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Equidistribution of Heegner points and the partition function

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Abstract Let $p(n)$ denote the number of partitions of a positive integer n. In this paper we study the asymptotic growth of $p(n)$ using the equidistribution of Galois orbits of Heegner points on the modular curve $X_0(6)$. We obtain a new asymptotic formula for $p(n)$ with an effective error term which is $O(n^{-(\frac{1}{2}+\delta)})$ for some $\delta > 0$. We then use this asymptotic formula to sharpen the classical bounds of Hardy and Ramanujan, Rademacher, and Lehmer on the error term in Rademacher's exact formula for $p(n)$.

1 Introduction and statements of results

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. It is a classical problem in number theory to study the asymptotic growth of $p(n)$ as $n \to \infty$. In a celebrated paper published in 1917, Hardy and Ramanujan [12] invented what is now known as the *circle method* in analytic number theory and used it to obtain the asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{2\pi\sqrt{n/6}}.$$

They also gave an asymptotic expansion with an estimate for the error term which is $O(n^{-1/4})$. In 1937, Rademacher refined the method of Hardy and Ramanujan to obtain

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the following remarkable *exact* formula for $p(n)$ (see e.g. [21], [22, p. 276–282])

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} k^{\frac{1}{2}} A_k(n) \frac{d}{dn} \frac{\sinh\left(\frac{\pi\lambda_n}{k}\sqrt{\frac{2}{3}}\right)}{\lambda_n}, \tag{1.1}$$

where $\lambda_n := \sqrt{n - \frac{1}{24}}$ and $A_k(n)$ is the Kloosterman type sum

$$A_k(n) := \sum_{h \pmod{k}^\times} \omega_{h,k} e^{-2\pi i hn/k},$$

where $\omega_{h,k}$ is a certain root of unity and $h \pmod{k}^\times$ means the summation is taken over those $h \mod k$ such that $(h, k) = 1$. Rademacher truncated the series (1.1) and obtained the expression

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{N} k^{1/2} A_k(n) \frac{d}{dn} \frac{\exp\left(\frac{\pi\lambda_n}{k}\sqrt{\frac{2}{3}}\right)}{\lambda_n} + R(n, N),$$

and upon choosing $N = \lfloor \alpha n^{\frac{1}{2}} \rfloor$ for any $\alpha > 0$, he showed that the error term satisfies the bound

$$R_{n,\alpha} := R\left(n, \left\lfloor \alpha n^{\frac{1}{2}} \right\rfloor\right) = O\left(n^{-\frac{3}{8}}\right).$$

Here $\lfloor \cdot \rfloor$ is the greatest integer function. Soon thereafter, Lehmer [16,17] established the sharper bound

$$R_{n,\alpha} = O\left(\log(n) n^{-\frac{1}{2}}\right).$$

Both Rademacher and Lehmer obtained their bounds by estimating $A_k(n)$ directly. Lehmer's improvement resulted from a sharper estimate for $A_k(n)$. Of course, bounds for the individual sums $A_k(n)$ lead to weaker estimates, and to obtain the best results one must take advantage of cancellation which should occur when estimating an infinite sum of weighted Kloosterman sums. Detecting such cancellation has come to be a problem of central importance in number theory in the 70 years since the work of Rademacher and Lehmer.

In this paper we take a modern approach to this problem and study the asymptotic distribution of $p(n)$ using the equidistribution of Galois orbits of Heegner points on the modular curve $X_0(6)$. For any sequence of positive integers n such that $D_n := 24n - 1$ is square-free, we obtain a new asymptotic formula for $p(n)$ with an effective error term which is $O(n^{-(\frac{1}{2}+\delta)})$ for some $\delta > 0$. We then use our asymptotic formula to sharpen the bounds of Hardy and Ramanujan, Rademacher, and Lehmer on $R_{n,\alpha}$ when $\alpha = 1/\sqrt{6}$.

In order to state our main results we fix the following notation and assumptions. Let $n \in \mathbb{Z}^+$ be a positive integer such that $-D_n = -24n + 1$ is square-free. Let $\mathcal{Q}_{D_n,6} := \{Q = [a, b, c]\}$ be the set of positive definite, primitive, integral binary quadratic forms $Q(x, y) = [a, b, c](x, y) = ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = -D_n$ with $6|a$. The set $\mathcal{Q}_{D_n,6}$ is stable under the action of $\Gamma_0(6)$. Define the map

$$\chi_{12} : \mathcal{Q}_{D_n,6} \to \{\pm 1\}$$

by $\chi_{12}(Q) := (\frac{12}{b})$ where $(\frac{12}{\cdot})$ is the Legendre symbol. This induces a map

$$\chi_{12} : \mathcal{Q}_{D_n,6}/\,\Gamma_0(6) \to \{\pm 1\}.$$

Let $\Lambda_{D_n}(6)$ be the set of Heegner points of discriminant $-D_n$ on $X_0(6)$. There is a bijection

$$\mathcal{Q}_{D_n,6}/\,\Gamma_0(6) \to \Lambda_{D_n}(6)$$

given by $[Q] \mapsto z_Q$ where

$$z_Q = \frac{-b + \sqrt{-D_n}}{12a} \in \mathbb{H}$$

is the unique root in the complex upper half-plane $\mathbb{H}$ of the dehomogenized form $Q(x, 1) = 6ax^2 + bx + c$.

Throughout this paper we let $e(z) := e^{2\pi i z}$. Our main result is the following theorem.

Theorem 1.1 *Let n be a positive integer such that $D_n = 24n - 1$ is square-free. Then there exists an effective constant $c > 0$ such that for all $\epsilon > 0$ and $0 < b < 1/12c$,*

$$p(n) = \frac{1}{(24n-1)} \sum_{\substack{z_Q \in \Lambda_{D_n}(6) \\ \mathrm{Im}(z_Q) > 1 + (24n-1)^{-b}}} \chi_{12}(Q) \left(1 - \frac{1}{2\pi \mathrm{Im}(z_Q)}\right) e(-z_Q)$$
$$+ O_\epsilon\left(n^{-(\frac{7}{12} - bc) + \epsilon}\right) + O_\epsilon\left(n^{-(\frac{1}{2} + b) + \epsilon}\right)$$

as $n \to \infty$. The implied constants in the error terms O_ϵ are effective.

In the following corollary we use Theorem 1.1 to sharpen the bounds of Hardy and Ramanujan, Rademacher, and Lehmer on the error term $R_{n,\alpha}$ when $\alpha = 1/\sqrt{6}$.

Corollary 1.2 *Let $n \in \mathbb{Z}^+$ and $c > 0$ be as in Theorem 1.1. Then for all $\epsilon > 0$ and $0 < b < 1/12c$,*

$$R_{n,1/\sqrt{6}} = O_\epsilon\left(n^{-(\frac{7}{12} - bc) + \epsilon}\right) + O_\epsilon\left(n^{-(\frac{1}{2} + b) + \epsilon}\right)$$

as $n \to \infty$. The implied constants in the error terms O_ϵ are effective.

Remark 1.3 If the subconvexity bound of Conrey and Iwaniec [3] in the proof of Theorem 3.1 is replaced with the corresponding bound under the generalized Riemann hypothesis, one finds that there exists an effective constant $c' > 0$ such that Theorem 1.1 and Corollary 1.2 hold for all $\epsilon > 0$ and $0 < b < 1/4c'$ where the first error term in both statements is $O_\epsilon(n^{-(\frac{3}{4}-bc')+\epsilon})$ with an effective implied constant.

Remark 1.4 The restriction to square-free discriminants $-D_n$ in Theorem 1.1 and Corollary 1.2 results from our assumption in Theorem 3.1 that $-D < -4$ be an odd fundamental discriminant. This assumption is made so that we can decompose the set of Heegner points of discriminant $-D$ on the modular curve $X_0(N)$ into Galois orbits under the action of $\mathrm{Gal}(H/K)$ where H is the Hilbert class field of $K = \mathbb{Q}(\sqrt{-D})$. If $-D$ is not assumed to be fundamental, we must deal with Heegner points over ring class fields, and we then need period relations for the corresponding hyperbolic Weyl sums in an explicit enough form to estimate their contribution to the spectral decomposition as in the proof of Theorem 3.1. Because such period relations exist, it is just a technical matter to remove the restriction to square-free discriminants $-D_n$ in Theorem 1.1 and Corollary 1.2.

We now outline the proof of Theorem 1.1. Rademacher used an expression for $A_k(n)$ as a sum over solutions to a quadratic congruence due to Selberg [23, p. 706, eq. (18)] to obtain an alternate form of his exact formula as essentially an infinite sum of the product of $A_k(n)$ and values of the Bessel function $I_{3/2}$ (see [22, p. 273, pp. 280-282]). Such sums can be described in terms of orbits of CM points under the action of congruence subgroups. Using this fact, Bringmann and Ono [1] obtained an arithmetic reformulation of Rademacher's exact formula which under the assumptions in this paper is given by

$$p(n) = \frac{1}{2D_n} \sum_{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q) P(z_Q) \tag{1.2}$$

where $P(z)$ is the Maass–Poincaré series

$$P(z) := 4\pi \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(6)} \mathrm{Im}(\gamma z)^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi \mathrm{Im}(\gamma z)) e(-\mathrm{Re}(\gamma z)), \quad z \in \mathbb{H}.$$

Here $\Gamma_\infty < \Gamma_0(6)$ is the stabilizer of the cusp at ∞ of $X_0(6)$.

In [5], Duke proved that the Heegner points of discriminant $-D$ on $X_0(N)$ are equidistributed with respect to the hyperbolic measure as $D \to \infty$. In Theorem 3.1 we prove a variant of Duke's theorem for test functions with moderate growth in the cusps of $X_0(N)$. The proof involves a careful spectral regularization, and ultimately rests on deep subconvexity bounds of Conrey and Iwaniec [3] obtained using estimates for infinite sums of twisted Kloosterman sums. Our strategy is to combine Theorem 3.1 with (1.2) to obtain an asymptotic formula for $p(n)$ as $n \to \infty$. However, we cannot apply Theorem 3.1 directly to (1.2) because the Maass–Poincaré series $P(z)$ grows *exponentially* in the cusps of $X_0(6)$. To overcome this difficulty, we adapt a construction of Duke [6] used to regularize the pole at ∞ of the modular j-function. Roughly speaking, for each $\varepsilon > 0$ we construct a smooth Poincaré series $\mathscr{P}_\varepsilon(z)$ such that $F_\varepsilon(z) := P(z) - \mathscr{P}_\varepsilon(z)$ has moderate growth in the cusps of $X_0(6)$. We then apply Theorem 3.1 to

$$\frac{1}{2D_n} \sum_{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q) F_\varepsilon(z_Q),$$

which leads to an asymptotic formula for $p(n)$ with main term given by the Salié type sum

$$\frac{1}{2D_n} \sum_{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q) \mathscr{P}_\varepsilon(z_Q).$$

To obtain Theorem 1.1 from this asymptotic formula we must analyze the contribution of the parameter ε to the error term. Part of this analysis requires an explicit upper bound for the L^2-norm of iterates of the hyperbolic Laplacian applied to a regularized version of $F_\varepsilon(z)$. For more details concerning these arguments, we refer the reader to [7].

Finally, we remark that the problem of estimating $p(n)$ can be thought of as a special case of the problem of estimating the coefficients of negative half-integral weight weakly holomorphic modular forms and harmonic Maass forms. Because the coefficients of such forms can often be written as sums of values of weight zero Maass–Poincaré series at suitably chosen CM points (see, e.g. [20, section 11]), the strategy in this paper should apply to the problem of bounding the analog of Rademacher's error term in these situations (see e.g. [7]).

2 Heegner points on $X_0(N)$

In this section we review some facts regarding Heegner points on $X_0(N)$ that will be needed throughout this paper. Let N be a positive integer, and let $-D < -4$ be an odd fundamental discriminant coprime to N such that every prime divisor p of N is split in $K = \mathbb{Q}(\sqrt{-D})$. Let $\mathcal{Q}_{D,N} := \{Q = [a, b, c]\}$ be the set of positive definite, primitive, integral binary quadratic forms $Q(x, y) = [a, b, c](x, y) = ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = -D$ with $N|a$. The set $\mathcal{Q}_{D,N}$ is stable under the action of $\Gamma_0(N)$.

Let $\Lambda_D(N)$ be the set of Heegner points of discriminant $-D$ on $X_0(N)$. There is a bijection

$$\mathcal{Q}_{D,N}/\Gamma_0(N) \to \Lambda_D(N)$$

given by $[Q] \mapsto z_Q$ where

$$z_Q = \frac{-b + \sqrt{-D}}{2Na} \in \mathbb{H}$$

is the unique root in the complex upper half-plane of the dehomogenized form $Q(x,1) = Nax^2 + bx + c$.

Fix a solution $r \bmod 2N$ of $r^2 \equiv -D \bmod 4N$. Note that there are exactly $2^{t(N)}$ such solutions r, where $t(N)$ is the number of distinct prime divisors of N. Define the subset of forms

$$\mathcal{Q}_{D,N,r} = \left\{ Q = [a,b,c] \in \mathcal{Q}_{D,N} : \ b \equiv r \mod 2N \right\}.$$

The set $\mathcal{Q}_{D,N,r}$ is also stable under the action of $\Gamma_0(N)$. There is a decomposition (see [8, p. 507])

$$\mathcal{Q}_{D,N}/\Gamma_0(N) = \bigcup_{\substack{r \mod 2N \\ r^2 \equiv -D \mod 4N}} \mathcal{Q}_{D,N,r}/\Gamma_0(N). \tag{2.1}$$

The natural map

$$\mathcal{Q}_{D,N,r}/\Gamma_0(N) \to \mathcal{Q}_{D,1}/\mathrm{SL}_2(\mathbb{Z})$$

is a bijection which makes the set $\mathcal{Q}_{D,N,r}/\Gamma_0(N)$ into a group of order $h(-D)$ via the Gauss law of composition on $\mathcal{Q}_{D,1}/\mathrm{SL}_2(\mathbb{Z})$. Here, $h(-D)$ is the ideal class number of $K = \mathbb{Q}(\sqrt{-D})$. Moreover, by class field theory $\mathcal{Q}_{D,N,r}/\Gamma_0(N) \cong \mathrm{CL}_K \cong \mathrm{Gal}(H/K)$ where CL_K is the ideal class group of K and H is the Hilbert class field of K. For details concerning these facts see [4].

The Heegner points $\Lambda_D(N)$ are divided into $2^{t(N)}$ simple, transitive $\mathrm{Gal}(H/K)$-orbits of size $h(-D)$ (see [9, pp. 235–236]). Define the Galois orbit

$$\mathcal{O}_r := \{ z_{Q_r}^{\sigma} : \ \sigma \in \mathrm{Gal}(H/K) \}$$

where $[Q_r]$ is any class in $\mathcal{Q}_{D,N,r}/\Gamma_0(N)$. Then one has a bijection

$$\mathcal{Q}_{D,N,r}/\Gamma_0(N) \to \mathcal{O}_r. \tag{2.2}$$

3 Regularization

Let $F : \mathbb{H} \to \mathbb{C}$ be a C^∞, $\Gamma_0(N)$-invariant function. We say that F has *cuspidal growth of power* α for some $\alpha \in \mathbb{R}$ if for every cusp $\mathfrak{a}$ of $\Gamma_0(N)$ there exists a constant $c_{\mathfrak{a}} \in \mathbb{C}$ (possibly equal to 0) such that for each integer $a \geq 0$,

$$\Delta^a \left(F(\sigma_{\mathfrak{a}} z) - c_{\mathfrak{a}} y^{\alpha} \right) = O(e^{-cy}) \quad \text{as } y = \mathrm{Im}(z) \to \infty$$

for some $c = c_{\mathfrak{a}}(a) > 0$. Here $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ is the hyperbolic Laplacian where Δ^a means we apply the Laplacian a-times, and $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ is a scaling matrix such that $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$.

Theorem 3.1 *Let N be a fixed positive, square-free integer. Let $-D < -4$ be an odd fundamental discriminant coprime to N such that every prime divisor p of N is split in $K = \mathbb{Q}(\sqrt{-D})$. Let*

$$\chi : \mathcal{Q}_{D,N}/\Gamma_0(N) \to \mathbb{C}$$

be a function satisfying the following 3 conditions:

(1) χ *is constant on* $\mathcal{Q}_{D,N,r}/\Gamma_0(N)$ *for each solution* $r \mod 2N$ *of* $r^2 \equiv -D \mod 4N$.

(2) *For some* $\delta_1 \geq 0$,

$$\sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q) \ll D^{\delta_1}.$$

(3) *For some* $\delta_2 \geq 0$,

$$\max_{\substack{r \mod 2N \\ r^2 \equiv -D \mod 4N}} |\chi(Q_r)| \ll D^{\delta_2}.$$

Let $F : \mathbb{H} \to \mathbb{C}$ be a C^∞, $\Gamma_0(N)$-invariant function with cuspidal growth of power α for some $\alpha \leq 1/2$. Then there exists an integer $a_0 > 0$ such that for all $\epsilon > 0$,

$$\sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)F(z_Q) = O_N\left(||F||_1\, D^{\delta_1}\right) + O_{\epsilon,N}\left(\left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2 D^{\frac{5}{12}+\delta_2+\epsilon}\right)$$
$$+ O_{\epsilon,N}\left(D^{\max\{\frac{5}{12}+\delta_2+\epsilon,\,\delta_1+\frac{\alpha-1}{2}\}}\right)$$

as $D \to \infty$. Here F_{T_0} is a regularized version of F where $T_0 \gg 1$ is a fixed cutoff parameter (*see* (3.1))*, and the implied constants are effective.*

Proof of Theorem 3.1 We begin by constructing a C^∞, $\Gamma_0(N)$-invariant function and which has growth coinciding precisely with that of F.

Lemma 3.2 *Let $T > 1$. There exists a C^∞, $\Gamma_0(N)$-invariant function $\eta_T : \mathbb{H} \to \mathbb{C}$ such that*

$$\eta_T(\sigma_{\mathfrak{b}} z) = \begin{cases} 0, & 1 < y < T \\ c_{\mathfrak{b}} y^\alpha \psi(y/T), & T \leq y \leq 2T \\ c_{\mathfrak{b}} y^\alpha, & y > 2T. \end{cases}$$

where $\psi : \mathbb{R}^+ \to [0, 1]$ is a C^∞ function such that

$$\psi(t) = \begin{cases} 0, & t < 1 \\ 1, & t > 2. \end{cases}$$

Proof Let F be a fundamental polygon for $\Gamma_0(N)$. The strip

$$P(T) := \{z = x + iy : 0 < x < 1,\ y = \mathrm{Im}(z) \geq T\}$$

is mapped by the scaling matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ into the cuspidal zone

$$F_{\mathfrak{a}}(T) := \sigma_{\mathfrak{a}} P(T).$$

For T sufficiently large the cuspidal zones are disjoint and the set

$$F(T) := F \setminus \bigcup_{\mathfrak{a}} F_{\mathfrak{a}}(T)$$

has compact closure and is adjacent to each $F_{\mathfrak{a}}(T)$ along the horocycles

$$\sigma_{\mathfrak{a}} L(T), \quad L(T) := \{z = x + iT :\ 0 < x < 1\}$$

(see [13, Sect. 2.2]). Thus F is partitioned into the compact central part $F(T)$ and the cuspidal zones $F_{\mathfrak{a}}(T)$ so that

$$F = F(T) \cup \bigcup_{\mathfrak{a}} F_{\mathfrak{a}}(T).$$

Let $\psi_{T,\mathfrak{a}} : \mathbb{R}^+ \to \mathbb{C}$ be a C^∞ function such that

$$\psi_{T,\mathfrak{a}}(t) = \begin{cases} 0, & t < 1 \\ c_{\mathfrak{a}} t^{\alpha}, & t > T. \end{cases}$$

Define the incomplete Eisenstein series

$$E_{\mathfrak{a}}(z|\psi_{T,\mathfrak{a}}) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \psi_{T,\mathfrak{a}} \left(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z) \right).$$

For a cusp $\mathfrak{b}$ of $\Gamma_0(N)$, $E_{\mathfrak{a}}(z|\psi_{T,\mathfrak{a}})$ has a Fourier expansion of the form (see [13, p. 60, eq. (3.17)])

$$\begin{aligned} &E_{\mathfrak{a}} \left(\sigma_{\mathfrak{b}} z | \psi_{T,\mathfrak{a}} \right) \\ &\quad = \delta_{\mathfrak{a}\mathfrak{b}} \psi_{T,\mathfrak{a}}(y) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c > 0} S_{\mathfrak{a}\mathfrak{b}}(0, n; c) \int_{-\infty}^{\infty} \psi_{T,\mathfrak{a}} \left(\frac{c^{-2} y}{t^2 + y^2} \right) e(-nt) dt \end{aligned}$$

where $\delta_{\mathfrak{a}\mathfrak{b}} = 1$ if $\mathfrak{a} = \mathfrak{b}$ and 0 if $\mathfrak{a} \neq \mathfrak{b}$ and $S_{\mathfrak{a}\mathfrak{b}}(m, n; c)$ is the Kloosterman sum

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; c) := \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \sigma_a^{-1} \Gamma_0(N) \sigma_{\mathfrak{b}} / \Gamma_\infty} e\left(m\frac{d}{c} + n\frac{a}{c}\right).$$

If $y > T$ then by definition of $\psi_{T,\mathfrak{a}}$ we have $E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z|\psi_{T,\mathfrak{a}}) = \delta_{\mathfrak{a}\mathfrak{b}} c_{\mathfrak{a}} y^\alpha$. Define

$$E_T(z) := \sum_{\mathfrak{a}} E_{\mathfrak{a}}(z|\psi_{T,\mathfrak{a}}).$$

Then for $y > T$,

$$E_T(\sigma_{\mathfrak{b}} z) = \sum_{\mathfrak{a}} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z|\psi_{T,\mathfrak{a}}) = \sum_{\mathfrak{a}} \delta_{\mathfrak{a}\mathfrak{b}} c_{\mathfrak{a}} y^\alpha = c_{\mathfrak{b}} y^\alpha.$$

Next, let $\psi : \mathbb{R}^+ \to [0, 1]$ be a C^∞ function such that

$$\psi(t) = \begin{cases} 0, & t < 1 \\ 1, & t > 2. \end{cases}$$

Define the incomplete Eisenstein series

$$E_{\mathfrak{a}}(z|\psi) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)).$$

Arguing with Fourier expansions as above, we find that for $y > 1$, $E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z|\psi) = \delta_{\mathfrak{a}\mathfrak{b}} \psi(y)$. Define

$$\phi(z) := \sum_{\mathfrak{a}} E_{\mathfrak{a}}(z|\psi).$$

Then for $y > 1$,

$$\phi(\sigma_{\mathfrak{b}} z) = \sum_{\mathfrak{a}} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z|\psi) = \sum_{\mathfrak{a}} \delta_{\mathfrak{a}\mathfrak{b}} \psi(y) = \psi(y).$$

Finally, define $\eta_T(z) := E_T(z)\phi(z/T)$. Then $\eta_T : \mathbb{H} \to \mathbb{C}$ is a C^∞, $\Gamma_0(N)$-invariant function which satisfies

$$\eta_T(\sigma_{\mathfrak{b}} z) = \begin{cases} 0, & 1 < y < T \\ c_{\mathfrak{b}} y^\alpha \psi(y/T), & T \leq y \leq 2T \\ c_{\mathfrak{b}} y^\alpha, & y > 2T. \end{cases}$$

□

Lemma 3.3 *For $T \gg \sqrt{D}$, the function $\eta_T(z)$ vanishes on the Heegner points $\Lambda_D(N)$.*

Proof For a Heegner point $z_Q \in \Lambda_D(N)$ write $z_Q = \sigma_{\mathfrak{a}} z$ for some $z \in \mathbb{H}$ with $0 < \mathrm{Re}(z) < 1$. Then if $\sigma_{\mathfrak{a}}^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$,

$$\mathrm{Im}(z) = \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z_Q) = \frac{\mathrm{Im}(z_Q)}{\left|\gamma z_Q + \delta\right|^2}.$$

Since $z_Q = \frac{-b+\sqrt{-D}}{2Na}$ we have

$$\mathrm{Im}(z_Q) = \frac{\sqrt{D}}{2Na} \leq \frac{\sqrt{D}}{2N}.$$

If $\gamma = 0$ then

$$\mathrm{Im}(z) \leq \frac{\sqrt{D}}{2N\delta^2}.$$

If $\gamma \neq 0$ then

$$\left|\gamma z_Q + \delta\right|^2 \geq \frac{\gamma^2}{4N^2}\frac{D}{a^2} > \frac{\gamma^2}{4N^2}$$

where for the last inequality we used Minkowski's bound, which says that every ideal class C of CL_K contains an ideal $\mathfrak{A}$ of norm $N_{K/\mathbb{Q}}(\mathfrak{A}) = a < (2/\pi)\sqrt{D}$. Thus

$$\mathrm{Im}(z) \leq \frac{2N}{\gamma^2}\sqrt{D}.$$

In either case, we see that for $T \gg \sqrt{D}$, the function $\eta_T(z)$ vanishes at z_Q. □

Define the regularized function

$$F_T(z) := F(z) - \eta_T(z).$$

In light of the preceding two lemmas, to prove Theorem 3.1 it suffices to prove the following proposition.

Proposition 3.4 *As $T \gg \sqrt{D}$ and $D \to \infty$ we have*

$$\sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q) F_T(z_Q) = O_N\left(||F||_1\, D^{\delta_1}\right) + O_{\epsilon,N}\left(\left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2\, D^{\frac{5}{12}+\delta_2+\epsilon}\right)$$
$$+ O_{\epsilon,N}\left(D^{\max\{\frac{5}{12}+\delta_2+\epsilon,\, \delta_1+\frac{\alpha-1}{2}\}}\right).$$

Proof Let $T \geq T_0 \gg 1$. Here T_0 is a fixed cutoff parameter which is independent of D. We introduce T_0 in order to decompose F_T into a sum of two functions so that we can isolate the contribution of η_T to the spectral decomposition. Consider the decomposition

$$F_T(z) = F_{T_0}(z) + \tilde{\eta}_T(z) \tag{3.1}$$

where

$$\tilde{\eta}_T(z) := \eta_{T_0}(z) - \eta_T(z).$$

We will first show that

$$\sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q) F_{T_0}(z_Q) = O_N\left(||F||_1\, D^{\delta_1}\right) - \sum_{Q \in \mathcal{Q}_{D,n}/\Gamma_0(N)} \chi(Q)\langle \eta_{T_0}, 1\rangle_2 + O_{\epsilon,N}\left(\left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2 D^{\frac{5}{12}+\delta_2+\epsilon}\right)$$

as $D \to \infty$. By definition of η_{T_0} and our assumption that F has cuspidal growth of power α we have

$$\Delta^a F_{T_0}(\sigma_{\mathfrak{a}} z) = O(e^{-cy}) \quad \text{as } y \to \infty$$

for each integer $a \geq 0$. The spectral decomposition of $L^2(Y_0(N))$ with respect to the hyperbolic Laplacian Δ then yields the expansion

$$F_{T_0}(z) = \langle F_{T_0}, 1\rangle_2 + \sum_{n=1}^{\infty} \langle F_{T_0}, u_n\rangle_2 u_n(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2 E_{\mathfrak{a}}\left(z, \frac{1}{2} + it\right) dt$$

which converges pointwise absolutely and uniformly on compact subsets of $Y_0(N)$ since $\Delta^a F_{T_0}$ is C^∞ with exponential decay in each cusp $\mathfrak{a}$ of $Y_0(N)$. Here $u_0(z) = 1$ is the constant eigenfunction for Δ corresponding to the eigenvalue $\lambda_0 = 0$, $\{u_n(z)\}_{n=1}^{\infty}$ is an orthonormal basis of Maass cusp forms satisfying $\Delta u_n = \lambda_n u_n$ for $n \in \mathbb{Z}^+$ where the eigenvalues $\lambda_n = 1/4 + t_n^2$ are ordered so that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and $E_{\mathfrak{a}}(z, s)$ is the real-analytic Eisenstein series

$$E_{\mathfrak{a}}(z, s) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \operatorname{Im}\left(\sigma_{\mathfrak{a}}^{-1} \gamma z\right)^s, \quad z \in \mathbb{H}, \quad \operatorname{Re}(s) > 1.$$

Summing the spectral expansion yields

$$\sum_{Q\in\mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)F_{T_0}(z_Q) = \sum_{Q\in\mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)\langle F_{T_0},1\rangle_2 + \sum_{n=1}^{\infty}\langle F_{T_0},u_n\rangle_2 W_n + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot,\frac{1}{2}+it\right)\right\rangle_2 W_{\mathfrak{a}}(t)dt, \tag{3.2}$$

where the twisted hyperbolic Weyl sums are defined by

$$W_n := \sum_{Q\in\mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)u_n(z_Q)$$

and

$$W_{\mathfrak{a}}(t) := \sum_{Q\in\mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)E_{\mathfrak{a}}\left(z_Q,\frac{1}{2}+it\right).$$

Using (2.1) and condition (1) we have

$$W_n = \sum_{\substack{r \mod 2N \\ r^2\equiv -D \mod 4N}} \chi(Q_r) \sum_{Q\in\mathcal{Q}_{D,N,r}/\Gamma_0(N)} u_n(z_Q).$$

Thus

$$|W_n| \le \max_{\substack{r \mod 2N \\ r^2\equiv -D \mod 4N}} |\chi(Q_r)| \sum_{\substack{r \mod 2N \\ r^2\equiv -D \mod 4N}} \left| \sum_{Q\in\mathcal{Q}_{D,N,r}/\Gamma_0(N)} u_n(z_Q)\right|. \tag{3.3}$$

Using the bijection (2.2) we have

$$\sum_{Q\in\mathcal{Q}_{D,N,r}/\Gamma_0(N)} u_n(z_Q) = \sum_{\sigma\in \mathrm{Gal}(H/K)} u_n(z_{Q_r}^{\sigma}).$$

As a consequence of work of Waldspurger [24] and Zhang [26], one obtains an identity of the form (see for example [10, eq. (17)], [11, eq. (30)], and [18, p. 99, eq. (5.11)])

$$\left|\sum_{\sigma\in\mathrm{Gal}(H/K)} u_n(z_{Q_r}^{\sigma})\right|^2 = C_{u_n}\sqrt{D}L\left(u_n\otimes\left(\frac{-D}{\cdot}\right),\frac{1}{2}\right) \tag{3.4}$$

where

$$C_{u_n} \ll_N (1+|t_n|)^{A_1}$$

for some fixed constant $A_1 > 0$, and $L(u_n \otimes (\frac{-D}{\cdot}), s)$ is the quadratic twist of the L-function of u_n by the Dirichlet character $(\frac{-D}{\cdot})$. Conrey and Iwaniec [3] established the following subconvexity bound, valid for any $\epsilon > 0$ and some fixed constant $A_2 > 0$,

$$L\left(u_n \otimes \left(\frac{-D}{\cdot}\right), \frac{1}{2}\right) \ll_{\epsilon,N} (1+|t_n|)^{A_2} D^{\frac{1}{3}+\epsilon}. \tag{3.5}$$

Combining (3.3), (3.4), (3.5) and condition (3) yields the estimate

$$W_n \ll_{\epsilon,N} (1+|t_n|)^{A_3} D^{\frac{5}{12}+\delta_2+\epsilon}. \tag{3.6}$$

For $W_{\mathfrak{a}}(t)$ we proceed as above to obtain the estimate

$$|W_{\mathfrak{a}}(t)| \leq \max_{\substack{r \mod 2N \\ r^2 \equiv -D \mod 4N}} |\chi(Q_r)| \sum_{\substack{r \mod 2N \\ r^2 \equiv -D \mod 4N}} \left| \sum_{Q \in \mathcal{Q}_{D,N,r}/\Gamma_0(N)} E_{\mathfrak{a}}\left(z_Q, \frac{1}{2}+it\right)\right|. \tag{3.7}$$

Using the bijection (2.2) we have

$$\sum_{Q \in \mathcal{Q}_{D,N,r}/\Gamma_0(N)} E_{\mathfrak{a}}\left(z_Q, \frac{1}{2}+it\right) = \sum_{\sigma \in \mathrm{Gal}(H/K)} E_{\mathfrak{a}}\left(z_{Q_r}^{\sigma}, \frac{1}{2}+it\right). \tag{3.8}$$

Following the argument in [11, Section 6], one can reduce the estimate of (3.8) to an analogous estimate for

$$\sum_{\sigma \in \mathrm{Gal}(H/K)} E\left(z^{\sigma}, \frac{1}{2}+it\right)$$

where $E(z,s)$ is the full-level Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ and $\Lambda_D(1) = \{z^{\sigma} : \sigma \in \mathrm{Gal}(H/K)\}$ is the set of Heegner points of discriminant $-D$ on the modular curve $X_0(1)$. By a classical formula of Dirichlet one has an identity of the form (see [9, p. 248])

$$\left|\sum_{\sigma \in \mathrm{Gal}(H/K)} E\left(z^{\sigma}, \frac{1}{2}+it\right)\right|^2 = \frac{\sqrt{D}}{2}\left|L\left(\left(\frac{-D}{\cdot}\right), \frac{1}{2}+it\right)\right|^2 \tag{3.9}$$

where $L((\frac{-D}{\cdot}), s)$ is the Dirichlet L-function of $(\frac{-D}{\cdot})$. By Conrey and Iwaniec [3], one has the following subconvexity bound, valid for any $\epsilon > 0$ and some fixed constant $A_4 > 0$,

$$\left| L\left(\left(\frac{-D}{\cdot}\right), \frac{1}{2} + it\right) \right|^2 \ll_\epsilon (1 + |t|)^{A_4} D^{\frac{1}{3}+\epsilon}. \tag{3.10}$$

Combining (3.7), (3.9), (3.10) and condition (3) yields the estimate

$$W_{\mathfrak{a}}(t) \ll_{\epsilon,N} (1 + |t|)^{A_5} D^{\frac{5}{12}+\delta_2+\epsilon}. \tag{3.11}$$

Using Stokes' theorem, the Cauchy–Schwartz inequality, and the Parseval formula (see for example [13, Lemma 1.18], [2, Proposition 1.4], and [15, eq. (15.17)]), one can establish the following estimates for some sufficiently large integer $a_0 > 0$,

$$\sum_{n=1}^{\infty} \langle F_{T_0}, u_n \rangle_2 (1 + |t_n|)^{A_3} \ll \left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2 \tag{3.12}$$

and

$$\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2 (1 + |t|)^{A_5} dt \ll \left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2. \tag{3.13}$$

Combine (3.6), (3.11), (3.12) and (3.13) to obtain the estimates

$$\sum_{n=1}^{\infty} \langle F_{T_0}, u_n \rangle_2 W_n \ll_{\epsilon,N} \left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2 D^{\frac{5}{12}+\delta_2+\epsilon}$$

and

$$\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2 W_{\mathfrak{a}}(t) dt \ll_{\epsilon,N} \left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2 D^{\frac{5}{12}+\delta_2+\epsilon}. \tag{3.14}$$

Then by combining these estimates with (3.2) and condition (2), we find that for all $\epsilon > 0$,

$$\sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q) F_{T_0}(z_Q) = O_N(||F||_1 D^{\delta_1}) - \sum_{Q \in \mathcal{Q}_{D,n}/\Gamma_0(N)} \chi(Q) \langle \eta_{T_0}, 1 \rangle_2 + O_{\epsilon,N}\left(\left|\left|\Delta^{a_0} F_{T_0}\right|\right|_2^2 D^{\frac{5}{12}+\delta_2+\epsilon}\right)$$

as $D \to \infty$.

To finish the proof of Proposition 3.4 we will show that for all $\epsilon > 0$,

$$\sum_{Q\in\mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)\tilde{\eta}_T(z_Q) = \sum_{Q\in\mathcal{Q}_{D,n}/\Gamma_0(N)} \chi(Q)\langle \eta_{T_0}, 1\rangle_2 + O_{\epsilon,N}\left(D^{\max\{\frac{5}{12}+\delta_2+\epsilon,\delta_1+\frac{\alpha-1}{2}\}}\right)$$

as $T \gg \sqrt{D}$ and $D \to \infty$.

By definition of $\tilde{\eta}_T$ we have

$$\tilde{\eta}_T(\sigma_{\mathfrak{b}} z) = \begin{cases} 0, & 1 < y < T_0 \\ c_{\mathfrak{b}} y^{\alpha}(\psi(y/T_0) - \psi(y/T)), & T_0 \le y \le 2T \\ 0, & y > 2T. \end{cases}$$

The spectral decomposition of $L^2(Y_0(N))$ with respect to Δ yields the expansion

$$\tilde{\eta}_T(z) = \langle \tilde{\eta}_T, 1\rangle_2 + \sum_{n=1}^{\infty} \langle \tilde{\eta}_T, u_n\rangle_2 u_n(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 E_{\mathfrak{a}}\left(z, \frac{1}{2}+it\right)$$

which converges pointwise absolutely and uniformly on compact subsets of $Y_0(N)$ since $\tilde{\eta}_T$ is in $C_c^{\infty}(Y_0(N))$.

A straightforward calculation shows that $\tilde{\eta}_T(z)$ is orthogonal to Maass cusp forms for $\Gamma_0(N)$. Hence the spectral expansion becomes

$$\tilde{\eta}_T(z) = \langle \tilde{\eta}_T, 1\rangle_2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 E_{\mathfrak{a}}\left(z, \frac{1}{2}+it\right) dt.$$

Summing the spectral expansion yields

$$\sum_{Q\in\mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)\tilde{\eta}_T(z_Q) = \sum_{Q\in\mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)\langle \tilde{\eta}_T, 1\rangle_2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 W_{\mathfrak{a}}(t) dt. \tag{3.15}$$

In Lemma 3.5 we establish the following estimate, valid for each $B > 0$,

$$\int_{-\infty}^{\infty} \left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 (1+|t|)^B dt \ll \log(T). \tag{3.16}$$

Take $\sqrt{D} \ll T \ll D$ and combine (3.16) with an argument similar to that in the proof of (3.14) to obtain the estimate

$$\sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 W_{\mathfrak{a}}(t) dt \ll_{\epsilon,N} \log(D) D^{\frac{5}{12}+\delta_2+\epsilon}.$$

Moreover, for $T \gg \sqrt{D}$ a straightforward estimate yields

$$\langle \eta_T, 1\rangle_2 = O_N\left(D^{\frac{\alpha-1}{2}}\right).$$

Finally, by combining these estimates with (3.15) and condition (2) we conclude that for $\sqrt{D} \ll T \ll D$ and all $\epsilon > 0$,

$$\begin{aligned} &\sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)\tilde{\eta}_T(z_Q) \\ &\quad = \sum_{Q \in \mathcal{Q}_{D,N}/\Gamma_0(N)} \chi(Q)\langle \eta_{T_0}, 1\rangle_2 + O_{\epsilon,N}\left(D^{\max\{\frac{5}{12}+\delta_2+\epsilon, \delta_1+\frac{\alpha-1}{2}\}}\right) \end{aligned}$$

as $D \to \infty$. □

Lemma 3.5 *For each $B > 0$ we have*

$$\int_{-\infty}^{\infty} \left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 (1+|t|)^B dt \ll \log(T).$$

Proof First suppose that $|t| \geq 1$. The Fourier expansion of the Eisenstein series $E_{\mathfrak{a}}(z,s)$ in the cusp $\mathfrak{b}$ is of the form (see [13, Theorem 3.4])

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{ab}} y^s + \phi_{\mathfrak{ab}}(s) y^{1-s} + 2\sqrt{y} \sum_{n \neq 0} \sqrt{|n|} \phi_{\mathfrak{ab}}(n,s) K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$$

where $\phi_{\mathfrak{ab}}(s)$ is the $\mathfrak{ab}$th entry in the scattering matrix for $\Gamma_0(N)$ and K_ν is the order ν Bessel function. Because $\tilde{\eta}_T(z)$ depends only on the 'height' of z in the cusps, substituting the Fourier expansion into the spectral coefficient and expanding yields

$$\left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 = \sum_{\mathfrak{b}} \left\{ \delta_{\mathfrak{a}\mathfrak{b}} \int_{T_0}^{2T} \tilde{\eta}_T(\sigma_{\mathfrak{b}} z) y^{\frac{1}{2}+it} \frac{dy}{y^2} + \phi_{\mathfrak{a}\mathfrak{b}}(1/2+it) \int_{T_0}^{2T} \tilde{\eta}_T(\sigma_{\mathfrak{b}} z) y^{\frac{1}{2}-it} \frac{dy}{y^2} \right\}. \tag{3.17}$$

Define

$$f_{T,\mathfrak{b}}(t) := \int_{T_0}^{2T} \tilde{\eta}_T(\sigma_{\mathfrak{b}} z) y^{\frac{1}{2}+it} \frac{dy}{y^2} = \int_{T_0}^{2T} c_{\mathfrak{b}} g_{T,T_0}(y) y^{\frac{1}{2}+it+(\alpha-1)} \frac{dy}{y}$$

where

$$g_{T,T_0}(y) := \psi(y/T_0) - \psi(y/T).$$

For all integers $k \geq 1$,

$$\frac{d^{k-1}}{dy^{k-1}} g_{T,T_0}(y)$$

is supported in $(T_0, 2T)$, and

$$\frac{d^k}{dy^k} \psi(y)$$

is supported in $(1, 2)$. Then integrating by parts k-times yields

$$(-1)^k \prod_{j=0}^{k-1} \left(it + \frac{1}{2} + (\alpha-1) + j\right) f_{T,\mathfrak{b}}(t) = c_{\mathfrak{b}} \left(T_0^{it+\frac{1}{2}+(\alpha-1)} - T^{it+\frac{1}{2}+(\alpha-1)}\right) \int_1^2 \psi^{(k)}(y) y^{it+\frac{1}{2}+(\alpha-1)+k} \frac{dy}{y}.$$

Since $|t| \geq 1$ we have the estimate

$$\left| (-1)^k \prod_{j=0}^{k-1} (it + 1/2 + (\alpha-1) + j) \right| \leq \prod_{j=0}^{k-1} (1/2 - (\alpha-1) + j + 1) \, |t|^k .$$

Furthermore, we have the estimate

$$\left|T_0^{it+1/2+(\alpha-1)} - T^{it+1/2+(\alpha-1)}\right| \le 2T_0^{\alpha-\frac{1}{2}}.$$

By combining the preceding facts we find that

$$\left|f_{T,\mathfrak{b}}(t)\right| \le |c_{\mathfrak{b}}| \frac{2}{T_0^{\frac{1}{2}-\alpha}} \frac{\max_{1\le y\le 2}\left|\psi^{(k)}(y)\right|}{\prod_{j=0}^{k-1}(1/2-(\alpha-1)+j+1)} \frac{\left|2^{k+\alpha-\frac{1}{2}}-1\right|}{\left|k+\alpha-\frac{1}{2}\right|} \cdot |t|^{-k}. \tag{3.18}$$

Finally, because $B > 0$ is fixed and $k \ge 1$ is arbitrary, we use (3.17), the inequality

$$|\phi_{\mathfrak{ab}}(1/2 - it)| \le 1$$

(see [13, Theorem 6.6]) and (3.18) to conclude that

$$\begin{aligned}
&\int\limits_{|t|\ge 1} \langle \tilde{\eta}_T, E_{\mathfrak{a}}(\cdot, \frac{1}{2}+it)\rangle_2 (1+|t|)^B dt \\
&= \sum_{\mathfrak{b}} \int\limits_{|t|\ge 1} (\delta_{\mathfrak{ab}} + \phi_{\mathfrak{ab}}(1/2-it)) f_{T,\mathfrak{b}}(t)(1+|t|)^B dt \\
&\ll \sum_{\mathfrak{b}} |c_{\mathfrak{b}}| \int\limits_{|t|\ge 1} \left|f_{T,\mathfrak{b}}(t)\right| (1+|t|)^B dt \\
&\ll T_0^{\alpha-\frac{1}{2}}.
\end{aligned}$$

Next suppose that $|t| < 1$. We have the estimate

$$\left|f_{T,\mathfrak{b}}(t)\right| \le 2\,|c_{\mathfrak{b}}| \sup_{y\in\mathbb{R}^+} |\psi(y)| \int\limits_{T_0}^{2T} y^{-1} dy \ll \log(T).$$

Because $(1+|t|)^B \ll_B 1$ for $|t| < 1$, we proceed as above to conclude that

$$\int\limits_{|t|<1} \left\langle \tilde{\eta}_T, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 (1+|t|)^B dt \ll \log(T).$$

□

4 Poincaré series

Recall that the Poincaré series $P(z)$ is defined by

$$P(z) = 4\pi \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(6)} \mathrm{Im}(\gamma z)^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi \mathrm{Im}(\gamma z)) e(-\mathrm{Re}(\gamma z)).$$

Because

$$y^{\frac{1}{2}} I_{\frac{3}{2}}(y) = O(y^2) \quad \text{as } y \to 0,$$

the series $P(z)$ is absolutely convergent on $\mathbb{H}$ (see [19, p. 134]).

In the following proposition we compute the Fourier expansion of $P(z)$ in each cusp of $\Gamma_0(6)$.

Proposition 4.1 *The Fourier expansion of $P(z)$ in the cusp $\mathfrak{b}$ of $\Gamma_0(6)$ is given by*

$$P(\sigma_{\mathfrak{b}} z) = \delta_{\infty,\mathfrak{b}} 4\pi y^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi y) e(-x) + c_{\mathfrak{b}} \frac{8\pi^3}{3} y^{-1} + 8\pi \sum_{n \neq 0} b_{\mathfrak{b}}(-1,n) y^{\frac{1}{2}} K_{\frac{3}{2}}(2\pi |n| y) e(nx)$$

where

$$c_{\mathfrak{b}} = \sum_{c=1}^{\infty} \frac{S_{\infty,\mathfrak{b}}(-1,0;c)}{c^4},$$

and

$$b_{\mathfrak{b}}(-1,n) = \begin{cases} \sum_{c=1}^{\infty} \frac{S_{\infty,\mathfrak{b}}(-1,n;c)}{c} I_3\left(\frac{4\pi\sqrt{n}}{c}\right), & n > 0 \\ \sum_{c=1}^{\infty} \frac{S_{\infty,\mathfrak{b}}(-1,n;c)}{c} J_3\left(\frac{4\pi\sqrt{|n|}}{c}\right), & n < 0, \end{cases}$$

where I_ν, J_ν are the order ν Bessel functions.

Proof The Fourier expansion of $P(z)$ in the cusp $\mathfrak{b}$ is of the form

$$P(\sigma_{\mathfrak{b}} z) = \delta_{\infty,\mathfrak{b}} 4\pi y^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi y) e(-x) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c>0} S_{\infty,\mathfrak{b}}(-1,n;c) A(n,c,y)$$

where

$$A(n,c,y) := \int_{-\infty}^{\infty} \psi\left(\frac{c^{-2}y}{t^2+y^2}\right) e\left(\frac{c^{-2}t}{t^2+y^2} - nt\right) dt$$

and

$$\psi(y) := 4\pi y^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi y).$$

The result follows from the following integral evaluations, which may be derived using results in [19, Sect. 2]:

$$A(0,c,y) = \frac{8\pi^3}{3c^4} y^{-1},$$

and

$$A(n,c,y) = \begin{cases} \frac{8\pi}{c} y^{\frac{1}{2}} K_{\frac{3}{2}}(2\pi n y) I_3\left(\frac{4\pi\sqrt{n}}{c}\right), & n > 0 \\ \frac{8\pi}{c} y^{\frac{1}{2}} K_{\frac{3}{2}}(2\pi |n| y) J_3\left(\frac{4\pi\sqrt{|n|}}{c}\right), & n < 0. \end{cases}$$

□

For $\varepsilon > 0$ [1] let $\psi_\varepsilon : \mathbb{R}^+ \to [0,2]$ be a suitable C^∞ function such that

$$\psi_\varepsilon(t) = \begin{cases} 0, & t \leq 1 \\ 2\left(1 - \frac{1}{2\pi t}\right), & t > 1+\varepsilon. \end{cases}$$

Define the Poincaré series

$$\mathscr{P}_\varepsilon(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(6)} \psi_\varepsilon(\mathrm{Im}(\gamma z)) e(-\gamma z).$$

Proposition 4.2 *For $y > 1+\varepsilon$ we have*

$$P(\sigma_{\mathfrak{b}} z) - \mathscr{P}_\varepsilon(\sigma_{\mathfrak{b}} z) = c_{\mathfrak{b}} \frac{8\pi^3}{3y} + \delta_{\infty,\mathfrak{b}} \cdot e^{-2\pi y} \cdot e(-x)\left(2 + \frac{1}{\pi y}\right)$$
$$+4\pi \sum_{n \neq 0} e^{-2\pi |n| y} \cdot b_{\mathfrak{b}}(-1,n) |n|^{-1/2} e(nx)\left(1 + \frac{1}{2\pi |n| y}\right).$$

Proof The Fourier expansion of $\mathscr{P}_\varepsilon(z)$ in the cusp $\mathfrak{b}$ is of the form

$$\mathscr{P}_\varepsilon(\sigma_{\mathfrak{b}} z) = \delta_{\infty,\mathfrak{b}} \psi_\varepsilon(y) e(-z) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c>0} S_{\infty,\mathfrak{b}}(-1,n;c)$$
$$\times \int_{-\infty}^{\infty} \psi_\varepsilon\left(\frac{c^{-2} y}{t^2 + y^2}\right) e\left(\frac{c^{-2}}{t+iy} - nt\right) dt.$$

[1] We emphasize that ε is to be distinguished from ϵ used previously.

Note that by definition of ψ_ε, if $y \geq 1$ then $\mathscr{P}_\varepsilon(\sigma_{\mathfrak{b}} z) = \delta_{\infty,\mathfrak{b}} \psi_\varepsilon(y) e(-z)$ (where $\mathscr{P}_\varepsilon(\sigma_{\mathfrak{b}} z) = 0$ if $\mathfrak{b} \neq \infty$), and if $y > 1 + \varepsilon$ then

$$\mathscr{P}_\varepsilon(\sigma_{\mathfrak{b}} z) = \delta_{\infty,\mathfrak{b}} \left(2 - \frac{1}{\pi y}\right) e(-z). \tag{4.1}$$

The Bessel functions $I_{n+\frac{1}{2}}$ and $K_{n+\frac{1}{2}}$ of half-integral order have expansions given as follows (see [25, Section 3.71]): for non-negative integers n we have

$$\begin{aligned} & y^{1/2} I_{n+\frac{1}{2}}(y) \\ & \quad = \frac{1}{\sqrt{2\pi}} \left(e^y \sum_{j=0}^{n} \frac{(-1)^j (n+j)!}{j!(n-j)!(2y)^j} + (-1)^{n+1} e^{-y} \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!(2y)^j} \right) \end{aligned} \tag{4.2}$$

and

$$y^{1/2} K_{n+\frac{1}{2}}(y) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-y} \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!(2y)^j}. \tag{4.3}$$

We substitute (4.2) and (4.3) with $n = 1$ into the Fourier expansion for $P(\sigma_{\mathfrak{b}} z)$ given in Proposition 4.1 to obtain

$$\begin{aligned} P(\sigma_{\mathfrak{b}} z) = c_{\mathfrak{b}} \frac{8\pi^3}{3y} + \delta_{\infty,\mathfrak{b}} \left(2 - \frac{1}{\pi y}\right) e(-z) + \delta_{\infty,b} \cdot e^{-2\pi y} \cdot e(-x) \left(2 + \frac{1}{\pi y}\right) \\ + 4\pi \sum_{n \neq 0} e^{-2\pi |n| y} \cdot b_{\mathfrak{b}}(-1, n) |n|^{-1/2} e(nx) \left(1 + \frac{1}{2\pi |n| y}\right). \end{aligned} \tag{4.4}$$

Combining (4.1) and (4.4), we find that for $y > 1 + \varepsilon$,

$$\begin{aligned} P(\sigma_{\mathfrak{b}} z) - \mathscr{P}_\varepsilon(\sigma_{\mathfrak{b}} z) = c_{\mathfrak{b}} \frac{8\pi^3}{3y} + \delta_{\infty,b} \cdot e^{-2\pi y} \cdot e(-x) \left(2 + \frac{1}{\pi y}\right) \\ + 4\pi \sum_{n \neq 0} e^{-2\pi |n| y} \cdot b_{\mathfrak{b}}(-1, n) |n|^{-1/2} e(nx) \left(1 + \frac{1}{2\pi |n| y}\right). \end{aligned}$$

□

5 Proof of Theorem 1.1

Theorem 1.1 follows immediately from (1.2) and the following theorem.

Theorem 5.1 *Let n be a positive integer such that $D_n = 24n - 1$ is square-free. Then there exists an effective constant $c > 0$ such that for all $\epsilon > 0$ and $0 < b < 1/12c$,*

$$\sum_{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q) P(z_Q) - 2 \sum_{\substack{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6) \\ \mathrm{Im}(z_Q) > 1 + D_n^{-b}}} \chi_{12}(Q) \left(1 - \frac{1}{2\pi \mathrm{Im}(z_Q)}\right) e(-z_Q)$$
$$= O_\epsilon\left(D_n^{\frac{5}{12} + bc + \epsilon}\right) + O_\epsilon\left(D_n^{\frac{1}{2} - b + \epsilon}\right) \tag{5.1}$$

as $n \to \infty$.

Proof Recall that

$$\chi_{12} : \mathcal{Q}_{D_n,6} \to \{\pm 1\}$$

is defined by $\chi_{12}(Q) := \left(\frac{12}{b}\right)$ where $Q = [a, b, c]$ and $\left(\frac{12}{\cdot}\right)$ is the Legendre symbol. Fix a solution r mod 12 of $r^2 \equiv -D_n$ mod 24. Then it is clear that the induced map

$$\chi_{12} : \mathcal{Q}_{D_n,6}/\Gamma_0(6) \to \{\pm 1\}$$

is constant on $\mathcal{Q}_{D_n,6,r}/\Gamma_0(6)$. Furthermore, because $\chi_{12}(Q_r) = 1$ if $r \equiv 1, 11$ mod 12 and $\chi(Q_r) = -1$ if $r \equiv 5, 7$ mod 12 where $[Q_r]$ is any class in $\mathcal{Q}_{D_n,6,r}/\Gamma_0(6)$, it follows from (2.1) that

$$\sum_{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q) = \sum_{\substack{r \mod 12 \\ r^2 \equiv -D_n \mod 24}} \sum_{Q \in \mathcal{Q}_{D_n,6,r}/\Gamma_0(6)} \chi_{12}(Q_r) = 0. \tag{5.2}$$

Let $\varepsilon < 1/4$ and define the function $F_\varepsilon(z) := P(z) - \mathscr{P}_\varepsilon(z)$ where $\mathscr{P}_\varepsilon(z)$ is the Poincaré series defined in Sect. 4. Then $F_\varepsilon(z)$ is C^∞ and $\Gamma_0(6)$-invariant, and by Proposition 4.2 we see that $F_\varepsilon(z)$ has cuspidal growth of power $\alpha = -1$. Here we emphasize that this growth is *uniform* in ε for $y > 5/4$, hence the same choice of cutoff parameter $T_0 = 2$ and corresponding cutoff function $\eta_2(z)$ given by[2]

$$\eta_2(\sigma_{\mathfrak{b}} z) = \begin{cases} 0, & 1 < y < 2 \\ c_{\mathfrak{b}} \frac{8\pi^3}{3} y^{-1} \psi(y/2), & 2 \le y \le 4 \\ c_{\mathfrak{b}} \frac{8\pi^3}{3} y^{-1}, & y > 4 \end{cases}$$

will work for each function $F_\varepsilon(z)$. We now substitute $F_\varepsilon(z)$ into Theorem 3.1 with the choices $D = D_n$, $N = 6$, $\chi = \chi_{12}$, $\delta_1 = \delta_2 = 0$, $\alpha = -1$, and $T_0 = 2$, and find that

[2] The function ψ in the definition of $\eta_2(z)$ is not related to the function ψ_ε in the definition of $\mathscr{P}_\varepsilon(z)$.

for all $\epsilon > 0$,

$$\sum_{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q)P(z_Q) - \sum_{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q)\mathscr{P}_\varepsilon(z_Q)$$
$$= O_\epsilon\left(\left|\left|\Delta^{a_0}F_{\varepsilon,2}\right|\right|_2^2 D_n^{\frac{5}{12}+\epsilon}\right) + O_\epsilon\left(D_n^{\frac{5}{12}+\epsilon}\right) \tag{5.3}$$

as $n \to \infty$. There is no error term of the form $O_N(||F_\varepsilon||_1)$ in (5.3) because the condition (5.2) implies that the first term on the right hand side of (3.2) is zero for each ε.

We now show how to deduce (5.1) from (5.3) and thus complete the proof. Because $\psi_\varepsilon(y) = 0$ for $y \le 1$, it follows from the definitions of $\mathscr{P}_\varepsilon(z)$ and ψ_ε that for n sufficiently large,

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q)\mathscr{P}_\varepsilon(z_Q) &= \sum_{\mathrm{Im}(z_Q)>1} \chi_{12}(Q)\psi_\varepsilon(\mathrm{Im}(z_Q))e(-z_Q)\\ &= \sum_{1<\mathrm{Im}(z_Q)\le 1+\varepsilon} \chi_{12}(Q)\psi_\varepsilon(\mathrm{Im}(z_Q))e(-z_Q)\\ &\quad +2\sum_{\mathrm{Im}(z_Q)>1+\varepsilon} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right)e(-z_Q).\end{aligned}$$

For notational convenience define

$$R_\varepsilon(D_n) := \sum_{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)} \chi_{12}(Q)P(z_Q) - 2\sum_{\mathrm{Im}(z_Q)>1+\varepsilon} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right)e(-z_Q).$$

Then we can write (5.3) in the equivalent form

$$\begin{aligned}R_\varepsilon(D_n) &= O_\epsilon\left(\left|\left|\Delta^{a_0}F_{\varepsilon,2}\right|\right|_2^2 D_n^{\frac{5}{12}+\epsilon}\right) + O_\epsilon\left(D_n^{\frac{5}{12}+\epsilon}\right)\\ &\quad + \sum_{1<\mathrm{Im}(z_Q)\le 1+\varepsilon} \chi_{12}(Q)\psi_\varepsilon(\mathrm{Im}(z_Q))e(-z_Q).\end{aligned}$$

We have the estimate

$$\left|\sum_{1<\mathrm{Im}(z_Q)\le 1+\varepsilon} \chi_{12}(Q)\psi_\varepsilon(\mathrm{Im}(z_Q))e(-z_Q)\right| \le 2e^{2\pi(1+\varepsilon)}\#\Lambda_\varepsilon(D_n) \tag{5.4}$$

where

$$\Lambda_\varepsilon(D_n) := \left\{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6) :\ 1 < \mathrm{Im}(z_Q) \le 1+\varepsilon\right\}.$$

By a refinement of the argument in [6, p. 248–249], one can show that for $\varepsilon < 1/4$,

$$\#\Lambda_\varepsilon(D_n) \leq \frac{16\varepsilon h(-D_n)}{\mathrm{vol}(X_0(6))} + O_\epsilon\left(\varepsilon^{-a_1} D_n^{\frac{5}{12}+\epsilon}\right) \tag{5.5}$$

for some sufficiently large integer $a_1 > 0$ (see [7]). Then by combining the estimates (5.4) and (5.5), we obtain

$$\sum_{1<\mathrm{Im}(z_Q)\leq 1+\varepsilon} \chi_{12}(Q)\psi_\varepsilon(\mathrm{Im}(z_Q))e(-z_Q) = O(\varepsilon h(-D_n)) + O_\epsilon\left(\varepsilon^{-a_1} D_n^{\frac{5}{12}+\epsilon}\right).$$

Moreover, one can show that for $\varepsilon < 1/4$,

$$\left|\left|\Delta^{a_0} F_{\varepsilon,2}\right|\right|_2^2 = O_{a_0}(\varepsilon^{-a_2})$$

for some sufficiently large integer $a_2 > 0$ depending on a_0 (see [7]).

It follows that

$$R_\varepsilon(D_n) = O_\epsilon\left(\varepsilon^{-a_2} D_n^{\frac{5}{12}+\epsilon}\right) + O_\epsilon\left(D_n^{\frac{5}{12}+\epsilon}\right) + O(\varepsilon h(-D_n)) + O_\epsilon\left(\varepsilon^{-a_1} D_n^{\frac{5}{12}+\epsilon}\right).$$

Let $c = \max\{a_1, a_2\}$ and choose $b > 0$ such that $b < 1/12c$. Because ε is independent of D_n in the preceding estimates, we can set $\varepsilon = D_n^{-b}$. Then using the (effective) upper bound (see [14, eq. (11.12)])

$$h(-D_n) \ll_\epsilon D_n^{\frac{1}{2}+\epsilon}$$

we obtain

$$R_\varepsilon(D_n) = O_\epsilon\left(D_n^{\frac{5}{12}+bc+\epsilon}\right) + O_\epsilon\left(D_n^{\frac{1}{2}-b+\epsilon}\right)$$

as $n \to \infty$. □

6 Proof of Corollary 1.2

We begin with the following proposition.

Proposition 6.1 *Let n be a positive integer such that $D_n = 24n - 1$ is square-free. Then*

$$\frac{1}{(24n-1)} \sum_{\substack{z_Q \in \Lambda_{D_n}(6) \\ \mathrm{Im}(z_Q)>1}} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right) e(-z_Q)$$
$$= \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\left\lfloor \frac{1}{\sqrt{6}}\sqrt{n-\frac{1}{24}} \right\rfloor} k^{1/2} A_k(n) \frac{d}{dn} \frac{\exp\left(\frac{\pi\lambda_n}{k}\sqrt{\frac{2}{3}}\right)}{\lambda_n}.$$

Proof The bijective correspondence between $\mathcal{Q}_{D_n,6}/\Gamma_0(6)$ and $\Lambda_{D_n}(6)$ associates to each class $[Q]$ the Heegner point

$$z_Q = \frac{-x+\sqrt{-D_n}}{12c}$$

which is the unique root in $\mathbb{H}$ of the quadratic form $6cX^2 + xX + a$ with $x^2 - 24ca = -D_n$. Those z_Q with $\mathrm{Im}(z_Q) > 1$ can be parameterized as integer pairs $(6c, x)$ with $0 < 6c < \sqrt{D_n}/2$ and $x^2 \equiv -D_n \mod 24c$, where we take only 1/2 the number of such pairs since $\mathcal{Q}_{D_n,6}$ contains only positive definite forms. Using (2.1) and letting $24c \mapsto c$, we find that

$$\frac{1}{D_n} \sum_{\substack{Q \in \mathcal{Q}_{D_n,6}/\Gamma_0(6) \\ \mathrm{Im}(z_Q)>1}} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right) e(-z_Q)$$
$$= \frac{1}{D_n} \sum_{\substack{r \pmod{12} \\ r^2 \equiv -D_n \pmod{24}}} \chi_{12}(r)$$
$$\cdot \frac{1}{2} \sum_{\substack{0<c<2\sqrt{D_n} \\ c\equiv 0 \pmod{24}}} \sum_{\substack{x^2 \equiv -D_n \pmod{c} \\ x \equiv r \pmod{12}}} \left(1-\frac{c}{4\pi\sqrt{D_n}}\right) e(2x/c) e^{4\pi\sqrt{D_n}/c}$$
$$= \frac{1}{2D_n} \sum_{\substack{0<c<2\sqrt{D_n} \\ c\equiv 0 \pmod{24}}} \sum_{x^2 \equiv -D_n \pmod{c}} \chi_{12}(x) e(2x/c) \left(1-\frac{c}{4\pi\sqrt{D_n}}\right) e^{4\pi\sqrt{D_n}/c}, \tag{6.1}$$

where for the last equality we used the following lemma whose proof is left to the reader.

Lemma 6.2 *For $c \equiv 0 \pmod{24}$,*

$$\sum_{\substack{r \pmod{12} \\ r^2 \equiv -D_n \pmod{24}}} \chi_{12}(r) \sum_{\substack{x^2 \equiv -D_n \pmod{c} \\ x \equiv r \pmod{12}}} e(2x/c) = \sum_{x^2 \equiv -D_n \pmod{c}} \chi_{12}(x) e(2x/c).$$

To complete the proof, we observe that (6.1) can be expressed as

$$\frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{\left\lfloor\frac{\sqrt{24n-1}}{12}\right\rfloor} A_k(n)k^{\frac{1}{2}}\left(\frac{e^{\pi\sqrt{\left(n-\frac{1}{24}\right)\frac{2}{3}}/k}\pi}{k\sqrt{6}\left(n-\frac{1}{24}\right)} - \frac{e^{\pi\sqrt{\left(n-\frac{1}{24}\right)\frac{2}{3}}/k}}{2\left(n-\frac{1}{24}\right)^{\frac{3}{2}}}\right)$$
$$= \frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{\left\lfloor\frac{1}{\sqrt{6}}\sqrt{n-\frac{1}{24}}\right\rfloor} A_k(n)k^{\frac{1}{2}}\frac{d}{dn}\frac{\exp\left(\frac{\pi\lambda_n}{k}\sqrt{\frac{2}{3}}\right)}{\lambda_n}.$$

□

Proof of Corollary 1.2 First we write

$$\sum_{\substack{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)\\ \mathrm{Im}(z_Q)>1+D_n^{-b}}} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right)e(-z_Q)$$
$$= \sum_{\substack{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)\\ \mathrm{Im}(z_Q)>1}} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right)e(-z_Q)$$
$$- \sum_{\substack{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)\\ 1<\mathrm{Im}(z_Q)\le 1+D_n^{-b}}} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right)e(-z_Q).$$

We have the estimate

$$\left|\sum_{\substack{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)\\ 1<\mathrm{Im}(z_Q)\le 1+D_n^{-b}}} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right)e(-z_Q)\right| \le 2e^{2\pi(1+D_n^{-b})}\#\Lambda_{D_n^{-b}}(D_n).$$

Then by an analysis similar to that in the proof of Theorem 5.1, we find that

$$\frac{1}{(24n-1)}\sum_{\substack{Q\in\mathcal{Q}_{D_n,6}/\Gamma_0(6)\\ 1<\mathrm{Im}(z_Q)\le 1+D_n^{-b}}} \chi_{12}(Q)\left(1-\frac{1}{2\pi\,\mathrm{Im}(z_Q)}\right)e(-z_Q)$$
$$= O_\epsilon\left(D_n^{-(\frac{1}{2}+b)+\epsilon}\right) + O_\epsilon\left(D_n^{-(\frac{7}{12}-a_1 b)+\epsilon}\right).$$

Hence Theorem 1.1 implies that

$$p(n) = \frac{1}{(24n-1)} \sum_{\substack{z_Q \in \Lambda_{D_n}(6) \\ \mathrm{Im}(z_Q)>1}} \chi_{12}(Q) \left(1 - \frac{1}{2\pi \mathrm{Im}(z_Q)}\right) e(-z_Q) \\ + O_\epsilon\left(n^{-(\frac{7}{12}-bc)+\epsilon}\right) + O_\epsilon\left(n^{-(\frac{1}{2}+b)+\epsilon}\right) \tag{6.2}$$

as $n \to \infty$.

To complete the proof, we show that the main term in (6.2) differs from the main term in Rademacher's asymptotic formula (1.1) when $N = \lfloor \frac{1}{\sqrt{6}} \sqrt{n} \rfloor$ by an effective error term which is $O(n^{-\frac{5}{6}})$.

By Proposition 6.1 we have

$$\frac{1}{(24n-1)} \sum_{\substack{z_Q \in \Lambda_{D_n}(6) \\ \mathrm{Im}(z_Q)>1}} \chi_{12}(Q) \left(1 - \frac{1}{2\pi \mathrm{Im}(z_Q)}\right) e(-z_Q) \\ = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\left\lfloor \frac{1}{\sqrt{6}}\sqrt{n-\frac{1}{24}} \right\rfloor} A_k(n) k^{\frac{1}{2}} \frac{d}{dn} \frac{\exp\left(\frac{\pi \lambda_n}{k}\sqrt{\frac{2}{3}}\right)}{\lambda_n}.$$

For any n such that $n \neq 6m^2$, $m \in \mathbb{Z}$, it is not difficult to show that $\lfloor \frac{1}{\sqrt{6}} \sqrt{n - \frac{1}{24}} \rfloor = \lfloor \frac{1}{\sqrt{6}} \sqrt{n} \rfloor$. On the other hand, for those n such that $n = 6m^2$ for some $m \in \mathbb{Z}$, it is not difficult to show that $\lfloor \frac{1}{\sqrt{6}} \sqrt{6m^2 - \frac{1}{24}} \rfloor = m - 1 = \lfloor \frac{1}{\sqrt{6}} \sqrt{6m^2} \rfloor - 1$. Then for such n, Corollary 1.2 follows from the following lemma.

Lemma 6.3 *Let* $n^* := \lfloor \frac{1}{\sqrt{6}} \sqrt{n - \frac{1}{24}} \rfloor + 1$. *Then*

$$\frac{1}{2\pi\sqrt{2}} A_{n^*}(n) {n^*}^{\frac{1}{2}} \frac{d}{dn} \frac{\exp\left(\frac{\pi \lambda_n}{n^*}\sqrt{\frac{2}{3}}\right)}{\lambda_n} = O(n^{-\frac{5}{6}}).$$

Proof We differentiate

$$\frac{\exp\left(\frac{\pi \lambda_n}{n^*}\sqrt{\frac{2}{3}}\right)}{\lambda_n}$$

with respect to n and apply the bound $|A_k(n)| < 2k^{\frac{5}{6}}$ (see [16]) to obtain

$$\begin{aligned}
\frac{1}{2\pi\sqrt{2}} A_{n^*}(n) n^{*\frac{1}{2}} \frac{d}{dn} \frac{\exp\left(\frac{\pi\lambda_n}{n^*}\sqrt{\frac{2}{3}}\right)}{\lambda_n} &\ll n^{*\left(\frac{5}{6}+\frac{1}{2}\right)} \left(\frac{\pi}{\sqrt{6}n^*\left(n-\frac{1}{24}\right)} + \frac{1}{2\left(n-\frac{1}{24}\right)^{\frac{3}{2}}} \right) \\
&\ll \left(\frac{\pi n^{*\frac{1}{3}}}{\sqrt{6}\left(n-\frac{1}{24}\right)} + \frac{n^{*\frac{4}{3}}}{2\left(n-\frac{1}{24}\right)^{\frac{3}{2}}} \right) \\
&\ll n^{-\frac{5}{6}}.
\end{aligned}$$

□

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