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Mock modular forms and d-distinct partitions

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ABSTRACT

The generating function for partitions into $(d+1)$-distinct parts $(d+1 \in \mathbb{N})$ may be expressed as the q-hypergeometric series

$$R_{d+1}(1;q) := \sum_{n\geqslant 0} q_{d+1}(n)q^n = \sum_{n\geqslant 0} \frac{q^{(d+1)\binom{n}{2}+n}}{(q;q)_n},$$

where $q_{d+1}(n) := p(n \mid \text{parts differ by at least } d+1)$. Within combinatorial number theory, this function has long since been of historical importance: a famous identity of Euler and Sylvester implies that the distinct parts function $q^{1/24}R_1(1;q)$ is an ordinary modular form, and the celebrated Rogers–Ramanujan identities imply that the 2-distinct parts function $q^{-1/60}R_2(1;q)$ is an ordinary modular form. More recently Alder–Andrews, Zagier, and others have studied the general series $R_{d+1}(1;q)$ for $d \in \mathbb{N}_0$. In particular, Zagier has proved for $d>1$ that these series are never ordinary modular forms; however, other than the cases $d=0$ (Euler) and $d=1$ (Rogers–Ramanujan), the precise modular properties of these combinatorial series remain unknown. Here, we prove for integers $d \geqslant 1$, that the combinatorial q-series $R_{d+1}(1;q)$, and relevant generalizations, are natural denominators of a new class of mixed mock modular forms. As such, we also obtain many new results as corollaries, including new expressions for Zwegers's lauded μ-function, and many of the well-known combinatorial q-series in the subject, confirming the central role that the q-hypergeometric series $R_{d+1}(1;q)$ play within the theory. Once armed with this realization, we also obtain general theorems on the analytic behavior of $R_{d+1}(1;q)$ and

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related series near the unit disk, a priori an impenetrable barrier littered with exponential singularities.

1. Introduction and statement of results

1.1. Introduction

The generating function for partitions into $(d+1)$-distinct parts ($d+1 \in \mathbb{N}$) is given by

$$R_{d+1}(1;q) := \sum_{n\geqslant 0} \frac{q^{(d+1)\binom{n}{2}+n}}{(q;q)_n} = \sum_{n\geqslant 0} q_{d+1}(n)q^n,$$

where

$$q_{d+1}(n) := p(n \mid \text{parts differ by at least } d+1),$$

with $p(n|S) := \#\{\text{partitions of } n \text{ subject to condition } S\}$, and the q-Pochhammer symbol is defined for any $m \in \mathbb{Z}$ by $(x;q)_m := (x;q)_\infty/(xq^m;q)_\infty$, with $(x;q)_\infty := \prod_{j=0}^{\infty}(1-xq^j)$. Within combinatorial number theory, this function has long since been of interest. In the simplest case when $d=0$, $R_1(1;q)$ is the generating function for partitions with distinct parts. Early works of Euler [17] and Sylvester [32] prove the famous combinatorial identity that $q_1(n)$ also equals the number of partitions into odd parts, a fact which can be expressed using generating functions as follows:

$$R_1(1;q) := \sum_{n\geqslant 0} \frac{q^{\binom{n+1}{2}}}{(q;q)_n} = \frac{1}{(q;q^2)_\infty}. \tag{1.1}$$

Not only does (1.1) provide a beautiful combinatorial identity, it also shows that the two associated combinatorial generating functions are (up to multiplication by $q^{1/24}$) modular forms when $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, as the right most expression in (1.1) is easily recognized to be the modular η-quotient $q^{-1/24}\eta(2\tau)/\eta(\tau)$. The next simplest case of the function $R_{d+1}(1;q)$ to study is the case in which $d=1$. Similar to (1.1), Rogers and Ramanujan proved that

$$R_2(1;q) := \sum_{n\geqslant 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}. \tag{1.2}$$

The second equality in (1.2) is one of the famous Rogers–Ramanujan identities (see for example [2,31]). Interpreted combinatorially, (1.2) yields the identity that the number of partitions into parts with minimal difference two equals the number of partitions into parts congruent to 1 and 4 modulo 5. Moreover, as was the case with Euler's product in (1.1), the infinite product on the right hand side of (1.2) is a modular form when $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, after multiplication by $q^{-1/60}$.

Given the combinatorial and modular properties of the function $R_{d+1}(1;q)$ in the cases $d = 0$, $d = 1$ exhibited in (1.1) and (1.2), it is thus a natural question to ask whether similar properties hold for the functions $R_{d+1}(1;q)$ for all non-negative integers d. For general d, these questions are much more subtle, and questions remain to this day. Let us turn first to combinatorial properties of $R_{d+1}(1;q)$ for general d. In more recent years since the time of Rogers–Ramanujan, Euler, and Sylvester, a conjecture of Alder, refined by Andrews, relates the coefficients $q_{d+1}(n)$ to the partition numbers

$$Q_{d+1}(n) := p\big(n \bigm| \text{parts} \equiv \pm 1 \pmod{d+4}\big).$$

Rather than a combinatorial equality (as was the case for $d = 0$, $d = 1$ discussed above), the general conjecture for all $d \geqslant 0$ is expressed as an inequality.

Conjecture A *(Alder–Andrews). (See [3].) For any $d \geqslant 0$, for any $n \geqslant 1$, we have that*

$$q_{d+1}(n) \geqslant Q_{d+1}(n).$$

The refined version of this conjecture states that for most n and $d \geqslant 3$, the inequality is in fact strict. This was proved to be true by the work of Andrews [3], Yee [34], and Alfes, Jameson, and Lemke Oliver [1].

Similarly, the precise modular properties of $R_{d+1}(1;q)$ for general d remain unknown. In fact, the modularity of the series $R_{d+1}(1;q)$ comprises a part of the Nahm conjecture [28], which purports that the modularity of certain generalized q-hypergeometric series is dictated by algebraic properties of the Bloch group. We summarize the Nahm conjecture in the rank 1 case as follows. (In what follows, $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$.)

Conjecture B. *(See Nahm [28].) Let $\alpha \in \mathbb{Q}^+$. Then there exist $\beta, \gamma \in \mathbb{Q}$, such that the q-hypergeometric series*

$$\sum_{n\geqslant 0} \frac{q^{\alpha\frac{n^2}{2}+\beta n+\gamma}}{(q;q)_n}$$

is a modular form if and only if a corresponding algebraic solution is torsion in the Bloch group.

Recently, by studying asymptotic properties, Zagier [35] has given a finite list of all triples (α, β, γ) for which the Nahm series in Conjecture B are modular forms. We summarize his result as follows.

Proposition C. *(See Zagier [35].) For $\alpha > 2$, the Nahm series of Conjecture B are never modular forms for any pairs $(\beta, \gamma) \in \mathbb{Q}^2$.*

Zagier's result in Proposition C implies that for $d > 1$ (and any $\gamma \in \mathbb{Q}$) the distinct parts generating function $q^\gamma R_{d+1}(1;q)$ is in fact *not* an ordinary modular form. With this in mind, one of our two main purposes here is to answer the following natural question.

Question D. What roles, if any, do the $(d+1)$-distinct partition generating functions

$$R_{d+1}(b;q) := \sum_{n \geqslant 0} \frac{q^{(d+1)\binom{n}{2}+n} b^n}{(q;q)_n} \tag{1.3}$$

play within the theory of *mock* modular forms?

We will show that the answer to this question is unexpectedly nice. All of these series for integers $d \geqslant 1$ (see Theorem 1 (i)) arise naturally as "denominators" of a new class of mixed mock modular forms. Moreover, we produce a key identity which gives the celebrated μ-functions of Zwegers, most generically, in the special case $d = 1$ (see Theorem 1 (ii)).

To describe this more precisely, recall the ubiquitous mock Jacobi forms $\mu(u, v; \tau)$ studied by Zwegers in his thesis [38] under Zagier, defined for $\tau \in \mathbb{H}$, $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, by

$$\mu(u,v;\tau) := \frac{a^{\frac{1}{2}}}{\vartheta(v;\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{n(n+1)}{2}} b^n}{1 - aq^n}, \tag{1.4}$$

where we let $a := e(u)$ and $b := e(v)$ with $e(x) := e^{2\pi i x}$, and as usual $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$. Here, the Jacobi theta function is defined by

$$\begin{aligned} \vartheta(v;\tau) &:= i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} b^{n+\frac{1}{2}} \\ &= -i q^{\frac{1}{8}} b^{-\frac{1}{2}} \prod_{n=1}^{\infty} \left(1-q^n\right)\left(1-bq^{n-1}\right)\left(1-b^{-1}q^n\right). \end{aligned} \tag{1.5}$$

The functions $\mu(u, v; \tau)$, and their transformation properties, play central roles in explaining the modularity properties of Ramanujan's original "mock theta functions," a list of curious q-series Ramanujan defined in his last letter to Hardy [9]. An example of one of Ramanujan's mock theta functions is the q-hypergeometric series

$$f(q) := \sum_{n \geqslant 0} \frac{q^{n^2}}{(-q;q)_n^2}.$$

The precise roles played by these functions within the theory of modular forms were not well understood until recently. Namely, due to work of Zwegers [38], Bringmann and Ono [29,36], and others, we now know that Ramanujan's mock theta functions are[1] holomorphic parts of weight 1/2 harmonic weak Maass forms, non-holomorphic modular forms defined by Bruinier and Funke [12]. That is, alone the mock theta functions do not transform like modular forms, but can made to transform appropriately once "completed" by the addition of suitable non-holomorphic functions, at the expense of losing their holomorphic properties (see Section 2).

The mock theta function $f(q)$ is in fact the sole example Ramanujan provides in his last letter to Hardy to illustrate in more detail his notoriously vague definition of a mock theta function. In particular, from the definition of $f(q)$, it is not difficult to see that $f(q)$ converges for $|q| < 1$, but has exponential singularities when q is an even order root of unity. To cancel the exponential singularities at even order roots of unity, Ramanujan offers the function

$$\theta(q) := \left(q; q^2\right)_\infty \cdot \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2},$$

which (up to a power of q) is a modular form when $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$. Ramanujan makes the following remarkable claim.

Claim E *(Ramanujan). As q approaches an even order $2k$ root of unity radially from within the unit disk, we have that*

$$f(q) - (-1)^k\theta(q) = O(1).$$

See also the work of Berndt [8], who recently wrote in his survey of Ramanujan's "lost notebook" about this claim and Ramanujan's imprecise definition of a mock theta function. In very recent joint work by the author, Ono, and Rhoades [18], we prove Ramanujan's claim by establishing an exact formula for the suggested $O(1)$ constants, a formula which amounts to the special value of a "quantum modular form" [37].

Theorem F. *(See Folsom, Ono, and Rhoades [18, Theorem 1.1].) If ζ is a primitive even order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(f(q) - (-1)^k\theta(q)\right) = -4\sum_{n=0}^{k-1}(-\zeta;\zeta)_n^2\zeta^{n+1}.$$

In [18], we prove this as a special case of a more general theorem relating combinatorial mock modular and modular q-series. In particular, in Theorem 1.2 of [18], we give

[1] Note. Here and throughout when discussing modularity, as is standard in the subject, we may mean modular up to a suitable q-power and/or change of variable.

a generalization pertaining to $\mathcal{R}(\omega; q)$, Dyson's combinatorial partition rank generating function, defined by

$$\mathcal{R}(\omega; q) := \sum_{n \geqslant 0} \sum_{m=-\infty}^{\infty} N(m,n)\omega^m q^n = \sum_{n \geqslant 0} \frac{q^{n^2}}{(\zeta q; q)_n (\zeta^{-1} q; q)_n}.$$

Here $N(m,n)$ counts the number of partitions of n with rank m, where the rank of a partition is defined to be its largest part minus the number of its parts. If $\omega = -1$, then $\mathcal{R}(-1; q) = f(q)$, Ramanujan's mock theta function. Bringmann and Ono [11] more generally prove that $\mathcal{R}(\omega; q)$ is a mock modular form (i.e. holomorphic part of harmonic weak Maass form) for any root of unity $\omega \neq 1$.

With this in mind, the second main question we seek to answer is the following.

Question G. What can be said about the analytic behavior of the q-series $R_{d+1}(1; q)$ near the unit disk?

Remark. Armed with our positive answer to Question D, we will subsequently answer Question G, and obtain general theorems on the analytic behavior of $R_{d+1}(1; q)$ and related series near the unit disk, a priori an impenetrable barrier littered with exponential singularities. (See Theorem 2.)

1.2. Statement of results

Here our purposes are twofold. First, for all integers $d \geqslant 1$, we indeed positively answer Question D, and incorporate the $(d+1)$-distinct partition generating function $R_{d+1}(b; q)$ into the theory of mock modular forms by establishing a natural q-hypergeometric quotient identity for general mixed mock modular forms, in which the divisor is $R_{d+1}(b; q)$. Our first main result (Theorem 1) is roughly of the following shape:

$$\begin{bmatrix} d\text{-distinct partition} \\ \text{generating function} \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} \text{bilateral } q\text{-basic} \\ \text{hypergeometric series} \end{bmatrix} + \begin{bmatrix} \text{Hecke-type } q\text{-basic} \\ \text{hypergeometric series} \end{bmatrix} \right)$$
$$= \begin{bmatrix} \text{(mixed) mock} \\ \text{modular form} \end{bmatrix}.$$

More precisely, we define Zwegers's [39] "mixed mock" modular generalization of the mock Jacobi forms $\mu(u, v; \tau)$ for any integer $d \geqslant 1$

$$A_d(u, v; \tau) := a^{\frac{d}{2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{dn} q^{d\binom{n+1}{2}} b^n}{1 - aq^n}. \tag{1.6}$$

In particular, $A_1(u, v; \tau) = \vartheta(v; \tau)\mu(u, v; \tau)$. (See Section 2 for more on these functions and their modular properties.) Next we define the bilateral q-hypergeometric series

$$B_d(u,v;\tau) := \frac{a^{d/2}(q;q)_\infty}{(1-a)(qa^{-1};q)_\infty} \cdot \sum_{n=-\infty}^{\infty} (-1)^{dn} q^{d\binom{n}{2}} \left(qb^{-1}\right)^n \left(a^{-1};q\right)_n,$$

and the Hecke-type q-hypergeometric series for integers $0 \leqslant j \leqslant d-1$

$$\Delta_d^{(j)}(u,v;\tau) := \epsilon_d^{(j)}(a,b;q) \sum_{n\geqslant 0} \left(q^{n+1};q\right)_\infty (-1)^{n(d+1)} b^n \left(qa^{-1}\right)^{dn} q^{(d+1)\binom{n}{2}} \times \sum_{m=0}^{n-1} q^{-d\binom{m+1}{2}-jm}(-a)^{dm} b^{-m},$$

where

$$\epsilon_d^{(j)}(a,b;q) := q^{-\frac{d}{8}} i^{-d} b^{-\frac{1}{2}} q^{-\frac{j}{2}} a^j.$$

Using $\Delta_d^{(j)}(u,v;\tau)$, we define the "mixed" Hecke-type sum of length d

$$T_d(u,v;\tau) := \frac{a^{d/2}}{(q;q)_\infty} \sum_{j=0}^{d-1} \vartheta\left(v + j\tau + \frac{d-1}{2}; d\tau\right) \Delta_d^{(j)}(u,v;\tau).$$

Using the $(d+1)$-distinct partition generating functions $R_{d+1}(b;q)$ (defined in (1.3)), we define

$$Q_d(a,b;q) := \frac{B_d(u,v;\tau) + T_d(u,v;\tau)}{R_{d+1}(ba^{-d}(-q)^{d-1};q)}. \tag{1.7}$$

Answering Question D, our first main result is the following theorem.

Theorem 1. *Let $\tau \in \mathbb{H}$, $u,v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$.*

(i) *For $d \in \mathbb{N}$, we have that*

$$Q_d(a,b;q) := \frac{B_d(u,v;\tau) + T_d(u,v;\tau)}{R_{d+1}(ba^{-d}(-q)^{d-1};q)} = A_d(u,v;\tau).$$

In particular, $Q_d(a,b;\tau)$ is a mixed mock modular form.

(ii) *For $d = 1$, we have that*

$$\frac{Q_1(a,b;q)}{\vartheta(v;\tau)} = \frac{\left(\frac{B_1(u,v;\tau)}{\vartheta(v;\tau)} + \frac{a^{\frac{1}{2}}\Delta_1^{(0)}(u,v;\tau)}{(q;q)_\infty}\right)}{R_2(ba^{-1};q)} = \mu(u,v;\tau).$$

In particular, $Q_1(a,b;q)/\vartheta(v;\tau)$ is a mock modular form.

(iii) *For $d = 1$, and $b = a$, $b = aq$ (respectively), we have that*

$$\text{1)}\quad \frac{Q_1(a,a;q)}{\vartheta(u;\tau)} = \left(q;q^5\right)\left(q^4;q^5\right)_\infty\left(\frac{B_1(u,u;\tau)}{\vartheta(u;\tau)} + \frac{a^{\frac{1}{2}}\Delta_1^{(0)}(u,u;\tau)}{(q;q)_\infty}\right) = \mu(u,u;\tau),$$

$$\text{2)}\quad \frac{Q_1(a,aq;q)}{\vartheta(u+\tau;\tau)} = \left(q^2;q^5\right)_\infty\left(q^3;q^5\right)_\infty\left(\frac{B_1(u,u+\tau;\tau)}{\vartheta(u+\tau;\tau)} + \frac{a^{\frac{1}{2}}\Delta_1^{(0)}(u,u+\tau;\tau)}{(q;q)_\infty}\right) = \mu(u,u+\tau;\tau).$$

In particular, $Q_1(a,a;q)/\vartheta(u;\tau)$ and $Q_1(a,aq;q)/\vartheta(u+\tau;\tau)$ are mock modular forms.

Remark. The non-holomorphic “completions” of the functions $Q_d(a,b;q)$ defined in (1.7) are explicitly given in Section 2 (2.7), by making use of Theorem 1 (i).

Remark. Theorem 1 yields many new q-series identities, involving $R_{d+1}(b;q)$, and many of the known examples of combinatorial (mixed) mock modular forms, including Zwegers's general mock Jacobi forms $\mu(u,v;\tau)$, Dyson's partition rank generating function, the “universal” mock theta functions, and all of Ramanujan's original mock theta functions. We illustrate this with Propositions 1–4 in Sections 1.3–1.5, and Examples 1–5 in Section 1.6.

Our second main result addresses Question G, and establishes that certain “radial limits” in the sense of Ramanujan as discussed above, as q approaches roots of unity ζ, involving $R_{d+1}(b;q)$ and $A_d(u,v;\tau)$, are given by explicit finite polynomials in roots of unity. Our second main result is roughly of the following shape:

$$\lim_{q\to\zeta}\left(\begin{bmatrix}\text{(mixed) mock}\\ \text{modular form}\end{bmatrix}\cdot\begin{bmatrix}(d+1)\text{-distinct partition}\\ \text{generating function}\end{bmatrix} - \begin{bmatrix}\text{combinatorial } q\text{-basic}\\ \text{hypergeometric series}\end{bmatrix}\right) = \begin{bmatrix}\text{finite poly-}\\ \text{nomial in } \zeta\end{bmatrix}.$$

In order to state this more precisely, we define for positive integers d the function

$$P_{d+1}(u,v;\tau) := \sum_{n\geqslant 0}\frac{(-1)^{(d+1)n}q^{(d+1)\binom{n}{2}+dn}a^n}{(bq;q)_n},$$

and for positive integers d, r, s with $s|k$ i.e. $ss' = k$, the constant (with $\zeta_N := e^{2\pi i/N}$)

$$\lambda_d = \lambda_d\binom{r}{s} := \left(1-\zeta_s^r\right)\zeta_{2s}^{-dr}.$$

Answering Question G, we have the following theorem.

Theorem 2. *Let h, k be positive integers satisfying* $\gcd(h,k)=1$, *and let s, m be positive integers with $s|k$ ($ss'=k$). Further, let ℓ, r be integers, and let $1 \leqslant h' \leqslant k-1$ be such that $hh' \equiv -1 \pmod{k}$. For $d \in \mathbb{N}$, as $q \to \zeta_k^h$ radially from within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\left(\frac{\lambda_d\cdot(\zeta_s^{-r}q;q)_\infty}{(q;q)_\infty}\left(A_d\left(\frac{r}{s},\frac{\ell}{m};\tau\right)\cdot R_{d+1}\left((-1)^{d+1}\zeta_s^{-dr}\zeta_m^{\ell}q^{d-1};q\right)-T_d\left(\frac{r}{s},\frac{\ell}{m};\tau\right)\right)\right.$$
$$\left.-P_{d+1}\left(\frac{r}{s}+\frac{\ell}{m},\frac{r}{s};\tau\right)\right)$$
$$=\sum_{n=1}^{rs'(k-h')}(-1)^{dn}\left(\zeta_m^{-\ell}\zeta_k^h\right)^n\zeta_k^{hd\binom{n}{2}}\left(\zeta_s^{-r};\zeta_k^h\right)_n.$$

It is of interest to compare these "radial limits" of Theorem 2 above, and Propositions 1–4 and Examples 1–5 below, to Theorem F and its generalization in [18], which express asymptotic differences of combinatorial mock modular and modular forms as finite polynomials in roots of unity, which arise from quantum modular forms.

Problem H. Conjecture A is a weak inequality, thus, it is natural to seek more precise asymptotics than have been previously obtained, keeping in mind Theorem 2. We leave this as a problem for future study. We also point the interested reader to a related recent work of Bringmann and Mahlburg [10], on Schur's partition theorem and mixed mock modular forms.

Remark. As the case with Theorem 1 (see Remark following), Theorem 2 too may be applied to many of the known combinatorial (mixed)-mock modular examples, as we will show in Propositions 1–4 and Examples 1–5 below.

1.3. 2-Distinct partitions and mock modular forms

Theorem 2 ultimately leads to the following propositions in the case $d=1$, pertaining to the 2-distinct partition generating functions. We point out that these results do not follow directly from Theorem 2, and give their proofs in Section 5. Proposition 1 gives a finite radial limit involving $R_2(b;q)$ and the mock Jacobi forms $\mu(u,v;\tau)$.

Proposition 1. *Let h, k be positive integers such that* $\gcd(h,k)=1$, *and let s, m be positive integers such that $s|k$ ($ss'=k$) and $m|k$ ($mm'=k$). Further, let ℓ, r be integers, and let $1 \leqslant h' \leqslant k-1$ be such that $hh' \equiv -1 \pmod{k}$. As $q \to \zeta_k^h$ radially from within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\left(\frac{\lambda_1\cdot\vartheta(\frac{\ell}{m};\tau)\cdot(q\zeta_s^{-r};q)_\infty}{(q;q)_\infty}\cdot R_2\left(\zeta_s^{-r}\zeta_m^{\ell};q\right)\cdot\mu\left(\frac{r}{s},\frac{\ell}{m};\tau\right)-P_2\left(\frac{r}{s}+\frac{\ell}{m},\frac{r}{s};\tau\right)\right)$$
$$=\sum_{n=1}^{rs'(k-h')}(-1)^n\zeta_m^{-\ell n}\zeta_k^{h\binom{n+1}{2}}\left(\zeta_s^{-r};\zeta_k^h\right)_n.$$

Proposition 2 shows that the 2-distinct rank function is asymptotically "close" to a related q-hypergeometric combinatorial series.

Proposition 2. *Let h, k be positive integers such that $\gcd(h,k)=1$, and let s, m be positive integers such that $s|k$ $(ss'=k)$ and $m|k$ $(mm'=k)$. Further, let ℓ, r be integers, and let $1 \leqslant h' \leqslant k-1$ be such that $hh' \equiv -1 \pmod{k}$. As $q \to \zeta_k^h$ radially from within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\left(\frac{\zeta_s^{\ell m' r h'}\cdot(q;q)_\infty}{(\zeta_s^r q;q)_\infty}\cdot R_2\big(\zeta_s^{-r}\zeta_m^{\ell};q\big)-P_2\left(\frac{r}{s}+\frac{\ell}{m},\frac{r}{s};\tau\right)\right)$$
$$=\sum_{n=1}^{rs'(k-h')}(-1)^n\zeta_m^{-\ell n}\zeta_k^{h\binom{n+1}{2}}\big(\zeta_s^{-r};\zeta_k^h\big)_n.$$

1.4. Universal mock theta functions

Next we give another proposition that follows using Theorem 2, pertaining to the "universal" mock theta functions of Gordon and McIntosh [20]

$$g_2(\omega;q):=\sum_{n\geqslant 0}\frac{(-q;q)_n q^{n(n+1)/2}}{(\omega;q)_{n+1}(q/\omega;q)_{n+1}},$$
$$g_3(\omega;q):=\sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(\omega;q)_{n+1}(q/\omega;q)_{n+1}},$$

aptly named since any of Ramanujan's original mock theta functions may be expressed in terms of $g_2(\omega;q)$ or $g_3(\omega;q)$ upon suitable specialization of the parameters ω and q. For example, the mock theta function $f(q)$ satisfies

$$f(-q)=-4qg_3\big(q;q^4\big)+\frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3(q^4;q^4)_\infty^3}.$$

To describe our results, we define the constants

$$\sigma\begin{pmatrix}h & r\\ k & s\end{pmatrix}:=\sum_{n=1}^{(k-h)rs'}\zeta_k^{hn}\big(\zeta_s^{-r};\zeta_k^h\big)_n\big(\zeta_s^{r};\zeta_k^h\big)_n,$$
$$\beta\begin{pmatrix}h & r\\ k & s\end{pmatrix}:=\sigma\begin{pmatrix}h & r\\ k & s\end{pmatrix}+\big(1-\zeta_s^r\big). \tag{1.8}$$

Proposition 3. *Let h, k, s be positive integers such that $\gcd(h,k)=1$, and $s|k$ $(ss'=k)$, and r an integer. Further, in part* (i), *suppose that $2|h$ (hence k odd). Let $1 \leqslant h' \leqslant k-1$ be such that $hh' \equiv -1 \pmod{k}$. Then as $q \to \zeta_k^h$ radially from within the unit disk, we have that*

(i)
$$\lim_{q\to\zeta_k^h}\left(\frac{i\lambda_1 q^{\frac{1}{8}}\cdot(\zeta_s^{-r}q;q)_\infty\cdot\vartheta(\frac{h}{2k}-\frac{r}{s};\tau)}{(q;q)_\infty}\cdot g_2\big(\zeta_s^r;q^{1/2}\big)\cdot R_2\big(\zeta_k^{h/2-2rs'};q\big)\right.$$
$$\left.-P_2\left(\frac{h}{2k},\frac{r}{s};\tau\right)\right)$$
$$=\sum_{n=1}^{rs'(k-h')}(-1)^n\zeta_s^{rn}\zeta_{2k}^{hn^2}\big(\zeta_s^{-r};\zeta_k^h\big)_n,$$

(ii)
$$\lim_{q\to\zeta_k^h}\left(\big(\zeta_s^{-r}q;q\big)_\infty\cdot\left(g_3\big(\zeta_s^r;q\big)\cdot\zeta_s^r\big(1-\zeta_s^r\big)+\beta\begin{pmatrix}h & r\\ k & s\end{pmatrix}\right)\cdot R_2\big(\zeta_s^{-2r};q\big)\right.$$
$$\left.-P_2\left(0,\frac{r}{s};\tau\right)\right)$$
$$=\sum_{n=1}^{rs'(k-h')}(-1)^n\zeta_s^{rn}\zeta_k^{h\binom{n+1}{2}}\big(\zeta_s^{-r};\zeta_k^h\big)_n.$$

1.5. Dyson's rank function

Theorem 1 and Theorem 2 may also be used to relate Dyson's combinatorial rank generating function $\mathcal{R}(\omega;q)$ to the $(d+1)$-distinct partition functions. Note that in Proposition 4 we relate $\mathcal{R}(\omega;q)$ to both the 2-distinct and 4-distinct partition generating functions.

Proposition 4. *With notation as above, we have that*

$$\mathcal{R}(a;q)=\frac{(1-a)a^{-\frac{3}{2}}}{(q;q)_\infty}\cdot\frac{(B_3(u,-\tau;\tau)+T_3(u,-\tau;\tau))}{R_4(a^{-3}q;q)}.$$

Further, for positive integers h, k, s and integer r satisfying $\gcd(h,k)=1$, $s|k$ $(ss'=k)$, *with* $1\leqslant h'\leqslant k-1$ *such that* $hh'\equiv -1 \pmod k$, *as* $q\to\zeta_k^h$ *radially from within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\left(\big(\zeta_s^{-r}q;q\big)_\infty\cdot\left(\mathcal{R}\big(\zeta_s^r;q\big)+\sigma\begin{pmatrix}h & r\\ k & s\end{pmatrix}\right)\cdot R_2\big(\zeta_s^{-2r};q\big)-P_2\left(0,\frac{r}{s};\tau\right)\right)$$
$$=\sum_{n=1}^{rs'(k-h')}(-1)^n\zeta_s^{rn}\zeta_k^{h\binom{n+1}{2}}\big(\zeta_s^{-r};\zeta_k^h\big)_n.$$

1.6. Examples

In this section we further illustrate Theorem 1, Theorem 2, and Propositions 1–4, with a number of examples.

Example 1 *(The third mock theta function $f(q)$).* The identity for $\mathcal{R}(a;q)$ given in Proposition 4 in the case $u = 1/2$ shows that Ramanujan's mock theta function $f(q)$ is related to the 4-distinct partition generating function by the following quotient identity:

$$f(q) = \frac{2i}{(q;q)_\infty} \cdot \frac{(B_3(\frac{1}{2}, -\tau;\tau) + T_3(\frac{1}{2}, -\tau;\tau))}{R_4(-q;q)}.$$

Moreover, let $\gcd(h,k) = 1$, where h, k are positive integers with $2|k$, let $r = 1$, $s = 2$, and let $1 \leqslant h' \leqslant k-1$ be such that $hh' \equiv -1 \pmod{k}$. The radial limit established in Proposition 4 in this setting, together with the Rogers–Ramanujan identity (1.2), shows that as $q \to \zeta_k^h$ radially within the unit disk, we have that

$$\lim_{q \to \zeta_k^h} \left(\frac{(-q;q)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} \cdot \left(f(q) + \sigma \begin{pmatrix} h & 1 \\ k & 2 \end{pmatrix} \right) - \sum_{n \geqslant 0} \frac{q^{n^2}}{(-q;q)_n} \right)$$
$$= \sum_{n=1}^{\frac{k}{2}(k-h')} \zeta_k^{h\binom{n+1}{2}} \left(-1;\zeta_k^h\right)_n.$$

Example 2 *(Mathieu moonshine and concave compositions).* The function

$$v_2(q) := \frac{1}{(q;q)_\infty^3} \left(\sum_{n \neq 0} \frac{(-1)^{n+1} n q^{\binom{n+1}{2}}}{1-q^n} - \frac{1}{4} - 2 \sum_{n \geqslant 1} \frac{q^n}{(1+q^n)^2} \right)$$

has been studied by Andrews [5] and Andrews, Rhoades, and Zwegers [7] in connection with concave compositions. The function $v_2(q)$ also arises in a different setting in work of Eguchi, Ooguri, and Tachikawa [16], and Cheng [13], related to the Mathieu group M_{24} and "Mathieu moonshine." In another direction, Mathieu moonshine series appear in work of Griffin, Malmendier, and Ono [21] and Malmendier and Ono [26], as related to $\mathbb{C}P^2$ Donaldson invariants in gauge theory. To apply our results to this function, we use the fact established in [7] that $v_2(q) = iq^{1/8}\mu(\frac{1}{2}, \frac{1}{2};\tau)$. With this, Theorem 1 (iii) case 1) shows that

$$v_2(q) = iq^{1/8}\left(q;q^5\right)_\infty \left(q^4;q^5\right)_\infty \left(\frac{B_1(\frac{1}{2}, \frac{1}{2};\tau)}{\vartheta(\frac{1}{2};\tau)} + \frac{i\Delta_1^{(0)}(\frac{1}{2}, \frac{1}{2};\tau)}{(q;q)_\infty} \right).$$

Next, we let $\gcd(h,k) = 1$, where h, k are positive integers with $2|k$, and let $1 \leqslant h' \leqslant k-1$ be such that $hh' \equiv -1 \pmod{k}$. Then as $q \to \zeta_k^h$ radially within the unit disk, Proposition 1 with $r = \ell = 1$, $s = m = 2$ implies that

$$\lim_{q \to \zeta_k^h} \left(\frac{4 \cdot (-q;q)_\infty^3}{(q;q^5)_\infty (q^4;q^5)_\infty} \cdot v_2(q) - \sum_{n \geqslant 0} \frac{q^{n^2}}{(-q;q)_n} \right) = \sum_{n=1}^{\frac{k}{2}(k-h')} \zeta_k^{h\binom{n+1}{2}} \left(-1;\zeta_k^h\right)_n.$$

Example 3 *(The second order mock theta function $B(q)$).* One of the second order mock theta functions is the q-hypergeometric series defined by

$$B(q) := \sum_{n\geqslant 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2}.$$

In terms of the universal mock theta function $g_2(a;q)$, we have that $B(q^{1/4}) = g_2(q^{1/4};q^{1/2})$ (see [20]). Using this, we let h, k be positive integers with $\gcd(h,k)=1$, and $4|h$ (implying k is odd), and let $1\leqslant h'\leqslant k-1$ be such that $hh'\equiv -1 \pmod{k}$. Under these hypotheses, Proposition 3 part (i) with $r=h/4$ and $s=k$ shows that as $q\to\zeta_k^h$ radially from within the unit disk, we have that

$$\lim_{q\to\zeta_k^h}\left(\frac{i\lambda_1(\frac{\frac{h}{4}}{k})q^{\frac{1}{8}}\cdot(\zeta_{4k}^{-h}q;q)_\infty\cdot\vartheta(\frac{h}{4k};\tau)}{(q;q)_\infty(q;q^5)_\infty(q^4;q^5)_\infty}\cdot B(q^{1/4}) - \sum_{n\geqslant 0}\frac{q^{n^2}\zeta_{2k}^{hn}}{(\zeta_{4k}^h q;q)_n}\right)$$
$$= \sum_{n=1}^{\frac{h}{4}(k-h')} (-1)^n\zeta_{4k}^{hn}\zeta_{2k}^{hn^2}\left(\zeta_{4k}^{-h};\zeta_k^h\right)_n.$$

Example 4 *(The overpartition generating function).* The rank generating function for overpartitions (see [25]) is defined by

$$O_2(w;q) := 1+\sum_{n\geqslant 1}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)w^m q^n$$

where $\overline{N}(m,n) := \#\{\text{overpartitions of } n \text{ with rank } m\}$. Using a hypergeometric expression for the generating function $O_2(w;q)$ [25], it is not difficult to show under appropriate hypotheses that $O_2(w;q)(1+w)/(2w(1-w)) - 1/(2w) = g_2(w;q)$. With this, we let h, k, s be positive integers and r an integer satisfying $\gcd(h,k)=1$, $2|h$ (hence k is odd), and $s|k$ ($ss'=k$). Let $1\leqslant h'\leqslant k-1$ be such that $hh'\equiv -1 \pmod{k}$. Arguing as in the proof of Proposition 1 that the limit in (5.1) tends to zero, we have that $(\zeta_s^{-r}q;q)_\infty\vartheta(\frac{h}{2k}-\frac{r}{s};\tau)/(q;q)_\infty$ tends to zero as $q\to\zeta_k^h$ under the hypotheses given. Combining these facts with Proposition 3 (i), as $q\to\zeta_k^h$ radially from within the unit disk, we have that

$$\lim_{q\to\zeta_k^h}\left(\frac{i(1+\zeta_s^r)q^{1/8}\cdot(\zeta_s^{-r}q;q)_\infty\cdot\vartheta(\frac{h}{2k}-\frac{r}{s};\tau)}{2\zeta_{2s}^{3r}\cdot(q;q)_\infty}\cdot O_2\left(\zeta_s^r;q^{\frac{1}{2}}\right)\cdot R_2\left(\zeta_k^{h/2-2rs'};q\right)\right.$$
$$\left.-\sum_{n\geqslant 0}\frac{q^{n^2}\zeta_{2k}^{hn}}{(\zeta_s^r q;q)_n}\right)$$
$$= \sum_{n=1}^{rs'(k-h')}(-1)^n\zeta_s^{rn}\zeta_{2k}^{hn^2}\left(\zeta_s^{-r};\zeta_k^h\right)_n.$$

Example 5 *(Modular and mock modular examples).* Proposition 1 and Proposition 2 with $\ell = r$, $m = s$, together with the Rogers–Ramanujan identity (1.2), yield the following two examples (respectively). Let h, k, s be positive integers and r an integer satisfying $\gcd(h,k) = 1$, and $s|k$ ($ss' = k$). Let $1 \leqslant h' \leqslant k-1$ be such that $hh' \equiv -1 \pmod{k}$. As $q \to \zeta_k^h$ radially from within the unit disk, we have that

$$\text{(i)} \qquad \lim_{q \to \zeta_k^h} \left(\frac{\lambda_1 \cdot \vartheta(\frac{r}{s};\tau)(q\zeta_s^{-r};q)_\infty}{(q;q)_\infty (q;q^5)_\infty (q^4;q^5)_\infty} \cdot \mu\left(\frac{r}{s}, \frac{r}{s}; \tau\right) - \sum_{n \geqslant 0} \frac{q^{n^2}\zeta_s^{2rn}}{(\zeta_s^r q;q)_n} \right)$$
$$= \sum_{n=1}^{rs'(k-h')} (-1)^n \zeta_s^{-rn} \zeta_k^{h\binom{n+1}{2}} \left(\zeta_s^{-r}; \zeta_k^h\right)_n,$$

$$\text{(ii)} \qquad \lim_{q \to \zeta_k^h} \left(\frac{\zeta_s^{r^2 s' h'} \cdot (q;q)_\infty}{(\zeta_s^r q;q)_\infty (q;q^5)_\infty (q^4;q^5)_\infty} - \sum_{n \geqslant 0} \frac{q^{n^2}\zeta_s^{2rn}}{(\zeta_s^r q;q)_n} \right)$$
$$= \sum_{n=1}^{rs'(k-h')} (-1)^n \left(\zeta_s^{-r}\zeta_k^h\right)^n \zeta_k^{h\binom{n}{2}} \left(\zeta_s^{-r}; \zeta_k^h\right)_n.$$

The remainder of the paper is structured as follows. In Section 2 we give relevant background information on modular and mock modular forms, and q-hypergeometric series. In Sections 3–5 we prove Theorem 1, Theorem 2, and Propositions 1–4. We proceed by new methods to prove Theorem 1 and Theorem 2 (from which Propositions 1–4 are later derived), differing, for example, from the methods used by the author in recent joint work with Rhoades and Ono in [18].

2. Preliminaries

2.1. Modular and mock modular forms

A modular form we require is the Dedekind η-function, defined for $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^n\right). \tag{2.1}$$

It is well known [30] that $\eta(\tau)$ is modular of weight 1/2, and transforms under $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by

$$\eta(\gamma\tau) = \psi(\gamma)(c\tau + d)^{\frac{1}{2}}\eta(\tau), \tag{2.2}$$

where $\psi(\gamma)$ is a 24th root of unity. We also require the Jacobi form $\vartheta(v;\tau)$ defined in (1.5). This function transforms as follows [30]:

$$\vartheta(v+\alpha\tau+\beta;\tau)=(-1)^{\alpha+\beta}q^{-\frac{\alpha^2}{2}}e^{-2\pi i\alpha v}\vartheta(v;\tau), \tag{2.3}$$

$$\vartheta\left(\frac{v}{c\tau+d};\gamma\tau\right)=\rho(\gamma)(c\tau+d)^{\frac{1}{2}}e^{\frac{\pi icv^2}{c\tau+d}}\vartheta(v;\tau), \tag{2.4}$$

for all $\alpha,\beta\in\mathbb{Z}$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, where $\rho(\gamma):=(\psi(\gamma))^3$.

We will also use the modular Klein forms $\mathfrak{t}_{(r,s)}(\tau)=\mathfrak{t}_{(r,s)}^{(N)}(\tau)$ defined for pairs $(r,s)\in\mathbb{Z}^2$ with respect to a positive integer level N, such that $(r,s)\not\equiv(0,0)\pmod{N\times N}$. These functions are defined using the Weierstrass σ-function, and were studied originally by Klein and Fricke. Here we give some of their key properties as summarized in the more modern source [24]:

$$\mathfrak{t}_{(r,s)}(\gamma\tau)=(c\tau+d)^{-1}\mathfrak{t}_{(r,s)\gamma}(\tau),\quad \text{for all } \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}), \tag{2.5}$$

$$\mathfrak{t}_{(r,s)}(\tau)=-\frac{\zeta_{2N^2}^{s(r-N)}}{2\pi i}q^{\frac{r(r-N)}{2N^2}}\left(1-\zeta_N^s q^{\frac{r}{N}}\right)\prod_{n=1}^{\infty}\frac{(1-\zeta_N^s q^{n+\frac{r}{N}})(1-\zeta_N^{-s}q^{n-\frac{r}{N}})}{(1-q^n)^2}, \tag{2.6}$$

where $q=e^{2\pi i\tau}$, and $(r,s)\gamma$ denotes matrix multiplication.

Weak Maass forms are certain non-holomorphic extensions of ordinary modular forms, defined by Bruinier and Funke [12] using the weight $\kappa\in\frac{1}{2}\mathbb{Z}$ Laplacian operator Δ_κ $(\tau=x+iy)$

$$\Delta_\kappa:=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+i\kappa y\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Definition 3. (See Bruinier and Funke [12].) Let $\kappa\in\frac{1}{2}\mathbb{Z}$, N a positive integer, $\Gamma\subseteq\mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup, χ a Dirichlet character (mod N). A *harmonic weak Maass form of weight κ for Γ with Nebentypus character χ* is a smooth function $f:\mathbb{H}\to\mathbb{C}$ satisfying:

(1) For all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ and all $\tau\in\mathbb{H}$, we have $f(\gamma\tau)=\chi(d)(c\tau+d)^\kappa f(\tau)$.
(2) We have that $\Delta_\kappa f=0$.
(3) The function f has at most linear exponential growth at all cusps.

As discussed in Section 1, we now understand Ramanujan's mock theta functions to be "holomorphic parts" of harmonic weak Maass forms of weight $1/2$. In general, the holomorphic part of part of a harmonic weak Maass form is called a "mock modular form" [36]. Here, we shall use the transformation properties of the level d Appell–Lerch series $A_d(u,v;\tau)$, defined in (1.6). The $A_d(u,v;\tau)$ may be completed into the non-holomorphic Jacobi form [39]

$$\widehat{A}_d(u,v;\tau):=A_d(u,v;\tau)+S_d(u,v;\tau).$$

The non-holomorphic completion $S_d(u,v;\tau)$ is defined by

$$S_d(u,v;\tau) := \frac{i}{2}\sum_{k=1}^{d-1} e(ku)\vartheta\big(v+k\tau+(d-1)/2;d\tau\big)$$
$$\times S\big(du-v-k\tau-(d-1)/2;d\tau\big), \tag{2.7}$$

where

$$S(v;\tau) := \sum_{n\in\mathbb{Z}}\left\{\operatorname{sgn}\left(n+\frac{1}{2}\right) - E\left(\left(n+\frac{1}{2}+\frac{\operatorname{Im}(v)}{\operatorname{Im}(\tau)}\right)\sqrt{2\cdot\operatorname{Im}(\tau)}\right)\right\}$$
$$\times (-1)^n q^{-\frac{1}{2}(n+\frac{1}{2})^2} e^{-2\pi i v(n+\frac{1}{2})},$$

and for $w\in\mathbb{C}$ we have

$$E(w) := 2\int_0^w e^{-\pi u^2}\,du.$$

The functions $S(v;\tau)$ transform as follows [38] under the generators of $\mathrm{SL}_2(\mathbb{Z})$:

$$S(v;\tau+1) = e^{-\frac{\pi i}{4}} S(v;\tau), \tag{2.8}$$
$$\frac{1}{\sqrt{-i\tau}} e^{\frac{\pi i v^2}{\tau}} S\left(\frac{v}{\tau};-\frac{1}{\tau}\right) + S(v;\tau) = h(v;\tau), \tag{2.9}$$

where the Mordell integral $h(v;\tau)$ is defined by

$$h(v;\tau) := \int_{-\infty}^{\infty} \frac{e^{\pi i\tau u^2 - 2\pi v u}}{\cosh(\pi u)}\,du.$$

The completed level d Appell functions transform as follows.

Theorem 2.1. *(See Zwegers [39].) For $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and all $\gamma := \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$\widehat{A}_d(u+n_1\tau+m_1, v+n_2\tau+m_2)$$
$$= (-1)^{d(n_1+m_1)} e\big(u(dn_1-n_2)-n_1 v\big) q^{dn_1^2/2-n_1n_2}\widehat{A}_\ell(u,v;\tau),$$
$$\widehat{A}_d\left(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\gamma\tau\right) = (c\tau+d)e\left(\frac{c(-du^2+2uv)}{2(c\tau+d)}\right)\widehat{A}_d(u,v;\tau).$$

The “holomorphic part” $A_d(u,v;\tau)$ of $\widehat{A}_d(u,v;\tau)$ is shown in [39] to be equal to a finite linear combination of modular theta functions multiplied by mock μ-functions, so we will refer to it as a “mixed mock modular form.” Here, we slightly modify the definition of (weak) “mixed mock modular form” given in [15], and use the term to mean functions that lie in the tensor product of the general spaces of mock modular forms and weakly holomorphic modular forms (up to possible rational multiples of q powers, which need not necessarily be holomorphic at cusps). Mixed mock modular forms in this sense occur in a variety of areas [4,15,22,27,33].

2.2. q-Hypergeometric series

Let $r, s \in \mathbb{N}$. The *bilateral basic q-hypergeometric series* is defined for $r, s \in \mathbb{N}_0$ by

$$
{}_r\psi_s\left(\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{matrix}; q; z\right)
:= \sum_{n=-\infty}^{\infty} \frac{\prod_{j=1}^{r}(a_j;q)_n}{\prod_{i=1}^{s}(b_i;q)_n}(-1)^{(s-r)n} q^{(s-r)n(n-1)/2} z^n. \tag{2.10}
$$

Recall from Section 1 that the q-Pochhammer symbol is defined for all $m \in \mathbb{Z}$ by

$$
(x;q)_m := \frac{(x;q)_\infty}{(xq^m;q)_\infty}.
$$

Ramanujan established (see for example [19]) that for $r = s = 1$, the ${}_1\psi_1$ bilateral basic q-hypergeometric has the following beautiful expression as a quotient of infinite products. In what follows, assume $|q| < 1$, $|\beta/\alpha| < |z| < 1$.

Theorem 2.2 *(Ramanujan). We have that*

$$
{}_1\psi_1\left(\begin{matrix} \alpha \\ \beta \end{matrix}; q; z\right) := \sum_{n=-\infty}^{\infty} \frac{(\alpha;q)_n}{(b;q)_n} z^n = \frac{(q;q)_\infty(\beta/\alpha;q)_\infty(\alpha z;q)_\infty(q/\alpha z;q)_\infty}{(\beta;q)_\infty(q/\alpha;q)_\infty(z;q)_\infty(\beta/\alpha z;q)_\infty}.
$$

Another limiting case of bilateral summation gives the following beautiful bilateral result for the mock Jacobi forms $\mu(u, v; \tau)$, due to Choi [14] and Ramanujan (see p. 67 of [6]).

Theorem 2.3. *(See [6,14].) Let $q = e^{2\pi i\tau}$, where $\tau \in \mathbb{H}$. For suitable complex numbers $a = e^{2\pi iu}$ and $b = e^{2\pi iv}$, we have*

$$
\sum_{n\in\mathbb{Z}} \frac{(ab)^n q^{n^2}}{(aq;q)_n(bq;q)_n} = iq^{\frac{1}{8}}(1-a)\left(ba^{-1}\right)^{\frac{1}{2}}\left(qa^{-1};q\right)_\infty\left(b^{-1};q\right)_\infty \mu(u,v;\tau).
$$

We also recall the q-exponential identity (see [19], for example)

$$
\sum_{n\geqslant 0} \frac{(-z)^n q^{\binom{n}{2}}}{(q;q)_n} = (z;q)_\infty. \tag{2.11}
$$

3. Proof of Theorem 1

To prove part (i), we consider the product of bilateral basic q-hypergeometric series

$$\begin{aligned}&(z;q)_\infty \cdot {}_1\psi_1\left(\begin{matrix}a\\ c\end{matrix};q;z\right)\cdot {}_d\psi_d\left(\begin{matrix}y & y & \cdots & y\\ 1/y & 1/y & \cdots & 1/y\end{matrix};q;\frac{z}{by^d}\right)\\ &\quad=\sum_{n\geqslant 0}\frac{(-z)^n q^{\binom{n}{2}}}{(q;q)_n}\cdot\sum_{m=-\infty}^{\infty}\frac{(a;q)_m}{(c;q)_m}z^m\cdot\sum_{r=-\infty}^{\infty}\frac{(y;q)_r^d}{(1/y;q)_r^d}\left(\frac{z}{by^d}\right)^r\\ &\quad=\sum_{M=-\infty}^{\infty}\sum_{n\geqslant 0}\sum_{m=-\infty}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}\frac{(a;q)_m}{(c;q)_m}\frac{(y;q)_{M-n-m}^d}{(1/y;q)_{M-n-m}^d}\left(\frac{1}{by^d}\right)^{M-n-m}z^M,\end{aligned}$$

where we have used (2.10) and (2.11). We collect the coefficient of z^0, let $y\to\infty$, and let $c\to aq$, and find

$$(1-a)\sum_{n\geqslant 0}\frac{(-1)^{n(d+1)}q^{d\binom{n+1}{2}+\binom{n}{2}}b^n}{(q;q)_n}\cdot\sum_{m=-\infty}^{\infty}\frac{(-1)^{dm}}{1-aq^m}q^{d\binom{m+1}{2}+dnm}b^m. \tag{3.1}$$

Now $n\geqslant 0$, so that the auxiliary sum

$$\begin{aligned}&\sum_{m=-\infty}^{\infty}\frac{(-1)^{dm}q^{d\binom{m+1}{2}}b^m}{1-aq^m}\left(q^{dnm}-a^{-dn}\right)\\ &\quad=-a^{-dn}\sum_{j=0}^{dn-1}a^j\sum_{m=-\infty}^{\infty}(-1)^{dm}q^{d\binom{m+1}{2}+mj}b^m\\ &\quad=-a^{-dn}q^{-\frac{d}{8}}b^{-\frac12}(-i)^d\sum_{j=0}^{dn-1}a^jq^{-\frac{j}{2}}\vartheta\left(j\tau+v+\frac{d-1}{2};d\tau\right)\\ &\quad=-a^{-dn}q^{-\frac{d}{8}}b^{-\frac12}(-i)^d\sum_{j=0}^{d-1}a^jq^{-\frac{j}{2}}\vartheta\left(j\tau+v+\frac{d-1}{2};d\tau\right)\sum_{\ell=0}^{n-1}a^{\ell d}(-1)^{\ell d}q^{-\frac{d\ell^2}{2}-\frac{d\ell}{2}-j\ell}b^{-\ell},\end{aligned}$$

where we have replaced j by $j \pmod d$ and summed over residue classes, making use of (2.3). (As usual we take the empty sum to equal 0.) Thus, (3.1) becomes

$$\begin{aligned}&(1-a)\sum_{n\geqslant 0}\frac{(-1)^{n(d+1)}q^{d\binom{n+1}{2}+\binom{n}{2}}b^na^{-dn}}{(q;q)_n}\sum_{m=-\infty}^{\infty}\frac{(-1)^{dm}q^{d\binom{m+1}{2}}b^m}{1-aq^m}\\ &\quad-(1-a)q^{-\frac{d}{8}}b^{-\frac12}(-i)^d\sum_{n\geqslant 0}\frac{(-1)^{n(d+1)}q^{d\binom{n+1}{2}+\binom{n}{2}}b^na^{-dn}}{(q;q)_n}\\ &\quad\times\sum_{j=0}^{d-1}a^jq^{-\frac{j}{2}}\vartheta\left(j\tau+v+\frac{d-1}{2};d\tau\right)\sum_{\ell=0}^{n-1}a^{\ell d}(-1)^{\ell d}q^{-\frac{d\ell^2}{2}-\frac{d\ell}{2}-j\ell}b^{-\ell}\end{aligned}$$

$$= (1-a)a^{-d/2}R_{d+1}\big((-1)^{d+1}ba^{-d}q^{d-1};q\big)\cdot A_d(u,v;\tau)$$
$$- (1-a)a^{-d/2}T_d(u,v;\tau). \tag{3.2}$$

Next, we point out that Ramanujan's ${}_1\psi_1$ summation of Theorem 2.2 implies

$$(z;q)_\infty \cdot {}_1\psi_1\left(\begin{matrix}a\\c\end{matrix};q;z\right) = \frac{(ca^{-1};q)_\infty}{(qa^{-1};q)_\infty}\lim_{x\to\infty} {}_1\psi_1\left(\begin{matrix}ax\\c\end{matrix};q;\frac{z}{x}\right). \tag{3.3}$$

Thus, the product of bilateral basic q-hypergeometric series

$$(z;q)_\infty \cdot {}_1\psi_1\left(\begin{matrix}a\\c\end{matrix};q;z\right)\cdot {}_d\psi_d\left(\begin{matrix}y & y & \cdots & y\\ 1/y & 1/y & \cdots & 1/y\end{matrix};q;\frac{z}{by^d}\right)$$
$$= \lim_{x\to\infty}\frac{(ca^{-1};q)_\infty}{(qa^{-1};q)_\infty}\cdot {}_1\psi_1\left(\begin{matrix}ax\\c\end{matrix};q;\frac{z}{x}\right)\cdot {}_d\psi_d\left(\begin{matrix}y & y & \cdots & y\\ 1/y & 1/y & \cdots & 1/y\end{matrix};q;\frac{z}{by^d}\right)$$
$$= \lim_{x\to\infty}\frac{(ca^{-1};q)_\infty}{(qa^{-1};q)_\infty}\cdot\sum_{m=-\infty}^{\infty}\frac{(ax;q)_m}{(c;q)_m}\left(\frac{z}{x}\right)^m\cdot\sum_{r=-\infty}^{\infty}\frac{(y;q)_r^d}{(1/y;q)_r^d}\left(\frac{z}{by^d}\right)^r$$
$$= \lim_{x\to\infty}\frac{(ca^{-1};q)_\infty}{(qa^{-1};q)_\infty}\sum_{M=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(ax;q)_m}{(c;q)_m}\left(\frac{1}{x}\right)^m\frac{(y;q)_{M-m}^d}{(1/y;q)_{M-m}^d}\left(\frac{1}{by^d}\right)^{M-m}z^M.$$

We again collect the coefficient of z^0, let x and y tend to ∞, with $c = aq$, and find after some simplification

$$(1-a)a^{-d/2}B_d(u,v;\tau).$$

Comparing this with (3.2), we find Theorem 1 (i).

To prove part (ii) of Theorem 1, we use part (i) of Theorem 1 in the case $d = 1$, together with the fact that $A_1(u,v;\tau) = \vartheta(v;\tau)\mu(u,v;\tau)$. Part (iii) of Theorem 1 follows from part (ii) of Theorem 1, together with the Rogers–Ramanujan identity (1.2), and its companion

$$\sum_{n\geqslant 0}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{3.4}$$

4. Proof of Theorem 2

To prove Theorem 2, we begin with Theorem 1. A short calculation reveals that

$$\frac{(1-a)(qa^{-1};q)_\infty}{a^{d/2}(q;q)_\infty}B_d(u,v;\tau) = M_d(u,v;\tau) + P_{d+1}(u+v,u;\tau),$$

where

$$M_d(u,v;\tau) := \sum_{n\geqslant 1}(-1)^{dn}q^{d\binom{n}{2}}\left(qb^{-1}\right)^n\left(a^{-1};q\right)_n.$$

Thus we rewrite the identity of Theorem 1 (i) as

$$M_d(u,v;\tau) = \frac{(1-a)(qa^{-1};q)_\infty}{a^{d/2}(q;q)_\infty}\big(A_d(u,v;\tau)\cdot R_{d+1}\big(ba^{-d}(-q)^{d-1};q\big) - T_d(u,v;\tau)\big) - P_{d+1}(u+v,u;\tau). \tag{4.1}$$

With hypotheses as in the statement of Theorem 2, we set $u = r/s$ and $v = \ell/m$ in (4.1). Taking the limit as $q \to \zeta_k^h$ gives Theorem 2, where we note that $M_d(\frac{r}{s},\frac{\ell}{m};q)$ becomes the finite sum given in the statement of Theorem 2 under the hypotheses stated.

5. Proofs of Propositions 1–4

Proof of Proposition 1. We begin with Theorem 2 in the case $d = 1$, with the additional assumption that $m|k$ $(mm' = k)$, together with the fact that $A_1(u,v;\tau) = \vartheta(v;\tau)\mu(u,v;\tau)$. To prove Proposition 1, it now suffices to show that under the hypotheses stated,

$$\lim_{q\to\zeta_k^h} \frac{(\zeta_s^{-r}q;q)_\infty\cdot\vartheta(\frac{\ell}{m};\tau)}{\eta^2(\tau)}\cdot\Delta_1^{(0)}\left(\frac{r}{s},\frac{\ell}{m};\tau\right) \tag{5.1}$$

tends to zero. We begin with the quotient $F(\tau) := \vartheta(\frac{\ell}{m};\tau)/\eta^2(\tau)$ appearing in (5.1). We define for $z\in\mathbb{R}^+$,

$$Z := \frac{1}{k}(h+iz), \qquad Z' := \frac{1}{k}\left(h'+\frac{i}{z}\right), \qquad \gamma := \begin{pmatrix} h & -(\frac{hh'+1}{k}) \\ k & -h'\end{pmatrix},$$

so that $\gamma\in \mathrm{SL}_2(\mathbb{Z})$, $\gamma Z' = Z$, and $Z, Z'\in\mathbb{H}$. Thus, to study $\lim_{q\to\zeta_k^h} F(\tau)$ radially from within the unit disk, it suffices to study $\lim_{z\to0^+} F(Z)$. We have

$$F(Z) = F\big(\gamma Z'\big) = \frac{\vartheta(\frac{\ell}{m};\gamma Z')}{\eta^2(\gamma Z')} = \psi(\gamma)\sqrt{-iz}\cdot e^{-\frac{\pi}{zk}(\ell m')^2}\cdot\frac{\vartheta(\frac{i\ell m'}{zk};Z')}{\eta^2(Z')}, \tag{5.2}$$

where we have used (2.2) and (2.4). Now we use (2.3) to rewrite

$$\vartheta\left(\frac{i\ell m'}{zk};Z'\right) = \vartheta\left(\ell m'Z' - \ell m'\frac{h'}{k};Z'\right) = (-1)^{\ell m'}\zeta_{2k}^{(\ell m')^2h'}\cdot e^{\frac{\pi}{zk}(\ell m')^2}\cdot\vartheta\left(-\ell m'\frac{h'}{k};Z'\right).$$

We make this substitution in (5.2) and find that

$$\begin{aligned} F(Z) &= \psi(\gamma)(-1)^{\ell m'}\zeta_{2k}^{(\ell m')^2h'}\sqrt{-iz}\cdot\frac{\vartheta(-\ell m'\frac{h'}{k};Z')}{\eta^2(Z')} \\ &= \psi(\gamma)(-1)^{\ell m'+1}i\zeta_{2k}^{(\ell m')^2h'+\ell m'h'}\zeta_{24k}^{h'}\big(1-\zeta_k^{-\ell m'h'}\big)\cdot\sqrt{-iz}\cdot e^{-\frac{\pi}{12kz}}\big(1+O\big(e^{-\frac{2\pi\alpha}{kz}}\big)\big)\end{aligned}$$

for some $\alpha > 0$, where we have used the product expansion given in (1.5), as well as (2.1). Thus, we see that $\lim_{z\to 0^+} F(Z) = 0$. Next we examine $\Delta_1^{(0)}(\frac{r}{s}, \frac{\ell}{m}; \tau)$ in (5.1). For each $n \geqslant 0$, the nth summand in this function is of the form $(q^{n+1}; q)_\infty \cdot c_n(\zeta_s^r, \zeta_m^\ell; q)$, where $c_n(\zeta_s^r, \zeta_m^\ell; q)$ is finite and non-singular as $q \to \zeta_k^h$. For each n under the hypotheses given, we see that $(q^{n+1}; q)_\infty$ vanishes when $q \to \zeta_k^h$. Thus, $\Delta_1^{(0)}(\frac{r}{s}, \frac{\ell}{m}; \tau)$ tends to zero as $q \to \zeta_k^h$. Similarly, using that $s|k$, we also see that $(\zeta_s^{-r} q; q)_\infty$ vanishes when $q \to \zeta_k^h$. Combining these facts, we see that indeed the expression in (5.1) tends to zero. □

Proof of Proposition 2. We begin with Proposition 1, and first seek to understand the asymptotic properties of $A_1(\frac{r}{s}, \frac{\ell}{m}; \tau) = \vartheta(\frac{\ell}{m}; \tau)\mu(\frac{r}{s}; \frac{\ell}{m}; \tau)$ as $\tau \to \frac{h}{k}$ with hypotheses given as in the statement of Proposition 1. With notation as in the proof of Proposition 1 above, it thus suffices to study the asymptotic properties of $A_1(\frac{r}{s}, \frac{\ell}{m}; Z)$ as $z \to 0^+$. Our treatment here is very similar to our proof of Theorem 3.2 in [18], so for brevity, we provide a detailed sketch of proof. Using Theorem 2.1, we see that the completed level 1 Appell function satisfies

$$\begin{aligned}
\widehat{A}_1\left(\frac{r}{s}, \frac{\ell}{m}; Z\right) &= \widehat{A}_1\left(\frac{r}{s}, \frac{\ell}{m}; \gamma Z'\right) \\
&= \left(\frac{i}{z}\right) e^{\frac{\pi k}{z}(\frac{r^2}{s^2} - \frac{2r\ell}{sm})} \widehat{A}_1\left(rs'Z' - \frac{rs'h'}{k}, \ell m'Z' - \frac{\ell m'h'}{k}; Z'\right) \\
&= (-1)^{rs'} \zeta_k^{\ell m' rs' h'} \zeta_{2k}^{-(rs')^2 h'} \left(\frac{i}{z}\right) \widehat{A}_1\left(-\frac{rs'h'}{k}, -\frac{\ell m'h'}{k}; Z'\right)
\end{aligned} \tag{5.3}$$

where we have used the hypotheses that $ss' = mm' = k$ for some (positive) integers s', m'. Thus we may rewrite

$$A_1\left(\frac{r}{s}, \frac{\ell}{m}; Z\right) = \sigma_1 + \sigma_2 + \vartheta\left(-\frac{\ell m'h'}{k}; Z'\right) \cdot \sigma_3,$$

where

$$\begin{aligned}
\sigma_1 &:= (-1)^{rs'} \zeta_k^{\ell m' rs' h'} \zeta_{2k}^{-(rs')^2 h'} \left(\frac{i}{z}\right) A_1\left(-\frac{rs'h'}{k}, -\frac{\ell m'h'}{k}; Z'\right), \\
\sigma_2 &:= -\frac{i}{2}\vartheta\left(\frac{\ell}{m}; Z\right) S\left(\frac{r}{s} - \frac{\ell}{m}; Z\right), \\
\sigma_3 &:= \frac{i}{2}(-1)^{rs'} \zeta_k^{\ell m' rs' h'} \zeta_{2k}^{-(rs')^2 h'} \left(\frac{i}{z}\right) S\left(-\frac{rs'h'}{k} + \frac{\ell m'h'}{k}; Z'\right).
\end{aligned}$$

First we establish that

$$\vartheta\left(\frac{\ell}{m}; Z\right) = \rho(\gamma)\sqrt{iz^{-1}}(-1)^{\ell m'} \zeta_{2k}^{\ell^2 (m')^2 h'} \vartheta\left(-\frac{\ell m'h'}{k}; Z'\right), \tag{5.4}$$

which follows similarly to (5.3) using (2.3) and (2.4). Making the appropriate changes of variables in our proof of Lemma 3.4 in [18], we obtain the following result.

Lemma 5.4. *With hypotheses as above, we have that*

$$S\left(\frac{r}{s}-\frac{\ell}{m};Z\right)=\frac{e^{-\frac{\pi ihk}{4}+\pi i(k-1)(\frac{r}{s}-\frac{\ell}{m}+\frac{1}{2})}(-1)^{\frac{(k-1)(h-1)}{2}}}{\sqrt{kz}}\sum_{j=0}^{k-1}e^{-\frac{\pi ih}{k}(j-\frac{k-1}{2})^2}$$
$$\times\zeta_m^{\ell j}\zeta_s^{-rj}(-1)^{j(h+1)}\left(-S\left(\frac{j-\frac{k-1}{2}}{k};\frac{i}{kz}\right)+h\left(\frac{j-\frac{k-1}{2}}{k};\frac{i}{kz}\right)\right).$$

Thus, by combining (5.4) and Lemma 5.4, we have that

$$\sigma_2=\vartheta\left(-\frac{\ell m'h'}{k};Z'\right)(\sigma_{21}+\sigma_{22}),$$

where

$$\sigma_{21}:=\frac{i}{2}\rho(\gamma)\sqrt{\frac{i}{z}}(-1)^{\ell m'}\zeta_{2k}^{\ell^2(m')^2h'}\frac{e^{-\frac{\pi ihk}{4}+\pi i(k-1)(\frac{r}{s}-\frac{\ell}{m}+\frac{1}{2})}(-1)^{\frac{(k-1)(h-1)}{2}}}{\sqrt{kz}}$$
$$\times\sum_{j=0}^{k-1}e^{-\frac{\pi ih}{k}(j-\frac{k-1}{2})^2}\zeta_m^{\ell j}\zeta_s^{-rj}(-1)^{j(h+1)}S\left(\frac{j-\frac{k-1}{2}}{k};\frac{i}{kz}\right),$$
$$\sigma_{22}:=-\frac{i}{2}\rho(\gamma)\sqrt{\frac{i}{z}}(-1)^{\ell m'}\zeta_{2k}^{\ell^2(m')^2h'}\frac{e^{-\frac{\pi ihk}{4}+\pi i(k-1)(\frac{r}{s}-\frac{\ell}{m}+\frac{1}{2})}(-1)^{\frac{(k-1)(h-1)}{2}}}{\sqrt{kz}}$$
$$\times\sum_{j=0}^{k-1}e^{-\frac{\pi ih}{k}(j-\frac{k-1}{2})^2}\zeta_m^{\ell j}\zeta_s^{-rj}(-1)^{j(h+1)}h\left(\frac{j-\frac{k-1}{2}}{k};\frac{i}{kz}\right).$$

Next, making the appropriate changes of variables and arguing as in our proof of Lemma 3.5 in [18], we establish that $\sigma_{21}+\sigma_3=0$. Next, as argued in [18], using that $h(c;i/kz)=O(\sqrt{z})$ for $c\in\mathbb{Q}$, we find that

$$\vartheta\left(-\frac{\ell m'h'}{k};Z'\right)\cdot\sigma_{22}=\kappa\cdot q_1^{1/8}/\sqrt{z}\cdot\left(1+O\left(q_1^\alpha\right)\right)$$

where $q_1:=e(Z')$, for some $\alpha>0$ and constant κ. That is, $\vartheta(-\frac{\ell m'h'}{k};Z')\cdot\sigma_{22}$ tends to zero as $z\to 0^+$. Summarizing, we have shown that $A_1(\frac{r}{s},\frac{\ell}{m};Z)\sim\sigma_1$. Now it is not difficult to see using the definition of $A_1(u,v;\tau)$ that

$$A_1\left(-\frac{rs'h'}{k},-\frac{\ell m'h'}{k};\right)=\frac{\zeta_{2k}^{-rs'h'}}{1-\zeta_k^{-rs'h'}}\left(1+O\left(q_1^\alpha\right)\right)\tag{5.5}$$

for some $\alpha>0$ as $z\to 0^+$. On the other hand (see (3.26) of [18] for example), using the modular properties and of the functions $\mathfrak{t}_{(r,s)}^{(N)}$, it is not difficult to establish that

$$\left(\frac{i}{z}\right)\mathfrak{t}^{(s)}_{(0,r)}(Z) = \mathfrak{t}^{(s)}_{(rk,-rh')}(Z') = (-1)^{rs'}\frac{\zeta_{2s^2}^{r^2h'k}(\zeta_{2s}^{-rh'} - \zeta_{2s}^{rh'})(1 + O(q_1^{\alpha}))}{(2\pi i)}, \tag{5.6}$$

for some $\alpha > 0$ as $z \to 0^+$. To conclude the proof of Proposition 2, we use (5.5), (5.6) and the definition of σ_1, together with Proposition 1. □

Proof of Proposition 3. To prove (i), we begin with the fact (see (9.4) of [20]) that $g_2(a; q^{\frac{1}{2}}) = -iq^{-\frac{1}{8}}\mu(u, \frac{\tau}{2} - u; \tau)$. Using this, the result follows from Proposition 1 with $\ell = h/2 - rs'$ and $m = k$. To prove Proposition 3 part (ii), we begin with the fact (see [23], for example) that $\mathcal{R}(a; q) = (1 - a) + a(1 - a)g_3(a; q)$. Part (ii) of Proposition 3 then follows immediately from Proposition 4 whose proof is below, taking note of the definition of $\beta\left(\begin{smallmatrix} h & r \\ k & s \end{smallmatrix}\right)$ in (1.8). □

Proof of Proposition 4. To prove the identity given for $\mathcal{R}(a; q)$ in the statement of Proposition 4, we begin with the fact (see [23], for example) that

$$\mathcal{R}(a; q) = \frac{(1 - a)}{a^{3/2}(q; q)_\infty} A_3(u, -\tau; \tau). \tag{5.7}$$

The identity now follows immediately from Theorem 1.

To prove the radial limit given in Proposition 4, we begin with the fact that

$$A_1(u, -u; \tau) = \frac{a^{1/2}(q; q)_\infty}{(1 - a)}\left(\mathcal{R}(a; q) + \sum_{n>0} q^n \left(a^{-1}; q\right)_n (a; q)_n\right), \tag{5.8}$$

which follows from Theorem 2.3 with $b = a^{-1}$, after some simplification, using the fact that $A_1(u, v; \tau) = \vartheta(v; \tau)\mu(u, v; \tau)$, as well as the product expression in (1.5). Proposition 4 now follows from Proposition 1 with $\ell = -r$, $m = s$, taking note of the definition of $\sigma\left(\begin{smallmatrix} h & r \\ k & s \end{smallmatrix}\right)$ in (1.8).

Remark. We point out that using (5.7) instead of (5.8) together with Theorem 2 yields a different radial limit than the one stated in Proposition 4, namely one which relates $\mathcal{R}(a; q)$ and $R_4(b; q)$. □

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