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Mock and mixed mock modular forms in the lower half-plane

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Abstract. We study mock and mixed mock modular forms in the lower half-plane. In particular, our results apply to Zwegers' three-variable mock Jacobi form $\mu(u, v; \tau)$, three-variable generalizations of the universal mock modular partition rank generating function, and the quantum and mock modular strongly unimodal sequence rank generating function. We do not rely upon the analytic properties of these functions; we establish our results concisely using the theory of q-hypergeometric series and partial theta functions. We extend related results of Ramanujan, Hikami, and prior work of the author with Bringmann and Rhoades, and also incorporate more recent aspects of the theory pertaining to quantum modular forms and the behavior of these functions at rational numbers when viewed as functions of τ (or equivalently, at roots of unity when viewed as functions of q).

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1. Introduction and statement of results. Historically, partial theta functions have been a topic of interest within the theories of q-hypergeometric series and partitions, as studied by a number of authors including Alladi, Andrews, Berndt, Fine, Rogers, Ramanujan, and more [1–6, 8, 9, 17, 24]. For example, due to Rogers, we have for $|q| < 1$ that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(-q;q)_n} = \sum_{n=0}^{\infty} \left(\frac{-12}{n}\right) q^{\frac{n^2-1}{24}}, \tag{1.1}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. The function on the left-hand side of (1.1) is an example of a *q-hypergeometric series*, defined by using the q-pochhammer symbol $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$, $(n \in \mathbb{N}_0)$. This function also admits a combinatorial, partition theoretic interpretation due to work of Andrews [4],

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relating distinct parts to largest parts of partitions. The function on the right-hand side of (1.1) is an example of a *partial theta function*, aptly named as it resembles an ordinary (modular) theta function, save for the fact that it is a sum only over the partial lattice of integers $\mathbb{N}_0$ as opposed to the full lattice $\mathbb{Z}$.

Despite their similarities to modular theta functions, a full theory of partial theta functions has not been well understood. However, more recently, we have begun to understand their relationships to mock modular and quantum modular forms. To describe this, consider one of Ramanujan's mock theta functions from his last letter to Hardy defined by

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}.$$

Similar to the left-hand side of (1.1), this function is a combinatorial q-hypergeometric series, well known to carry information about partition ranks (see (1.5)). More recently, due to fundamental work of Zwegers [27,28], we also know how this function fits into the theory of modular forms: it is a *mock modular form*, the holomorphic part of a harmonic Maass form [13], as are all of Ramanujan's original mock theta functions. Loosely speaking, harmonic Maass forms transform like ordinary modular forms, but are non-holomorphic, are annihilated by a certain Laplacian operator, and satisfy relaxed growth conditions. With respect to partial theta functions, an interesting relationship was given by Ramanujan in his "lost" notebook [3] when q is replaced by q^{-1}. That is, for $|q| < 1$,

$$f(q^{-1}) = 2\sum_{n=0}^{\infty} \left(\frac{-12}{n}\right) q^{\frac{n^2-1}{24}} - (-q;q)_\infty^{-2} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}}. \tag{1.2}$$

The identity in (1.2) shows how Ramanujan's mock theta function f extends to a function outside of the unit disk ($|q| < 1 \Leftrightarrow |q^{-1}| > 1$), and there it can be expressed in terms of two partial theta functions. Identities similar to (1.2) have been studied more recently by Hikami [21], and in [10] in more generality.

The previous discussion begs the question of understanding these types of functions on the boundary of the unit disk, since we are sometimes able to understand functions like $f(q)$ both inside and outside the unit disk. Indeed, we have made recent progress in understanding this aspect as related to quantum modular forms, newly defined by Zagier [26]; however, in hindsight, some of the origins of this question date back to Ramanujan. To describe this in an example, in his last letter to Hardy, Ramanujan made the claim that as q tends radially from within the unit disk towards any even ordered $2k$-th root of unity ζ (on the boundary of the unit disk), then

$$\lim_{q\to\zeta} (f(q) - (-1)^k b(q)) = O(1), \tag{1.3}$$

where $b(q)$ is an explicitly given weakly holmorphic modular form of weight $1/2$ (when $q = e^{2\pi i\tau}, \tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, up to multiplication by $q^{-1/24}$). Note that the even ordered roots of unity are exponential singularities of the mock theta function $f(q)$. While Ramanujan's claim can be verified using later work of Watson [25], questions remained into the present day, such as understanding the implied constants in (1.3) and understanding (1.3) within a larger framework. Ono, Rhoades, and the author addressed these questions in [18], and established the following more general result, where ζ_b^a is a fixed root of unity, and q tends radially towards another suitable root of unity ζ_k^h from within the unit disk (here $hh' \equiv -1 \pmod{k}$):

$$\lim_{q\to\zeta_k^h}\left(R\left(\zeta_b^a;q\right)-\zeta_{b^2}^{-a^2h'k}C\left(\zeta_b^a;q\right)\right)=-(1-\zeta_b^a)(1-\zeta_b^{-a})\cdot U(\zeta_b^a;\zeta_k^h). \tag{1.4}$$

This result is stated in terms of the two-variable q-hypergeometric combinatorial generating functions

$$R(w;q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}=\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}}N(m,n)w^mq^n, \tag{1.5}$$

$$U(w;q):=\sum_{n=0}^{\infty}q^{n+1}(wq;q)_n(w^{-1}q;q)_n=\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}}u(m,n)(-w)^mq^n, \tag{1.6}$$

where $N(m,n)$ is the number of partitions of n with rank m, and $u(m,n)$ is the number of strongly unimodal sequences of size n with rank m. The function $C(w;q)$, similarly, is the two-variable generating function for partition cranks. Ramanujan's claim (1.3) pertains to the special case $\zeta_b^a = -1$. On one hand, (1.4) can be interpreted as an asymptotic statement between combinatorial generating functions. On the other hand, (1.4) admits a modular interpretation. Up to multiplication by a suitable q-power, due to work of Bringmann and Ono [11], for roots of unity w, the function $R(w;q)$ is a mock modular form. Moreover, it is not difficult to show that when specialized appropriately, $C(w;q)$ is essentially an ordinary modular form. Recently, the authors in [14] related the function $U(w;q)$ (with $q=e^{2\pi i\tau}$) to *quantum modular forms* [26], which are like modular forms, but are defined for $\tau\in\mathbb{Q}$ as opposed to $\tau\in\mathbb{H}$, and exhibit a modular transformation property on (a subset of) $\mathbb{Q}$ up to an error function, which should extend to a suitably continuous or analytic function in (a subset of) $\mathbb{R}$ [26]. Results from [12,22] later extended (1.4) from [18], and we now know that a similar relationship among mock theta functions and quantum modular forms exists even more generally.

In this paper, our goal is to study mock modular forms, in a general sense, as functions of τ in the lower half-plane (equivalently, as functions of $q=e^{2\pi i\tau}$ outside of the unit disk). Despite the generally intricate analytic nature of mock modular forms, we establish our results concisely using the theory of q-hypergeometric series and partial theta functions. Our results extend results in [3,10,21] to general mock and mixed mock modular forms, and incorporate more recent aspects of the theory from [12,14,18,22] as discussed above pertaining to the behavior of these functions at rational numbers (equivalently, at roots of unity on the boundary of the unit disk). To this end we study three three-variable functions. The first, $\mu(u,v;\tau)$, due to Zwegers [28], is defined for $\tau \in \mathbb{H}$, and $u, v \in \mathbb{C}\backslash(\mathbb{Z}\tau + \mathbb{Z})$:

$$\mu(u, v; \tau) := \frac{e^{\pi i u}}{\vartheta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i n\left(v+\frac{1}{2}\right)} q^{\frac{n}{2}(n+1)}}{1 - e^{2\pi i u} q^n}, \tag{1.7}$$

where $\vartheta(v; \tau) := \sum_{m \in \mathbb{Z}} e^{2\pi i\left(m+\frac{1}{2}\right)\left(v+\frac{1}{2}\right)} q^{\frac{1}{2}\left(m+\frac{1}{2}\right)^2}$ and $q = e^{2\pi i \tau}$. This function behaves like a mock Jacobi form, in that it may be completed by the addition of a suitable non-holomorphic function, such that the resulting sum transforms like a (non-holomorphic) modular Jacobi form. Upon suitable specializations of parameters, the μ-function becomes a mock modular form. The μ-function plays a ubiquitous role within the theory: Ramanujan's original mock theta functions, and a very large number of canonical examples and infinite families of mock modular forms, can be written and studied in terms of μ (see, for example, the comprehensive survey articles [23, 26]). The second function we study is a three-variable q-hypergeometric sum $\mathcal{R}(\alpha, \beta; q)$ ($|q| < 1$) that specializes to the mock modular partition rank generating function $R(w; q)$ in (1.5) when $\alpha = \beta^{-1} = w$:

$$\mathcal{R}(\alpha, \beta; q) := \sum_{n=0}^{\infty} \frac{(\alpha\beta)^n q^{n^2}}{(\alpha q; q)_n (\beta q; q)_n}.$$

The rank generating function $R(w; q)$ is also essentially a universal mock theta function in the sense of Gordon and McIntosh [20], meaning that many of Ramanujan's original mock theta functions can be expressed in terms of $R(w; q)$ upon suitable choices of parameters, up to the addition of a modular form (see also [10, Theorem 3.1]). The same is thus true for its generalization $\mathcal{R}(\alpha, \beta; q)$. The universal mock theta functions have recently been studied in the context of this paper in the aforementioned works [12, 22]. The third function we study is a three-variable generalization of the quantum modular form $U(w; q)$ in (1.6) ($|q| < 1$):

$$\mathcal{U}(\alpha, \beta; q) := \sum_{n=0}^{\infty} (\alpha; q)_n (\beta; q)_n q^n.$$

That is, $q\,\mathcal{U}(wq; w^{-1}q) = U(w; q)$. From $\mathcal{U}, \mathcal{R}$, and μ, we also define companion functions by multiplying or dividing by certain infinite products. From the perspective of modularity, these multiplications give rise to *mixed mock modular forms* [16]. Precisely, we define

$$\mathcal{U}_j(\alpha, \beta; q) := \begin{cases} \mathcal{U}(\alpha, \beta; q)\, (\alpha; q)_\infty^{-1} (\beta; q)_\infty^{-1} & \text{if } j = 1, \\ \mathcal{U}(\alpha, \beta; q) & \text{if } j = 2, \end{cases}$$

$$\mathcal{R}_j(\alpha, \beta; q) := \begin{cases} \mathcal{R}(\alpha, \beta; q)\, (\alpha^{-1}; q)_\infty^{-1} (\beta^{-1}; q)_\infty^{-1} & \text{if } j = 1, \\ \mathcal{R}(\alpha, \beta; q) & \text{if } j = 2, \end{cases}$$

$$\mathcal{M}(u, v; \tau) := -i q^{\frac{1}{8}} e^{\pi i (u+v)} (e^{-2\pi i u}; q)_\infty (e^{-2\pi i v}; q)_\infty \mu(u, v; \tau),$$

where in the definition of $\mathcal{M}$, we let $q = e^{2\pi i \tau}$. Here and throughout, we define $(a; q)_\infty := \lim_{n\to\infty} (a; q)_n$. In what follows, we show how to extend the functions μ, $\mathcal{M}$, $\mathcal{R}_j$, and $\mathcal{U}_j$ into the lower half-plane, using the following q-hypergeometric series and partial theta functions:

$$\chi(z;q) := \sum_{m=0}^{\infty}(-1)^m z^m q^{\frac{m(m+1)}{2}}, \quad X(z;q) := \sum_{n=0}^{\infty}\frac{(z^{\frac{1}{2}}q^{\frac{1}{4}};q^{\frac{1}{2}})_n z^{\frac{n}{2}}q^{\frac{n}{4}}}{(-z^{\frac{1}{2}}q^{\frac{1}{4}};q^{\frac{1}{2}})_{n+1}},$$

$$S_1(\alpha,\beta;q) := \sum_{n=0}^{\infty}(\alpha;q)_{n+1}\beta^n, \quad S_3(\alpha,\beta;q) := (1-\alpha)\sum_{n=0}^{\infty}\frac{q^{\frac{n^2+n}{2}}(-\alpha\beta)^n}{(\beta;q)_{n+1}},$$

$$S_2(\alpha,\beta;q) := \sum_{n=0}^{\infty}\frac{(\alpha;q)_{n+1}}{(\beta;q)_{n+1}}(1-\alpha\beta q^{2n+1})(-\alpha\beta^2)^n q^{\frac{3n^2+n}{2}}. \tag{1.8}$$

Analogously to the functions $\mathcal{R}_j$ and $\mathcal{U}_j$ ($j \in \{1,2\}$), we also define functions X_j, χ_j, and $S_{j,k}$ ($j \in \{1,2\}, k \in \{1,2,3\}$); their explicit definitions are given in (2.4). We offer multiple different extensions of the aforementioned generalized (mixed) mock and quantum modular forms into the lower half-plane using the five functions in (1.8) in Theorems 1.1, 1.3, and 1.4 below. Proposition 1.2 pertains to evaluation at rationals or on the boundary of the unit disk $\mathbb{D} := \{q \in \mathbb{C} \mid |q| < 1\}$, as does the remark following Theorem 1.4. Throughout, we set $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$.

Theorem 1.1. *Let $\tau \in \mathbb{H}$, and $u, v \in \mathbb{C}$ be such that $Im(\tau-u) > 0$, $Im(\tau-v) > 0$, and $u \neq a\tau + b, v \neq c\tau + d, (a, c \in \mathbb{N}, b, d \in \mathbb{Z})$.*

(i) *If additionally* $\mathrm{Im}(v) < 0$ *and $u, v \notin \mathbb{Z}\tau + \mathbb{Z}$, then for $k \in \{1,2,3\}$, we have that*

$$\mu(u,v;-\tau) = iq^{\frac{1}{8}}e^{-\pi i(u+v)}\big(-(e^{2\pi i(-u+\tau)};e^{2\pi i\tau})_\infty(e^{2\pi i(-v+\tau)};e^{2\pi i\tau})_\infty + (1-e^{-2\pi iv})S_{1,k}(e^{2\pi iu},e^{2\pi iv};e^{2\pi i\tau})\big).$$

(ii) *If additionally* $\mathrm{Im}(v) < 0$ *then for $k \in \{1,2,3\}$, we have that*

$$\mathcal{M}(u,v;-\tau) = -1 + (1-e^{-2\pi iv})S_{2,k}(e^{2\pi iu},e^{2\pi iv};e^{2\pi i\tau}).$$

(iii) *If additionally* $\mathrm{Im}(u) < 0$ *and $u, v \notin \mathbb{Z}\tau + \mathbb{Z}$, then for $k \in \{1,2,3\}$, we have that*

$$\mu(u,v;-\tau) = ie^{\frac{\pi i\tau}{4}}e^{-\pi i(u+v)}\big(-(e^{2\pi i(-u+\tau)};e^{2\pi i\tau})_\infty(e^{2\pi i(-v+\tau)};e^{2\pi i\tau})_\infty + (1-e^{-2\pi iu})S_{1,k}(e^{2\pi iv},e^{2\pi iu};e^{2\pi i\tau})\big).$$

(iv) *If additionally* $\mathrm{Im}(u) < 0$, *then for $k \in \{1,2,3\}$, we have that*

$$\mathcal{M}(u,v;-\tau) = -1 + (1-e^{-2\pi iu})S_{2,k}(e^{2\pi iv},e^{2\pi iu};e^{2\pi i\tau}).$$

As discussed, it is of particular interest to understand these functions on the boundary of the unit disk at roots of unity or, equivalently, at rational numbers in-between the two half complex planes. To this end, we introduce some notation, and let

$$\widetilde{S}(\alpha,\beta;q) := -1 + (1-\beta^{-1})S_2(\alpha,\beta^{-1};q).$$

As shown in Theorem 1.1 ($k = 2$), the functions $\widetilde{S}(\alpha,\beta;q)$ and $\widetilde{S}(\beta,\alpha;q)$ extend the mixed mock modular form $\mathcal{M}(u,v;\tau)$ into the lower half of the complex plane when $\alpha = e^{2\pi iu}$ and $\beta = e^{2\pi iv}$; the functions $\widetilde{S}$ are themselves defined for $|q| < 1$ or, equivalently, $\tau \in \mathbb{H}$ ($q = e^{2\pi i\tau}$). However, one can show that they are also defined—and can be explicitly evaluated as finite sums—for certain roots of unity q or, equivalently, rational numbers $\tau \in \mathbb{Q}$. We establish an extension to roots of unity on the unit circle or, equivalently, to $\mathbb{Q}$, in Proposition 1.2 below. To state it, we let $b, s, k \in \mathbb{N}$, $a, r, h \in \mathbb{Z}$, with $\gcd(a, b) = \gcd(r, s) = \gcd(h, k) = 1$. We let $\overline{h}$ be an integer such that $h\overline{h} \equiv 1 \pmod{k}$. For a fixed pair (h, k) of integers satisfying the hypotheses just given, suppose $c \in \mathbb{Z}, d \in \mathbb{N}$ are integers with $\gcd(c, d) = 1$, and suppose $d|k$, that is, $dd' = k$ for some integer d'. Then there exists a smallest integer $N_{c,d,h,k}$, $0 \leq N_{c,d,h,k} \leq k-1$, satisfying $N_{c,d,h,k} \equiv -c\overline{h}d' \pmod{k}$. This number will play a role in Proposition 1.2 below, which features the function $\widetilde{S}$ from Theorem 1.1, with $u \to a/b, v \to r/s$, and $\tau \to h/k$.

Proposition 1.2. *Assume the notation above. If $b|k$ and $s \nmid k$, then*

$$\widetilde{S}(\zeta_b^a, \zeta_s^r; \zeta_k^h) = -1 + \sum_{n=0}^{N_{a,b,h,k}-1} \frac{(\zeta_b^a; \zeta_k^h)_{n+1}}{(\zeta_s^{-r}\zeta_k^h; \zeta_k^h)_n} \times (1 - \zeta_b^a \zeta_s^{-r} \zeta_k^{h(2n+1)})(-\zeta_b^a \zeta_s^{-2r})^n \zeta_{2k}^{h(3n^2+n)}. \tag{1.9}$$

We note that (1.9) holds for $\widetilde{S}(\zeta_s^r, \zeta_b^a; \zeta_k^h)$ if $s|k$ and $b \nmid k$, by interchanging the roles of ζ_b^a and ζ_s^r in (1.9), and replacing $N_{a,b,h,k}$ by $N_{r,s,h,k}$.

Analogous to Theorem 1.1 for the mock Jacobi form μ, we have Theorems 1.3 and 1.4 below for the generalized (mixed) mock and quantum modular forms $\mathcal{R}_j$ and $\mathcal{U}_j$.

Theorem 1.3. *Let $q \in \mathbb{D}^*$ and $\alpha, \beta \in \mathbb{C}\backslash\{0\}$, $\alpha \neq q^\ell, \beta \neq q^m$ $(\ell, m \in \mathbb{N})$.*

(i) *If additionally one of the following holds*
   - $j = 1, k \in \{1, 2, 3\}, |\beta| > 1, \alpha \neq q^{-\ell}, \beta \neq q^{-m}$ $(\ell, m \in \mathbb{N}_0)$, *and* $\alpha^{-1}q, \beta^{-1}q \in \mathbb{D}^*$,
   - $j = 2, k \in \{1, 2, 3\}$, *and* $|\beta| > 1$,

   then we have that

$$\mathcal{R}_j(\alpha, \beta; q^{-1}) = \alpha\chi_j\left(\beta^{-1}, \alpha^{-1}; q\right) + (1 - \beta^{-1})S_{j,k}(\alpha, \beta; q).$$

(ii) *If additionally one of the following holds*
   - $j = 1, k \in \{1, 2, 3\}, |\alpha| > 1, \alpha \neq q^{-\ell}, \beta \neq q^{-m}$ $(\ell, m \in \mathbb{N}_0)$, *and* $\alpha^{-1}q, \beta^{-1}q \in \mathbb{D}^*$,
   - $j = 2, k \in \{1, 2, 3\}$, *and* $|\alpha| > 1$,

   then we have that

$$\mathcal{R}_j(\alpha, \beta; q^{-1}) = \beta\chi_j\left(\alpha^{-1}, \beta^{-1}; q\right) + (1 - \alpha^{-1})S_{j,k}(\beta, \alpha; q).$$

(iii) *If additionally $-\beta^{\frac{1}{2}} \neq q^{\frac{r}{4}}\alpha^{\frac{1}{2}} (r \in 2\mathbb{N}_0 + 1), \alpha\beta^{-1}q \in \mathbb{D}^*$, and one of the following holds*
   - $j = 1, k \in \{1, 2, 3\}, |\beta| > 1, \alpha \neq q^{-\ell}, \beta \neq q^{-m}$ $(\ell, m \in \mathbb{N}_0)$, *and* $\alpha^{-1}q, \beta^{-1}q \in \mathbb{D}^*$,
   - $j = 2, k \in \{1, 2, 3\}$, *and* $|\beta| > 1$,

then we have that

$$\mathcal{R}_j(\alpha,\beta;q^{-1}) = \alpha X_j\left(\beta^{-1},\alpha^{-1};q\right) + (1-\beta^{-1})S_{j,k}(\alpha,\beta;q).$$

(iv) *If additionally* $-\alpha^{\frac{1}{2}} \neq q^{\frac{r}{4}}\beta^{\frac{1}{2}}$ $(r \in 2\mathbb{N}_0+1)$, $\alpha^{-1}\beta q \in \mathbb{D}^*$, *and one of the following holds*
- $j = 1, k \in \{1,2,3\}, |\alpha| > 1, \alpha \neq q^{-\ell}, \beta \neq q^{-m}$ $(\ell, m \in \mathbb{N}_0)$, *and* $\alpha^{-1}q, \beta^{-1}q \in \mathbb{D}^*$,
- $j = 2, k \in \{1,2,3\},$ *and* $|\alpha| > 1$,

then we have that

$$\mathcal{R}_j(\alpha,\beta;q^{-1}) = \beta X_j\left(\alpha^{-1},\beta^{-1};q\right) + (1-\alpha^{-1})S_{j,k}(\beta,\alpha;q).$$

Remark. From the fact that the mock modular partition rank generating function R can be expressed in terms of $\mathcal{R}$ as $R(w;q) = \mathcal{R}(w,w^{-1};q)$, we immediately obtain results pertaining to $R(w;q)$ from Theorem 1.3. These extend the results on $R(w;q)$ given in [10, Theorem 3.2].

Theorem 1.4. *Let* $q \in \mathbb{D}^*$ *and* $\alpha, \beta \in \mathbb{C}\backslash\{0\}$.

(i) *If additionally one of the following holds*
- $j = 1$ *and* $\alpha \neq q^{\ell}, \beta \neq q^{m}$ $(\ell, m \in \mathbb{N}_0)$,
- $j = 2, \alpha \neq q^{-\ell}, \beta \neq q^{-m}$ $(\ell, m \in \mathbb{N})$, *and* $\alpha q, \beta q \in \mathbb{D}^*$,

then we have that

$$\mathcal{U}_j(\alpha,\beta;q^{-1}) = -\beta^{-1}\cdot\chi_j(\alpha,\beta;q) \quad \textit{or, equivalently, } \mathcal{U}_j(\alpha,\beta;q^{-1}) = -\alpha^{-1}\cdot\chi_j(\beta,\alpha;q).$$

(ii) *If additionally* $-\beta^{\frac{1}{2}} \neq q^{\frac{r}{4}}\alpha^{\frac{1}{2}}$ $(r \in 2\mathbb{N}_0+1), \alpha\beta^{-1}q \in \mathbb{D}^*$ *and one of the following holds*
- $j = 1$ *and* $\alpha \neq q^{\ell}, \beta \neq q^{m}$ $(\ell, m \in \mathbb{N}_0)$,
- $j = 2$, $\alpha \neq q^{-\ell}, \beta \neq q^{-m}$ $(\ell, m \in \mathbb{N})$, *and* $\alpha q, \beta q \in \mathbb{D}^*$,

then we have that

$$\mathcal{U}_j(\alpha,\beta;q^{-1}) = -\beta^{-1}\cdot X_j(\alpha,\beta;q).$$

(iii) *If additionally* $-\alpha^{\frac{1}{2}} \neq q^{\frac{r}{4}}\beta^{\frac{1}{2}}$ $(r \in 2\mathbb{N}_0+1)$, $\alpha^{-1}\beta q \in \mathbb{D}^*$, *and one of the following holds*
- $j = 1$ *and* $\alpha \neq q^{\ell}, \beta \neq q^{m}(\ell, m \in \mathbb{N}_0)$,
- $j = 2$, $\alpha \neq q^{-\ell}, \beta \neq q^{-m}(\ell, m \in \mathbb{N})$, *and* $\alpha q, \beta q \in \mathbb{D}^*$,

then we have that

$$\mathcal{U}_j(\alpha,\beta;q^{-1}) = -\alpha^{-1}\cdot X_j(\beta,\alpha;q).$$

Remark. We immediately obtain results pertaining to the quantum modular unimodal rank generating function $U(w;q)$ from Theorem 1.4, using the relationship $q\,\mathcal{U}(wq,w^{-1}q;q) = U(w;q)$. Moreover, when $w = 1 (\alpha = \beta = q)$, the function $\chi(1;q) = \sum_{m=0}^{\infty}(-1)^m q^{\frac{m(m+1)}{2}}$ arising from part (i) of Theorem 1.4 is essentially the Eichler integral of the *shadow* $\eta^3(\tau) = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{(2n+1)^2}{8}}$ appearing in [14, Theorem 1.3].[1] Asymptotic relationships of this

[1] The function $U(-w;q)$ as defined in [14, (1.1)] is equal to the function $U(w;q)$ defined in (1.6).

type have been shown to exist as q tends radially towards roots of unity ζ on the boundary of $\mathbb{D}$ [12,18,26], and these have been of particular interest when studying quantum modular forms arising from mock modular forms. Here, we see the function χ directly via the extension of $\mathcal{U}$ to the lower half-plane in Theorem 1.4. In this case, one is able to directly (as opposed to asymptotically) evaluate the function X from parts (ii) and (iii) of Theorem 1.4 at suitable roots of unity as an explicit finite sum, similar to Proposition 1.2.

2. Preliminaries. In this section, we establish some preliminary results and definitions for q-hypergeometric series and partial theta functions, which we will use in our proofs of Theorems 1.1, 1.3, and 1.4. The first such result is due to Andrews [3], extending similar results of Ramanujan. The identity in (2.1) holds for complex parameters a, b, A, B, and q such that the functions appearing converge.

Theorem [3, Theorem 1]. *We have that*

$$\sum_{n=0}^{\infty} \frac{(B;q)_n(-Abq;q)_n q^n}{(-aq;q)_n\,(-bq;q)_n} = \frac{-a^{-1}(B;q)_\infty(-Abq;q)_\infty}{(-bq;q)_\infty\,(-aq;q)_\infty} \sum_{m=0}^{\infty} \frac{(A^{-1};q)_m \left(\frac{Abq}{a}\right)^m}{\left(\frac{-B}{a};q\right)_{m+1}} + (1+b)\sum_{m=0}^{\infty} \frac{(-a^{-1};q)_{m+1}\left(\frac{-ABq}{a};q\right)_m (-b)^m}{\left(\frac{-B}{a};q\right)_{m+1}\left(\frac{Abq}{a};q\right)_{m+1}}. \tag{2.1}$$

We will also use the following lemma, which extends the infinite product $(\alpha;q)$ $(|q|<1)$ outside of the unit disk $(|q|>1)$.

Lemma 2.1. *For $|q|<1$ and $|\alpha q|<1$, we have that $(\alpha;q^{-1})_\infty = (\alpha q;q)_\infty^{-1}$.*

Proof. The following identities of Euler are well-known $(|q|<1)$ [19]

$$(\alpha q;q)_\infty^{-1} = \sum_{n=0}^{\infty} \frac{(\alpha q)^n}{(q;q)_n}, \quad (\alpha;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} \alpha^n}{(q;q)_n}, \tag{2.2}$$

The first holds under the additional hypothesis $|\alpha q|<1$. The sum in the second identity in (2.2) also makes sense for $|q|>1$, which can be seen by replacing $q \mapsto q^{-1}$, and applying

$$(a;q^{-1})_n = (a^{-1};q)_n(-a)^n q^{-\frac{n(n-1)}{2}}, \tag{2.3}$$

which is also well known [19], and holds for $n \in \mathbb{N}_0$. After some simplification, we find the sum appearing on the right-hand side of the first identity in (2.2). The result follows by applying the first identity in (2.2). $\square$

Next we define the partial theta functions and q-hypergeometric series appearing in the statements of Theorems 1.1, 1.3, and 1.4 in terms of the functions χ, X, and S_k $(1 \le k \le 3)$ defined in (1.8). We define for $j \in \{1,2\}$ and

$k \in \{1,2,3\}$ the partial theta and q-hypergeometric functions

$$\begin{aligned}\chi_j(\alpha,\beta;q) &:= \begin{cases}\chi\left(\frac{\alpha}{\beta};q\right) & \text{if } j=1,\\ \chi\left(\frac{\alpha}{\beta};q\right)(\alpha q;q)_\infty^{-1}(\beta q;q)_\infty^{-1} & \text{if } j=2,\end{cases}\\ X_j(\alpha,\beta;q) &:= \begin{cases}X\left(\frac{\alpha}{\beta};q\right) & \text{if } j=1,\\ X\left(\frac{\alpha}{\beta};q\right)(\alpha q;q)_\infty^{-1}(\beta q;q)_\infty^{-1} & \text{if } j=2,\end{cases}\\ S_{j,k}(\alpha,\beta;q) &:= \begin{cases}S_k\left(\alpha,\beta^{-1};q\right)(\alpha^{-1}q;q)_\infty(\beta^{-1}q;q)_\infty & \text{if } j=1,\\ S_k(\alpha,\beta^{-1};q) & \text{if } j=2.\end{cases}\end{aligned} \tag{2.4}$$

3. Proofs. We first prove Theorems 1.3 and 1.4 using limiting versions of Andrews' theorem (i.e. (2.1)), Lemma 2.1, and relationships between the functions χ, X, and $S_k (k \in \{1,2,3\})$. We then prove Theorem 1.1 in part from Theorems 1.3 and 1.4. For brevity, we omit the proof of Proposition 1.2, as it follows by a direct calculation using the definition of the function S_2, the numbers $N_{c,d,h,k}$, and the hypotheses given.

Proof of Theorem 1.4. In (2.1), we first let $b = \frac{-\alpha}{Aq}$. We then let $A \to \infty$, and use the fact that $\lim_{A\to\infty} A^{-m}(xA;q)_m = (-x)^m q^{\frac{m(m-1)}{2}}$. Thus, after the aforementioned substitution in b, letting $A \to \infty$, we have (using uniform convergence) that (2.1) becomes

$$\sum_{n=0}^{\infty} \frac{(B;q)_n(\alpha;q)_n q^n}{(-aq;q)_n} = \frac{-a^{-1}(B;q)_\infty(\alpha;q)_\infty}{(-aq;q)_\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{-\alpha}{a}\right)^m}{\left(\frac{-B}{a};q\right)_{m+1}} + \sum_{m=0}^{\infty} \frac{(-a^{-1};q)_{m+1}\left(\frac{\alpha B}{a}\right)^m q^{\frac{m(m-1)}{2}}}{\left(\frac{-B}{a};q\right)_{m+1}\left(\frac{-\alpha}{a};q\right)_{m+1}}. \tag{3.1}$$

Next we let $a \to 0$, using that $\lim_{a\to 0} a^m(xa^{-1};q)_m = (-x)^m q^{\frac{m(m-1)}{2}}$. This allows us to determine the limiting value of the first sum on the right-hand side of (3.1).

To evaluate the remaining limiting value of the second sum on the right-hand side of (3.1), we rewrite

$$\begin{aligned}\sum_{m=0}^{\infty} &\frac{(-a^{-1};q)_{m+1}\left(\frac{\alpha B}{a}\right)^m q^{\frac{m(m-1)}{2}}}{\left(\frac{-B}{a};q\right)_{m+1}\left(\frac{-\alpha}{a};q\right)_{m+1}}\\ &= a\sum_{m=0}^{\infty} \frac{\left(a^{m+1}(-a^{-1};q)_{m+1}\right)(\alpha B)^m q^{\frac{m(m-1)}{2}}}{\left(a^{m+1}\left(\frac{-B}{a};q\right)_{m+1}\right)\left(a^{m+1}\left(\frac{-\alpha}{a};q\right)_{m+1}\right)}.\end{aligned} \tag{3.2}$$

We proceed similarly and find that the second sum on the right-hand side of (3.1) converges to 0 as $a \to 0$. Thus, we have shown that

$$\sum_{n=0}^{\infty} (B;q)_n(\alpha;q)_n q^n = -B^{-1}(B;q)_\infty(\alpha;q)_\infty \sum_{m=0}^{\infty} \left(\frac{-\alpha}{B}\right)^m q^{-\frac{m(m+1)}{2}}.$$

We let $B = \beta$, then divide both sides by $(\alpha;q)_\infty(\beta;q)_\infty$, and let $q \mapsto q^{-1}$. Subject to the hypotheses given to ensure convergence, this proves the first statement in part (i) of Theorem 1.4 when $j = 1$. To prove the second assertion, we repeat the above, interchanging the roles of α and β. To prove part (i) when $j = 2$, we multiply both sides of the the result just established for $j = 1$ by $(\alpha; q^{-1})_\infty(\beta; q^{-1})_\infty$, and apply Lemma 2.1 on the right-hand side. Parts (ii) and (iii) follow from part (i), using that $X(z; q) = \chi(z; q)$ when $|q| < 1$ and $|zq| < 1$ where defined, which can be deduced from a result of Rogers and Fine ([17], see also [18, p. 7]). □

Proof of Theorem 1.3. We proceed in a similar manner to the proof of Theorem 1.4 above. We use (2.3), as well as the identity from (2.1), letting $A \to 0$, and $B \to 0$, with $a = -\alpha^{-1}, b = -\beta^{-1}$. This shows that $\mathcal{R}(\alpha, \beta; q^{-1})$ equals

$$\sum_{n=0}^{\infty} \frac{q^n}{(\alpha^{-1}q; q)_n(\beta^{-1}q; q)_n} = \frac{\alpha}{(\alpha^{-1}q; q)_\infty(\beta^{-1}q; q)_\infty}\chi\left(\frac{\alpha}{\beta}; q\right) + (1-\beta^{-1})\sum_{m=0}^{\infty}(\alpha; q)_{m+1}\beta^{-m}. \tag{3.3}$$

Subject to the hypotheses given to ensure convergence (from the definition of $S_{j,k}(\alpha, \beta; q)$ in (2.4), we require in particular $|\beta| > 1$), this proves part (i) of Theorem 1.3 when $j = 2$ and $k = 1$. To prove the results in (i) for $j = 2$ and $k \in \{2, 3\}$, we use that the three functions $S_k(\alpha, \beta; q)$ $k \in \{1, 2, 3\}$ are equal where they converge: the equality of S_1 and S_3 can be deduced from [17, (12.2)], and the equality of S_1 and S_2 follows from the "Rogers-Fine identity" (see [24, (1)] or [10, (2.2)]). This proves part (i) for $j = 2$. To prove part (i) for $j = 1$, we multiply both sides of the identity just established for $j = 2$ by $(\alpha^{-1}q; q)_\infty(\beta^{-1}q; q)_\infty$, and apply Lemma 2.1 on the left-hand side. Subject to the convergence conditions given, this proves part (i) for $j = 1$. As was the case in the proof of Theorem 1.4, we prove part (ii) of Theorem 1.3 by interchanging the roles of α and β. Also similar to the proof of Theorem 1.4, we deduce parts (iii) and (iv) of Theorem 1.3 from parts (i) and (ii). □

We are now prepared to prove Theorem 1.1, which follows, in part, from Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.1. We use Ramanujan's striking identity, recently revisited by Choi and in [18], (see [7, Entry 3.4.7, p. 67], [15], or [18, Theorem 3.1]), relating the functions $\mathcal{M}, \mathcal{R}$, and $\mathcal{U}$; this identity can be re-written as follows, with $\alpha = e^{2\pi i u}, \beta = e^{2\pi i v}$:

$$\mathcal{M}(u, v; \tau) = \mathcal{R}(\alpha, \beta; q) + \mathcal{U}(\alpha^{-1}, \beta^{-1}; q) - 1.$$

We let $q \mapsto q^{-1}$, and apply Theorem 1.3 part (i) with $j = 2$, and the second expression in Theorem 1.4 part (i) with $j = 2$ and $\alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}$. Subject to the hypotheses given, we have that $\mathcal{M}(\alpha, \beta; q^{-1}) = -1 + (1 - \beta^{-1})S_{2,k}(\alpha, \beta^{-1}; q)$, which proves part (ii) of Theorem 1.1. Part (iv) follows similarly. Parts (i) and (iii) of Theorem 1.1 also follow in a similar way, using Lemma 2.1, parts (i) and (ii) of Theorem 1.3 with $j = 1$, and part (i) of Theorem 1.4 with $j = 1$. □

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References

[1] K. Alladi, A partial theta identity of Ramanujan and its number-theoretic interpretation, Ramanujan J. **20** (2009), 329–339.

[2] K. Alladi, A combinatorial study and comparison of partial theta identities of Andrews and Ramanujan, Ramanujan J. **23** (2010), 227–241.

[3] G. Andrews, An introduction to Ramanujan's "lost" notebook, Amer. Math. Monthly **86** (1979), 89–108.

[4] G. Andrews, Partitions: Yesterday and Today, New Zealand Mathematical Society, Wellington, 1979.

[5] G. Andrews, Ramanujan's "lost" notebook. I. Partial θ functions, Adv. Math. **41** (1981), 137–172.

[6] G. Andrews and B. Berndt, Ramanujan's Lost Notebook, Part I, Springer, New York, 2005.

[7] G. Andrews and B. Berndt, Ramanujan's Lost Notebook, Part II, Springer, New York, 2009.

[8] B. Berndt and A. Yee, Combinatorial proofs of identities in Ramanujan's lost notebook associated with the Rogers-Fine identity and false theta functions, Ann. Comb. **7** (2003), 409–423.

[9] B. Berndt, B. Kim, and A. Yee, Ramanujan's lost notebook: combinatorial proofs of identities associated with Heine's transformation or partial theta functions, J. Combin. Theory Ser. A **117** (2010), 857–973.

[10] K. Bringmann, A. Folsom, and R.C. Rhoades, Partial theta functions and mock modular forms as q-hypergeometric series, Ramanujan J., special issue Ramanujan's 125th birthday, **29** (2012), 295–310.

[11] K. Bringmann and K. Ono, Dyson's ranks and Maass forms, Ann. of Math. (2) **171** (2010), 419–449.

[12] K. Bringmann and L. Rolen, Radial limits of mock theta functions, Res. Math. Sci. **2** (2015), 2–17.

[13] J.H. Bruinier and J. Funke, On two geometric theta lifts, Duke Math. J. **125** (2004), 45–90.

[14] J. Bryson, S. Pitman, K. Ono, and R.C. Rhoades, Unimodal sequences and quantum mock modular forms, Proc. Natl. Acad. Sci. USA **109** (2012), 16063–16067.

[15] Y-S Choi, The basic bilateral hypergeometric series and the mock theta functions, Ramanujan J. **24** (2011), 345–386.

[16] A. Dabholkar, S. Murthy, and D. Zagier, Quantum Black Holes, Wall Crossing, and Mock Modular Forms, arXiv:1208.4074.

[17] N.J. Fine, Basic Hypergeometric Series and Applications, Math. Surveys and Monographs, 27, American Mathematical Society, Providence, RI, 1988.

[18] A. Folsom, K. Ono, and R.C. Rhoades, Mock theta functions and quantum modular forms, Forum Math. Pi **1** (2013), e2, 27 p.

[19] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications, vol. 35, Cambridge University Press, Cambridge, 1990.

[20] B. Gordon and R. McIntosh, A survey of classical mock theta functions, Dev. Math., **23**, Springer, New York, 2012, 95–144.

[21] K. Hikami, Mock (false) theta functions as quantum invariants, Regul. Chaotic Dyn. **10** (2005), 509–530.

[22] M-J Jang and S. Löbrich, Radial Limits of the Universal Mock Theta Function g_3, Proc. Amer. Math. Soc., to appear.

[23] K. Ono, Unearthing the visions of a master: harmonic Maass forms and number theory, Current developments in mathematics, Int. Press, Somerville, MA, 2008, 347–454.

[24] L. Rogers, On two theorems of combinatory analysis and some allied identities, Proc. London Math. Soc. (2) **16** (1917), 315–336.

[25] G. Watson, The final problem: An account of the mock theta functions, J. London Math. Soc. **2** (1936), 55–80.

[26] D. Zagier, Quantum modular forms, In Quanta of Maths: Conference in honor of Alain Connes, Clay Math. Proc. **11** (2010), Amer. Math. Soc., Providence, RI, 659–675.

[27] S.P. Zwegers, Mock θ-functions and real analytic modular forms, q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), Contemp. Math., 291, American Mathematical Society, Providence, RI, 2001.

[28] S. Zwegers, Mock theta functions, Ph.D. Thesis, Universiteit Utrecht, 2002.

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