z;\tau)$ largely follow from those of $\widehat{A}_2$ (in Lemma 2.5), after a straightforward calculation. □

2.3 A combinatorial *q*-hypergeometric series

Recall from the introduction the q-hypergeometric series $P_n(w;q)$ which appears in our identities:

$$P_n(w;q) := \sum_{m=0}^{\infty}(-w^2)^m q^{\frac{m(m+1)}{2}}\begin{bmatrix} m+n \\ m \end{bmatrix}_q.$$

This function can be interpreted as the following combinatorial generating function.

Lemma 2.10 *We have that*

$$P_n(w;q) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\alpha_n(k,m)(-w^2)^m q^k,$$

where $\alpha_n(k,m) := p(k \mid m \text{ distinct parts, rank} \le n)$.

Proof We have that

$$\begin{aligned} P_n(w;q) &= \sum_{m=0}^{\infty}\begin{bmatrix} m+n \\ m \end{bmatrix}_q (-w^2)^m q^{\frac{m(m+1)}{2}} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} p(k \mid \le m \text{ parts, parts} \le n)(-w^2)^m q^{\frac{m(m+1)}{2}+k} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(k \mid m \text{ distinct parts, rank} \le n)(-w^2)^m q^k, \end{aligned}$$

which we justify as follows. To move from the first equality to the second above, we use the combinatorial interpretation of the q-binomial term $\begin{bmatrix} n+m \\ m \end{bmatrix}_q$ (see [3, §7.2]). To move from the second equality to the third above, we appeal to the evident bijection between partitions of $m(m+1)/2+k$ with at most m parts, parts at most n, and partitions of k into m distinct parts with rank at most n. Note that partitions into distinct parts have non-negative rank. Thus we have obtained the desired result. □

This interpretation of $P_n(w;q)$ as a generating function gives us an easy way to find its infinite counterpart.

Lemma 2.11

$$\lim_{n\to\infty} P_n(w;q) = (w^2q;q)_\infty,$$

Proof We let the rank n go to infinity in the combinatorial interpretation above in Lemma 2.10, obtaining

$$\lim_{n\to\infty} P_n(w;q) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(k,m)(-w^2)^m q^k,$$

where $p(k,m){:=}p(k \mid m \text{ distinct parts})$.

It is known that the generating function for partitions with distinct parts is

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} p(k,m)t^m q^k = (-tq;q)_\infty.$$

With the substitution $t=-w^2$ into the formula above, we obtain $(w^2q;q)_\infty$ as claimed. □

3 Sums of tails: proofs of Proposition 1.2 and Corollary 1.3.

Proof of Proposition 1.2 To prove the identity in (*i.*) of Proposition 1.2, it is not difficult to show that

$$(1-w)\sum_{n=0}^{\infty}\left(\frac{(wq;q)_\infty}{(-wq;q)_\infty} - \frac{(wq;q)_n}{(-wq;q)_n}\right)w^n = \frac{(wq;q)_\infty}{(-wq;q)_\infty} - (1-w)\sum_{n=0}^{\infty}\frac{(wq;q)_n}{(-wq;q)_n}w^n.$$

Applying [14, (14.31)] proves the result.

The proof of identity (*ii.*) begins similarly. We apply [14, (14.3)] with $q\mapsto q^2$, and then setting $a=t=w$, and $b=wq$, which yields that

$$(1-w)\sum_{n=0}^{\infty}\frac{(wq^2;q^2)_n}{(wq;q^2)_{n+1}}w^n = \sum_{n=0}^{\infty}(1+wq^{2n+1})w^{2n}q^{2n^2+n}.$$

Some additional rewriting shows that this function equals $\theta_2(w;q)$.

To prove (*iii.*), we begin similarly, apply [14, (7.7)], and do some rewriting to reveal $\theta_3(w;q)$.

To prove (*iv.*), we proceed similarly and apply the following identities [14, (6.1), (6.22), (7.7)]:

$$
\begin{aligned}
(1-w)\sum_{n=0}^{\infty}P_n(w;q)w^n &= \sum_{n=0}^{\infty}\frac{(-w^2)^n q^{\frac{n(n+1)}{2}}}{(wq;q)_n}\\
&= \sum_{n=0}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}}w^{3n} + \sum_{n=1}^{\infty}(-1)^n q^{\frac{3n^2-n}{2}}w^{3n-1}\\
&= \theta_3(w;q).
\end{aligned}
$$

The result now follows after a short calculation.

Lastly, to prove (*v.*), we begin by noting that applying [14, (6.1)] to the expression

$$\lim_{t\to 1^-}\frac{\partial}{\partial t}(1-t)\sum_{n=0}^{\infty}(w^2q)_n t^n, \tag{3.1}$$

then differentiating, gives us

$$\lim_{t\to 1^-}\frac{\partial}{\partial t}(1-t)\sum_{n=0}^{\infty}P_n(w;q)t^n + \widetilde{D}(w;q),$$

where

$$\widetilde{D}(w;q) := \lim_{t\to 1^-}\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{(-w^2t)^n q^{\frac{n(n+1)}{2}}}{(q;q)_n}.$$

By Proposition 2.1 of [4], we have that

$$\lim_{t\to 1^-}\frac{\partial}{\partial t}(1-t)\sum_{n=0}^{\infty}P_n(w;q)t^n = \sum_{n=0}^{\infty}((w^2q;q)_\infty - P_n(w;q)).$$

Applying the same proposition to equation (3.1) gives us

$$\sum_{n=0}^{\infty}((w^2q;q)_\infty - (w^2q;q)_n).$$

Lastly, letting $N\to\infty$ in [14, (6.32)], we have that $\widetilde{D}(w;q)$ is equal to $\lim_{t\to 1^-}\frac{\partial}{\partial t}(tw^2q;q)_\infty$. A direct calculation then reveals that $\widetilde{D}(w;q) = -D(w;q)$. Combining the above results proves the desired identity. □

Proof of Corollary 1.3 Identities (*i.*), (*ii.*), and (*iii.*) in Corollary 1.3 follow by differentiating in the variable w the sum of tails identities (*i.*), (*ii.*), and (*iii.*) in Proposition 1.2, respectively, and evaluating at $w=1$. In the case of Corollary 1.3 (*iii.*), we also use the well-known fact that

$$(q;q)_\infty = \sum_{n=0}^{\infty}\left(\frac{12}{n}\right)q^{\frac{n^2-1}{24}}. \tag{3.2}$$

Identity (*iv.*) in Corollary 1.3 can be deduced from Proposition 1.2 in two ways. As above, we differentiate identity (*iv.*) in Proposition 1.2 in w and evaluate at $w=1$ to obtain the result, also using (3.2). On the other hand, we may evaluate Proposition 1.2 identity (*v.*) at $w=1$, and apply (*iii.*) from Corollary 1.3 (just established from Proposition 1.2 in the preceding paragraph), to obtain the result. □

4 Quantum Jacobi forms

In this section, we make Theorem 1.4 more precise in Theorem 4.1, Theorem 4.3, and Theorem 4.6. We also establish their proofs.

4.1 Quantum Jacobi forms from $H_1(w;q)$ and $\theta_1(w;q)$

To state our results, using $\theta_1(w;q)$ and $H_1(w;q)$ as defined in (1.4) and (1.7), and with $w = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$, we define for any positive integers a and b satisfying $a < b$, where b is even and at least 4, the functions

$$\widetilde{\theta}_1(z;\tau) = \widetilde{\theta}_1(a,b;z;\tau) := q^{\frac{a^2}{b^2}} w^{\frac{a}{b}} \left(1 + \theta_1(-iw^{\frac{1}{2}} q^{\frac{a}{b}};q)\right),$$
$$\widetilde{H}_1(z;\tau) = \widetilde{H}_1(a,b;z;\tau) := q^{\frac{a^2}{b^2}} w^{\frac{a}{b}} \left(1 + (1 + iw^{\frac{1}{2}} q^{\frac{a}{b}}) H_1(-iw^{\frac{1}{2}} q^{\frac{a}{b}};q)\right).$$

Remark It follows from [14, (14.31)] that the functions $\widetilde{\theta}_1$ and $\widetilde{H}_1$ are equal.

For ease of notation, we will often omit the dependence of the functions $\widetilde{\theta}_1$ and $\widetilde{H}_1$ on a and b.

Theorem 4.1 *The following are true.*

(1) *The functions $\widetilde{\theta}_1(z;-\tau)$ and $\widetilde{H}_1(z;-\tau)$ are quantum Jacobi forms of weight* $1/2$ *and index* $-1/4$.
(2) *The functions $\widetilde{\theta}_1(z;-\tau)$ and $\widetilde{H}_1(z;-\tau)$ are mock Jacobi forms of weight* $1/2$ *and index* $-1/4$.

Remark The explicit transformation properties of these functions are ultimately deduced from results given in [11]. In particular, the functions exhibit quantum Jacobi transformation properties on the set $\mathcal{Q}_{a,b} \subseteq \mathbb{Q} \times \mathbb{Q}$ defined as follows:

$$\mathcal{Q}_{a,b} := \left\{ \left(\frac{r}{s}, \frac{h}{k}\right) \in \mathbb{Q} \times \mathbb{Q} : \begin{array}{l} \text{(i) } s > 0, k > 0, \gcd(r,s) = \gcd(h,k) = 1, \text{and } k \text{ even} \\ \text{(ii) for all } j \bmod k, hs(a+bj) \not\equiv \frac{bk}{4}(2r+s) \pmod{bks} \\ \text{(iii) if k} \equiv 0 \pmod{4} \text{ then h} \not\equiv \pm 1 \pmod{2b} \end{array} \right\}. \tag{4.1}$$

We refer the reader to [11] and also to the proof of Theorem 4.1 below, for additional details.

A key ingredient to our proof of Theorem 4.1 involves the universal mock theta function $g_2(w;q)$, defined by

$$g_2(w;q) := \sum_{n\geq 0} \frac{(-q;q)_n q^{n(n+1)/2}}{(w;q)_{n+1}(w^{-1}q;q)_{n+1}}.$$

The function g_2 is aptly named "universal," as specializations of this function in the variables w and q yield all of Ramanujan's mock theta functions (up to the addition of modular forms) [19].

As Proposition 4.2 reveals (and makes more precise), the function $\theta_1(w;q)$ is "dual" to the universal mock theta function $g_2(w;q)$ outside of the unit disk. (See for example [9,15,18] for more results along these lines.)

Proposition 4.2 *We have that*

$$\theta_1(w;q^{-1}) = -1 + 2w^{-1}g_2(-w^{-1};q).$$

Proof of Proposition 4.2 Using the identity stated in Theorem 4.1 (which follows from [14, (14.31)]), and the fact that $(a;q^{-1})_n = (-a)^n q^{-\frac{n(n-1)}{2}}(a^{-1};q)_n$, we have that

$$\begin{aligned}\theta_1(w;q^{-1}) &= (1-w)\sum_{n=0}^{\infty}\frac{(wq^{-1};q^{-1})_n}{(-wq^{-1};q^{-1})_n}w^n\\ &= (1-w)\sum_{n=0}^{\infty}\frac{(w^{-1}q;q)_n}{(-w^{-1}q;q)_n}(-w)^n\\ &= -1+2w^{-1}g_{2,3}(-w^{-1};q),\end{aligned}$$

where

$$g_{2,3}(w;q){:=}-\frac{1+w}{2w^2}\sum_{n=0}^{\infty}\frac{(-wq;q)_n}{(wq;q)_n}w^{-n}-\frac{1}{2w}.$$

The result now follows from [9, Theorem 4.1]. □

Proof of Theorem 4.1 From Proposition 4.2 with $w\mapsto -iw^{\frac{1}{2}}q^{-\frac{a}{b}}$, and $w=e^{2\pi iz}, q=e^{2\pi i\tau}$, we have that

$$\tfrac{1}{2}\widetilde{\theta}_1(z;-\tau)=iq^{-\frac{a^2}{b^2}+\frac{a}{b}}w^{\frac{a}{b}-\frac{1}{2}}g_2(-iq^{\frac{a}{b}}w^{-\frac{1}{2}};q),$$

which is the function $G_{a,b}(z;\tau)$ defined in [11]. From [11, Theorem 1.1] we have that $G_{a,b}(z;\tau)$ is a quantum Jacobi form of weight $1/2$, index $-1/4$ on $\mathcal{Q}_{a,b}$ (see (4.1)) and Jacobi group $\Gamma'_{b,1}\ltimes(2\mathbb{Z}\times 2\mathbb{Z})$ (see [11, Definition 3.3]). By Remark (2) following Theorem 1.1 in [11], we also have that this function is a mock Jacobi form when viewed as a function in $\mathbb{C}\times\mathbb{H}$. □

4.2 Quantum Jacobi forms from $H_2(w;q)$ and $\theta_2(w;q)$

Let $w=e^{2\pi iz}$ and $q=e^{2\pi i\tau}$. We define

$$\begin{aligned}\widetilde{\theta}_2(z;\tau)&{:=}q^{\frac{1}{8}}w^{\frac{1}{4}}\theta_2(w^{\frac{1}{2}};q),\\ \widetilde{H}_2(z;\tau)&{:=}q^{\frac{1}{8}}w^{\frac{1}{4}}(1-w^{\frac{1}{2}})H_2(w^{\frac{1}{2}};q).\end{aligned}$$

Remark It follows from [14, (14.4) and (6.3)] that the functions $\widetilde{\theta}_2$ and $\widetilde{H}_2$ are equal.

For $\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma_0(2)$, we define $\psi_\gamma{:=}\left(\frac{C}{D}\right)\left(\frac{C/2}{D}\right)^{-2}\zeta_8^{1-D-BD}$.

Theorem 4.3 *The following are true.*

(1) *The functions $\widetilde{\theta}_2(z;-\tau)$ and $\widetilde{H}_2(z;-\tau)$ are quantum Jacobi forms on S_2 of weight $1/2$ and index $-1/8$ with respect to $\Gamma_0(4)\ltimes(4\mathbb{Z}\times 2\mathbb{Z})$ and with character ψ_γ. In particular, for $z\in(-\frac{1}{4},0)$, $\tau\neq-\frac{1}{4}$, we have that*

$$\begin{aligned}&\widetilde{\theta}_2(z;-\tau)-(4\tau+1)^{-\frac{1}{2}}e\left(\frac{z^2}{2(4\tau+1)}\right)\widetilde{\theta}_2\left(\frac{z}{4\tau+1};\frac{-\tau}{4\tau+1}\right)\\ &\quad=-\frac{1}{2}\int_0^{\infty}\frac{\sum_{\pm}g_{\mp\frac{1}{4},-z}(1+it)}{\sqrt{-i(1+it+4\tau)}}dt, \qquad (4.2)\end{aligned}$$

and the difference in (4.2) extends to a C^∞ function on $(\mathbb{R}\backslash(\mathbb{Z}+\{0,\frac{1}{2},\pm\frac{1}{4}\}))\times(\mathbb{R}\backslash\{-\frac{1}{4}\})$.

(2) *The functions* $\widetilde{\theta}_2(z;-\tau)$ *and* $\widetilde{H}_2(z;-\tau)$ *are mock Jacobi forms of weight* $1/2$ *and index* $-1/8$, *on* $\Gamma_0(2)\ltimes(4\mathbb{Z}\times 2\mathbb{Z})$ *and character* ψ_γ.

Our proof of Theorem 4.3 is very similar to the proof of [15, Theorem 1], but the work done there does not directly apply due to the hypothesis there that $4\mid\beta$.

A key ingredient to the proof of Theorem 4.3 involves the Appell-Lerch sum $B_{1,2}$ (see Sect. 2.2). We recall that the functions N and J appearing in Proposition 4.4 are defined in Sect. 2.1. Similar to Proposition 4.2 used in our proof of Theorem 4.1, Proposition 4.4 below establishes a "dual" to $\theta_2(w;q)$ outside of the unit disk, in terms of a mock Jacobi form.

Proposition 4.4 *We have that*

$$w^{\frac{1}{2}}q^{-\frac{1}{8}}\theta_2(w;q^{-1})=-N(\tau)B_{1,2}(2z;\tau)-J(2z;\tau).$$

Proof of Proposition 4.4 Proceeding as in the proof of Theorem 4.1, we have that

$$\begin{aligned}\theta_2(w;q^{-1})&=(1-w)\sum_{n=0}^{\infty}\frac{(wq^{-2};q^{-2})_n}{(wq^{-1};q^{-2})_{n+1}}w^n\\&=(1-w^{-1})\sum_{n=0}^{\infty}\frac{q^{n+1}w^n(w^{-1}q^2;q^2)_n}{(w^{-1}q;q^2)_{n+1}}.\end{aligned}\tag{4.3}$$

Using [9, Theorem 5], after some simplification, we see that (4.3) is equal to

$$\frac{1}{(1-w)}K(w^{-1};q)-\frac{1}{(1-w)}\frac{(q;q^2)^3_\infty(q^2;q^2)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty},\tag{4.4}$$

where

$$K(w;q):=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}(q;q^2)_n}{(wq^2;q^2)_n(w^{-1}q^2;q^2)_n}.$$

By [15, Lemma 2], we have that

$$\frac{w^{\frac{1}{2}}q^{-\frac{1}{8}}}{(w-1)}K(w^{-1};q)=N(\tau)B_{1,2}(2z;\tau).$$

Using this with (4.4) and some simplifying completes the proof. □

Proof of Theorem 4.3 We have from Proposition 2.9 that

$$\begin{aligned}&B_{1,2}(z;\tau)-(4\tau+1)^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)B_{1,2}\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)\\&\quad=-B^-_{1,2}(z;\tau)+(4\tau+1)^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)B^-_{1,2}\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right),\end{aligned}$$

where

$$B^-_{1,2}(z;\tau):=e\left(\frac{z}{4}\right)A^-_2\left(-\frac{z}{2},-\tau;2\tau\right),$$

and $A^-_2(z_1,z_2;\tau)$ is the nonholomorphic part of the sum given in (2.6) (i.e., $i/2$ multiplied by the sum on j in (2.6)).

Using this and the transformation for N under $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$ given in Lemma 2.3 we have that

$$N(\tau)B_{1,2}(z;\tau)-(4\tau+1)^{-\frac{1}{2}}e\left(\frac{z^2}{2(4\tau+1)}\right)N\left(\frac{\tau}{4\tau+1}\right)B_{1,2}\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)$$
$$=-N(\tau)\left(B^-_{1,2}(z;\tau)-(4\tau+1)^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)B^-_{1,2}\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)\right).$$

Similar to the proof given in [15, p21], we compute that

$$B^-_{1,2}(z;\tau)-(4\tau+1)^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)B^-_{1,2}\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)$$
$$=T(\tau)\left[e\left(\frac{z}{4}\right)\sigma(z;\tau)-(4\tau+1)^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)\right.$$
$$\left.\times\,\varepsilon^3\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)(4\tau+1)^{\frac{1}{2}}e\left(\frac{(-\tau+\frac{1}{2}(4\tau+1))^2}{2(4\tau+1)}\right)\sigma\left(\frac{z}{4\tau+1};\gamma\tau\right)\right],$$

where

$$\sigma(z;\tau):=\sum_{\pm}f_\pm(z;\tau)r_\pm(z;\tau),$$
$$f_\pm(z;\tau):=\frac{i}{2}e\left(\left(\frac{1\mp1}{2}\right)\left(-\frac{z}{2}\right)\right),$$
$$r_\pm(z;\tau):=R\left(-z+\tau-(1\mp1)\tau-\tfrac{1}{2};4\tau\right).$$

The term in brackets $[\cdot]$ above is

$$e\left(\frac{z}{4}\right)\sum_{\pm}f_\pm(z;\tau)r_\pm(z;\tau) \tag{4.5}$$
$$-(4\tau+1)^{-\frac{1}{2}}\varepsilon^3\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)e\left(\frac{(-\tau+\frac{1}{2}(4\tau+1))^2}{2(4\tau+1)}\right) \tag{4.6}$$
$$\times\sum_{\pm}f_\pm\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)r_\pm\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right). \tag{4.7}$$

(Note. These last two large expressions shown above are the same expressions as in [15, (5.12)] and at the top of page 22, respectively, with $(\alpha,\beta)=(1,2)$, although we note again that [15] requires $4\mid\beta$ so not all of the work done there directly applies here. E.g., in what comes next, our proof proceeds somewhat differently than as in [15].)

Let

$$z_1^\pm:=\frac{-z+\tau}{4\tau+1}-(1\mp1)\frac{\tau}{4\tau+1}-\frac{1}{2}=-\frac{1}{2}+\frac{-z\pm\tau}{4\tau+1},$$
$$z_2^\pm:=\frac{1}{2}-\tau+(1\mp1)\tau+z=\frac{1}{2}+z\mp\tau,$$
$$\tau_1:=\frac{-1}{4\tau}-1.$$

Using Lemma 2.6, we establish Lemma 4.5. For brevity, we refer the reader to the analogous result and proof in [15] for more details.

Lemma 4.5 *We have that*

$$r_\pm\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)=a_\pm(z;\tau)h(z_1^\pm\tau_1;\tau_1)-\widetilde{b}_\pm(z;\tau)h(-z_2^\mp;4\tau)+\widetilde{b}_\pm(z;\tau)r_\mp(z;\tau),$$

where

$$\widetilde{b}_\pm(z;\tau):=\pm\, a_\pm(z;\tau)\zeta_8\sqrt{-4i\tau}\,e\left(\frac{-(z_2^\mp)^2}{8\tau}\right),$$

and

$$a_\pm(z;\tau):=\sqrt{-i\tau_1}\,e\left(\frac{-(z_1^\pm)^2\tau_1}{2}\right).$$

Resuming the proof of Theorem 4.3, we have that the expression in (4.5) equals

$$\frac{i}{2}\sum_\pm e\left(\frac{\pm z}{4}\right)r_\pm(z;\tau). \tag{4.8}$$

Further, using Lemma 4.5, we have that the expression in (4.6)-(4.7) equals

$$\begin{aligned}&-(4\tau+1)^{-\frac{1}{2}}\varepsilon^3\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)e\left(\frac{(-\tau+\frac{1}{2}(4\tau+1))^2}{2(4\tau+1)}\right)\\&\quad\times\sum_\pm f_\pm\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)\left(\widetilde{G}_\pm(z;\tau)+\widetilde{b}_\pm(z;\tau)r_\mp(z;\tau)\right),\end{aligned} \tag{4.9}$$

where

$$\widetilde{G}_\pm(z;\tau):=a_\pm(z;\tau)h(z_1^\pm\tau_1;\tau_1)-\widetilde{b}_\pm(z;\tau)h(-z_2^\mp;4\tau).$$

We now consider the portion of the multi-line expression in (4.9) involving $\widetilde{b}_\pm(z;\tau)$ $r_\mp(z;\tau)$, namely

$$\begin{aligned}&-(4\tau+1)^{-\frac{1}{2}}\varepsilon^3\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)e\left(\frac{(-\tau+\frac{1}{2}(4\tau+1))^2}{2(4\tau+1)}\right)\\&\quad\times\sum_\pm f_\pm\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)\widetilde{b}_\pm(z;\tau)r_\mp(z;\tau).\end{aligned} \tag{4.10}$$

This multi-line expression in (4.10) is equal to

$$\begin{aligned}&-(4\tau+1)^{-\frac{1}{2}}\zeta_8^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)e\left(\frac{(-\tau+\frac{1}{2}(4\tau+1))^2}{2(4\tau+1)}\right)\\&\sum_\pm\frac{i}{2}e\left(\left(\frac{1\mp1}{2}\right)\left(-\frac{z}{2(4\tau+1)}\right)\right)(\pm1)a_\pm(z;\tau)\zeta_8\sqrt{-4i\tau}\,e\left(\frac{-(z_2^\mp)^2}{8\tau}\right)r_\mp(z;\tau)\\&\quad=-(4\tau+1)^{-\frac{1}{2}}\zeta_8^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)e\left(\frac{(-\tau+\frac{1}{2}(4\tau+1))^2}{2(4\tau+1)}\right)\end{aligned}$$

$$\sum_{\pm}\frac{i}{2}e\left(\left(\frac{1\mp1}{2}\right)\left(-\frac{z}{2(4\tau+1)}\right)\right)(\pm1)\sqrt{-i\left(-\frac{1}{4\tau}-1\right)}e\left(\frac{-\left(\frac{-z\pm\tau}{4\tau+1}-\frac{1}{2}\right)^2\left(-\frac{1}{4\tau}-1\right)}{2}\right)$$
$$\zeta_8\sqrt{-4i\tau}e\left(\frac{-(\frac{1}{2}\pm\tau+z)^2}{8\tau}\right)r_{\mp}(z;\tau)$$
$$=-\frac{i}{2}\sum_{\pm}e\left(\frac{\mp z}{4}\right)r_{\mp}(z;\tau),$$

which we see cancels with the sum in (4.8).

Next we consider the function $\widetilde{G}_{\pm}(z;\tau)$ from (4.9). Let

$$a_1^{\pm}:=-\frac{1}{4}+\frac{(1\mp1)}{4}=\mp\frac{1}{4},\quad a_2:=-\frac{1}{2}-z,\quad b_1^{\pm}:=z+\frac{1}{4}-\frac{1}{4}(1\mp1):=z\pm\frac{1}{4}.$$

By the hypothesis given in Theorem 4.3 (1), we have that $z\in(-\frac{1}{4},0)$, so we may use Lemma 2.7, noting that $z_1^{\pm}\tau_1=a_2\tau_1-b_1^{\pm}$, and that $z_2^{\pm}=a_1^{\pm}4\tau-a_2$. Thus we have that

$$h(z_1^{\pm}\tau_1;\tau_1)=-e\left(\frac{a_2^2\tau_1}{2}-a_2(b_1^{\pm}+\frac{1}{2})\right)\int_0^{i\infty}\frac{g_{a_2+\frac{1}{2},b_1^{\pm}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau_1)}}du, \tag{4.11}$$
$$h(z_2^{\pm};4\tau)=-e\left(\frac{(a_1^{\pm})^2 4\tau}{2}-a_1^{\pm}(a_2+\frac{1}{2})\right)\int_0^{i\infty}\frac{g_{a_1^{\pm}+\frac{1}{2},a_2+\frac{1}{2}}(u)}{\sqrt{-i(u+4\tau)}}du. \tag{4.12}$$

In the integral in (4.11) we let $u=1-1/\rho$ so that (4.11) becomes

$$-e\left(\frac{a_2^2\tau_1}{2}-a_2(b_1^{\pm}+\frac{1}{2})\right)\int_1^0\frac{g_{a_2+\frac{1}{2},b_1^{\pm}+\frac{1}{2}}\left(1-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}}\frac{\sqrt{4\rho\tau}d\rho}{\rho^2}$$
$$=-e\left(\frac{a_2^2\tau_1}{2}-a_2(b_1^{\pm}+\frac{1}{2})\right)e\left(-\frac{(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right)$$
$$\times\int_1^0\frac{g_{a_2+\frac{1}{2},a_2+b_1^{\pm}+\frac{3}{2}}\left(-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}}\frac{\sqrt{4\rho\tau}d\rho}{\rho^2}, \tag{4.13}$$

where we used Lemma 2.8 (iii). We rewrite

$$a_2+b_1^{\pm}+\frac{3}{2}=\left(a_2+\frac{1}{2}\right)+b_1^{\pm}+1=-a_1^{\pm}+1,$$

and obtain using Lemma 2.7 that (4.13) is equal to

$$-e\left(\frac{a_2^2\tau_1}{2}-a_2(b_1^{\pm}+\frac{1}{2})\right)e\left(-\frac{(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right)\int_1^0\frac{g_{a_2+\frac{1}{2},-a_1^{\pm}+1}\left(-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}}\frac{\sqrt{4\rho\tau}d\rho}{\rho^2}$$
$$=e\left(\frac{a_2^2\tau_1}{2}-a_2(b_1^{\pm}+\frac{1}{2})\right)e\left(-\frac{(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right)$$
$$\times\sqrt{\frac{4\tau}{-1}}i(-i)^{\frac{3}{2}}e\left(\left(a_2+\frac{1}{2}\right)(-a_1^{\pm}+1)\right)\int_1^0\frac{g_{a_1^{\pm},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho.$$

After some additional simplifications, we find that

$$a_\pm(z;\tau)h(z_1^\pm\tau_1;\tau_1) = a_\pm(z;\tau)\sqrt{\frac{4\tau}{-1}}i(-i)^{\frac{3}{2}}e^{\pi i\frac{(1\pm1)}{4}}e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)$$
$$\times\int_1^0 \frac{g_{\mp\frac{1}{4},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho \tag{4.14}$$
$$= a_\pm(z;\tau)\sqrt{4\tau}\rho_8^\pm e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\int_1^0 \frac{g_{\mp\frac{1}{4},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho, \tag{4.15}$$

where $\rho_8^+:=\zeta_8^{-1}$ and $\rho_8^-:=\zeta_8^{-3}$.

Additionally, we have (after simplifying, and using Lemma 2.7) that

$$-\widetilde{b}_\pm(z;\tau)h(-z_2^\mp;4\tau)$$
$$=\pm a_\pm(z;\tau)\zeta_8\sqrt{-4i\tau}e\left(\frac{-(z_2^\mp)^2}{8\tau}\right)e\left(\frac{(a_1^\mp)^2 4\tau}{2}-a_1^\mp\left(a_2+\frac{1}{2}\right)\right)$$
$$\times\int_0^{i\infty}\frac{g_{a_1^\mp+\frac{1}{2},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(\rho+4\tau)}}d\rho$$
$$= a_\pm(z;\tau)\sqrt{4\tau}\rho_8^\pm e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\int_0^{i\infty}\frac{g_{\mp\frac{1}{4},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(\rho+4\tau)}}d\rho. \tag{4.16}$$

Adding (4.15) and (4.16) we obtain that

$$\widetilde{G}_\pm(z;\tau) = a_\pm(z;\tau)\sqrt{4\tau}\rho_8^\pm e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\int_1^{i\infty}\frac{g_{\mp\frac{1}{4},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho. \tag{4.17}$$

We now substitute this into the portion of the multi-line expression in (4.9) involving $\widetilde{G}_\pm(z;\tau)$. That is, from (4.9) we have the expression

$$-(4\tau+1)^{-\frac{1}{2}}\varepsilon^3\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)e\left(\frac{(-\tau+\frac{1}{2}(4\tau+1))^2}{2(4\tau+1)}\right)$$
$$\times\sum_\pm f_\pm\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)\widetilde{G}_\pm(z;\tau). \tag{4.18}$$

Using (4.17), we have that (4.18) equals

$$-(4\tau+1)^{-\frac{1}{2}}\zeta_8^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)$$
$$\times e\left(\frac{\left(-\tau+\frac{1}{2}(4\tau+1)\right)^2}{2(4\tau+1)}\right)\sum_\pm f_\pm\left(\frac{z}{4\tau+1},\frac{\tau}{4\tau+1}\right)\widetilde{G}_\pm(z;\tau)$$
$$=-(4\tau+1)^{-\frac{1}{2}}\zeta_8^{-1}e\left(\frac{z^2}{2(4\tau+1)}\right)e\left(\frac{z}{4(4\tau+1)}\right)$$
$$\times e\left(\frac{\left(-\tau+\frac{1}{2}(4\tau+1)\right)^2}{2(4\tau+1)}\right)\sum_\pm\frac{i}{2}e\left(\left(\frac{1\mp1}{2}\right)\left(\frac{-z}{2(4\tau+1)}\right)\right)$$

$$
\begin{aligned}
&\times \sqrt{-i\left(-\frac{1}{4\tau}-1\right)} e\left(\frac{-(\frac{-z\pm\tau}{4\tau+1}-\frac{1}{2})^2\left(-\frac{1}{4\tau}-1\right)}{2}\right)\\
&\times \sqrt{4\tau}\rho_8^{\pm} e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\int_1^{i\infty}\frac{g_{\mp\frac{1}{4},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho\\
&= -\frac{i}{2}\zeta_8^{-1}\sqrt{i} e\left(\frac{z^2}{2(4\tau+1)}\right) e\left(\frac{z}{4(4\tau+1)}\right) e\left(\frac{\left(-\tau+\frac{1}{2}(4\tau+1)\right)^2}{2(4\tau+1)}\right)\\
&\times e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\\
&\times \sum_{\pm}\rho_8^{\pm} e\left(\left(\frac{1\mp 1}{2}\right)\left(\frac{-z}{2(4\tau+1)}\right)\right) e\left(\frac{-(\frac{-z\pm\tau}{4\tau+1}-\frac{1}{2})^2\left(-\frac{1}{4\tau}-1\right)}{2}\right)\\
&\times \int_1^{i\infty}\frac{g_{\mp\frac{1}{4},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho\\
&= -\frac{i}{2}\zeta_8^{-1}\sqrt{i}q^{\frac{1}{8}}\sum_{\pm}\rho_8^{\pm}(\rho_8^{\pm})^{-1}\int_1^{i\infty}\frac{g_{\mp\frac{1}{4},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho\\
&= -\frac{i}{2}q^{\frac{1}{8}}\int_1^{i\infty}\frac{\sum_{\pm} g_{\mp\frac{1}{4},-z}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho.
\end{aligned}
$$

Overall, under the hypotheses given, we have shown that

$$
\begin{aligned}
&N(\tau)B_{1,2}(z;\tau)-(4\tau+1)^{-\frac{1}{2}} e\left(\frac{z^2}{2(4\tau+1)}\right) N\left(\frac{\tau}{4\tau+1}\right) B_{1,2}\left(\frac{z}{4\tau+1};\frac{\tau}{4\tau+1}\right)\\
&= N(\tau)T(\tau)\frac{i}{2}q^{\frac{1}{8}}\int_1^{i\infty}\frac{\sum_{\pm} g_{\mp\frac{1}{4},-z}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho\\
&= -\frac{i}{2}\int_1^{i\infty}\frac{\sum_{\pm} g_{\mp\frac{1}{4},-z}(\rho)}{\sqrt{-i(4\tau+\rho)}}d\rho = \frac{1}{2}\int_0^{\infty}\frac{\sum_{\pm} g_{\mp\frac{1}{4},-z}(1+it)}{\sqrt{-i(4\tau+1+it)}}dt, \qquad (4.19)
\end{aligned}
$$

also using that $N(\tau)T(\tau) = -q^{-\frac{1}{8}}$.

To complete the transformation and analytic properties stated in the theorem with respect to $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$, we observe that from Proposition 4.4, we have that $\widetilde{\vartheta}_2(z;-\tau) = -N(\tau)B_{1,2}(z;\tau) - J(z;\tau)$. We combine this with (4.19) and the Jacobi modular transformation properties for $J(z;\tau)$ under $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$ given in Lemma 2.4. Thus, to finish the proof (with respect to $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$), we must establish the claimed C^∞ properties in $\mathbb{R}\times\mathbb{R}$. We establish these in Sect. 4.2.1 below.

Next, we observe that $\Gamma_0(4)$ is generated by the matrices $\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$. The appropriate behavior under the second and third generators are easily deduced directly. Similarly, the Jacobi elliptic properties (and subsequently the required analytic properties in $\mathbb{R}\times\mathbb{R}$) may be directly checked. That is, we recall again that $\widetilde{\vartheta}_2(z;-\tau) = -N(\tau)B_{1,2}(z;\tau) - J(z;\tau)$ The elliptic Jacobi properties for $B_{1,2}$ are directly established using its definition and the elliptic Jacobi properties of A_2; the elliptic Jacobi properties of J may be deduced from its definition and those of ϑ (see also Lemma 2.4).

4.2.1 C^∞ properties

In this section we establish the claimed C^∞ properties in Theorem 4.3 (1). The proof follows in a similar manner to the proof of a related result in [7], and we refer the reader there for additional details.

For the region $(-\frac{1}{4}, 0) \times \left(\mathbb{R} \setminus \{-\frac{1}{4}\}\right)$, the result is obtained using the expression on the right-hand side of (4.2), using the Leibniz rule as explained in [7]. To finish the proof, we are left to establish the C^∞ nature of the error to Jacobi transformation in the larger region in $\mathbb{R} \times \mathbb{R}$ given in Theorem 4.3 (1). If it's not the case that $z \in \left(-\frac{1}{4}, 0\right)$, then we appeal to the following facts:

$$h(u+1;\tau) = -h(u;\tau) + \frac{2}{\sqrt{-i\tau}} e\left(\frac{(u+\frac{1}{2})^2}{2\tau}\right), \tag{4.20}$$

$$h(u+\tau;\tau) = -e\left(u+\frac{\tau}{2}\right) h(u;\tau) + 2e\left(\frac{u}{2}+\frac{3\tau}{8}\right), \tag{4.21}$$

which are established in [29]. We begin with three cases: $z \in (-\frac{1}{2}, -\frac{1}{4})$ (case 1), $z \in (0, \frac{1}{4})$ (case 2), and $z \in (\frac{1}{4}, \frac{1}{2})$ (case 3). In all cases, $a_1^\pm \in (-\frac{1}{2}, \frac{1}{2})$. In case 2, $b_1^\pm \in (-\frac{1}{2}, \frac{1}{2})$ and $a_2 + 1 \in (-\frac{1}{2}, \frac{1}{2})$. In case 3, b_1^-, $b_1^+ - 1$, and $a_2 + 1$ are all in $(-\frac{1}{2}, \frac{1}{2})$. In case 1, b_1^+, $b_1^- + 1$, and a_2 are all in $(-\frac{1}{2}, \frac{1}{2})$. With this established, the C^∞ properties follow from (4.20) and (4.21) and those already established (when $z \in (-\frac{1}{4}, 0)$). In the general setting, for $z \in \mathbb{R}\backslash(\mathbb{Z}+\{0, \frac{1}{2}, \pm\frac{1}{4}\})$, we can always find some $\ell \in \mathbb{Z}$ such that $z+\ell$ is in $(-\frac{1}{2}, \frac{1}{2})\backslash\{0, \pm\frac{1}{4}\}$. The result follows according to cases 1-3 above (and the previously established results on $(-\frac{1}{4}, 0)$), making repeated use of (4.20) and (4.21). (See [7] for a similar argument.) □

4.3 Quantum Jacobi forms from $H_{3,j}(w;q)$ and $\theta_3(w;q)$

Let $w = e^{2\pi i z}$, $q = e^{2\pi i \tau}$. Define

$$\widetilde{\theta}_3(z;\tau) := w^{\frac{1}{2}} q^{\frac{1}{24}} \theta_3(w;q),$$
$$\widetilde{H}_{3,j}(z;\tau) := w^{\frac{1}{2}} q^{\frac{1}{24}} (1-w) H_{3,j}(w;q),$$

$(j \in \{1, 2\})$.

Remark It follows from [14, (7.7)] after rewriting the right-hand side appearing there as $\theta_3(w;q)$, that the functions $\widetilde{\theta}_3$ and $\widetilde{H}_{3,1}$ are equal.

Theorem 4.6 *The following are true.*

(1) *The functions* $\widetilde{\theta}_3(z;\tau)$ *and* $\widetilde{H}_{3,j}(z;\tau)$ $(1 \leq j \leq 2)$ *are quantum Jacobi forms of weight* $1/2$ *and index* $-3/2$.
(2) *The functions* $\widetilde{\theta}_3(z;-\tau)$ *and* $\widetilde{H}_{3,1}(z;-\tau)$ *are mock Jacobi forms of weight* $1/2$ *and index* $-3/2$.

Remark The explicit transformation properties of these functions are ultimately deduced from results given in [15]. We refer the reader there, and also to the proof of Theorem 4.6 below, for more details.

Proof of Theorem 4.6 The results for $\widetilde{\theta}_3(z;\tau)$ and $\widetilde{H}_{3,1}(z;\tau)$ follows from work in [15]. Namely, we deduce from the proof of [15, Lemma 17] that

$$T_1(w;q) = \widetilde{\theta}_3(z;\tau) = \sum_{j=1}^{4} \chi_{12}(\alpha_1^{(j)}) C_{\alpha_1^{(j)},12}(12z;12\tau),$$

where T_1, $\alpha_1^{(j)}$, and $C_{\alpha,\beta}$ are as defined in [15]. We have also used that $\chi_{12}(\cdot)$ from [15] is equal to $\left(\frac{12}{\cdot}\right)$. The result then follows as argued on p. 30-31 of [15] in the proof of Theorem 4.

The results for $\widetilde{H}_{3,2}(z;\tau)$ follow from Proposition 1.2 (*iv.*) and the results just established for $\widetilde{\theta}_3(z;\tau)$. That is, by a similar argument to one given in [28], we see using Proposition 1.2 (*iv.*) asymptotically as $\tau \to h/k$ vertically for fixed a/b such that $(\zeta_b^{2a}\zeta_k^h;\zeta_k^h)_\infty$ vanishes (e.g., $(a/b, h/k) \in S_3$), we have that $\widetilde{H}_{3,2}(a/b, h/k)$ and $\widetilde{\theta}_3(a/b, h/k)$ are equal. The function $\widetilde{H}_{3,2}(z;\tau)$ now inherits its quantum properties from $\widetilde{\theta}_3(z;\tau)$ as established above. □

5 Asymptotics and *L*-values: Proof of Theorem 1.5

Proof of Theorem 1.5 Part (1). The proof follows in a similar manner as the proofs of [18, Theorem 1.3 (3)] and [16, Theorem 1.3 (ii)]. For brevity, we provide a detailed sketch of proof. Ultimately, we apply [21, Proposition p.98]. To do so, we show that the coefficients $c_1(n)$ have mean value 0 with period $2k$. We compute, using that $bb' = k$ for some integer b',

$$\begin{aligned}\sum_{n=0}^{2k-1} c_1(n) &= \sum_{n=0}^{2k-1}(-1)^n\zeta_k^{2ab'n+hn^2}\\ &= \left(\sum_{n=0}^{k-1}+\sum_{n=k}^{2k-1}\right)(-1)^n\zeta_k^{2ab'n+hn^2}\\ &= \sum_{n=0}^{k-1}(-1)^n\zeta_k^{2ab'n+hn^2}+\sum_{n=0}^{k-1}(-1)^{n+k}\zeta_k^{2ab'(n+k)+h(n+k)^2}\\ &= \sum_{n=0}^{k-1}(-1)^n\zeta_k^{2ab'n+hn^2}+\sum_{n=0}^{k-1}(-1)^{n+1}\zeta_k^{2ab'n+hn^2}\\ &= 0,\end{aligned}$$

where we have also used that k is odd.

The first way to evaluate the functions given in part (1) of the theorem follows from the asymptotic expansion just established. The second way to evaluate follows from fact that any summand of the sum defining $H_1(w;q)$ which is indexed by $n \geq \mathcal{N}_1$ vanishes, due to the conditions satisfied by a, b, h, k.

Part (2). The asymptotic expansion given part (2) of the theorem follows from [18, Theorem 1.3 (3)], noting that $q^{-\frac{1}{8}}H(-a,b;\tau) = \theta_2(\zeta_b^a;q)$, where $H(a,b;\tau)$ is as defined in [18]. The evaluations given in part (2) of the theorem follow by a similar method of proof as the one above for the functions in part (1) of the theorem.

Part (3). The asymptotic expansion given in part (3) of the theorem for $H_{3,1}(w;q)$ and $\theta_3(w;q)$ follows from [16, Theorem 1.3 (ii)], noting that $\widetilde{\vartheta}(\frac{a}{b};\tau) = \zeta_{2b}^a q^{\frac{1}{24}}\theta_3(\zeta_b^a;q)$, where $\widetilde{\vartheta}(z;\tau)$ is as defined in [16]. The evaluations given in part (3) of the theorem for $H_{3,1}(w;q)$ and $\theta_3(w;q)$ follow by a similar method of proof as the one above for the functions in part (1) of the theorem. The results for $H_{3,2}(w;q)$ follow from those just established as well as Proposition 1.2 (*iv.*): from Proposition 1.2 (*iv.*), we have that for $(a/b, h/k) \in S_3$, $H_{3,2}(\zeta_b^a;q)$ is asymptotic to $\theta_3(\zeta_b^a;q)$ as q tends radially towards ζ_k^h, also noting that $(\zeta_b^{2a}\zeta_k^h;\zeta_k^h)_\infty = 0$. □

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