WHAT IS... a Mock Modular Form?

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The tale has been told and retold over time. The year: 1913. An unlikely correspondence begins between prominent number theorist G. H. Hardy and (then) poor Indian clerk S. Ramanujan. A mathematical collaboration between the two persists for the remainder of Ramanujan's lifetime: a mere seven years, until his untimely death at the age of thirty-two. New to the tale yet rooted in the Hardy-Ramanujan era is the modern notion of "mock modular form", not defined in the literature until nearly a century later, in 2007, by D. Zagier. Today, the story seems to hail from the distant past, but its mathematical harvest, its intrigue, has not let up. For example, the Hardy-Ramanujan story has been described recently as "one of the most romantic stories in the history of mathematics" by Zagier (2007), and decades prior to this, G. N. Watson describes Ramanujan's mathematics as inspiring in him a great "thrill" (1936). Far too many mathematicians to list, including G. Andrews, B. Berndt, K. Bringmann, K. Ono, H. Rademacher, and S. Zwegers, have perpetuated the legacy of Ramanujan's mathematics. Ramanujan's life story has even been dramatically reinterpreted in the 2007 work of fiction *The Indian Clerk* by David Leavitt.

An aspect of the allure to the Hardy-Ramanujan story is the spawn of a mathematical mystery surrounding the content of Ramanujan's deathbed letter to Hardy, his last. Watson, in his presidential address to the London Mathematical Society, prophetically declared the subject of the last letter, Ramanujan's "mock theta functions", to be "the final problem". The mathematical visions of Ramanujan were telling, particularly the seventeen peculiar functions of the last letter, which he dubbed "mock theta functions" and which appeared along with various properties and relations between them, yet with little to no explanation. (A handful of other such functions appear in Ramanujan's so called "lost notebook", unearthed by G. Andrews in 1976, and in work of G. N. Watson.) Curiously, the "mock theta functions" were reminiscent of modular forms, which, loosely speaking, are holomorphic functions on the upper half complex plane equipped with certain symmetries. In fact, Ramanujan used the term "theta function" to refer to what we call a modular form, so his choice of terminology "mock theta function" implies he thought of his functions as "fake"' or "pseudo" modular forms.

For example, one of Ramanujan's mock theta functions is given by $f(q) := \sum_{n\geq 0} q^{n^2}/(-q;q)_n^2$, where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. Modular forms by nature are also equipped with q-series expansions, where $q := e^{2\pi i\tau}$, τ is the variable in the upper half complex plane, and Dedekind's η-function, a well-known modular form, satisfies $q^{1/24}\eta^{-1}(\tau) = \sum_{n\geq 0} q^{n^2}/(q;q)_n^2$. These series expansions for $f(q)$ and $q^{1/24}\eta^{-1}(\tau)$ barely differ. In fact, Ramanujan observed other analytic properties shared by his mock theta functions and modular forms, providing a *description* of what he calls a "mock theta function"—and a notoriously vague *definition*.

Historically, one reason why the mock theta functions became an object of fascination for so many lies in the theory of integer partitions. For any natural number n, a partition of n is defined to be any nonincreasing sequence of positive integers whose sum is n. So, for example, there are three

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partitions of $3: 1+1+1, 2+1$, and 3, and if $p(n)$ denotes the number of partitions of n, $p(3) = 3$. A useful tool for studying $p(n)$, as with many functions on $\mathbb{N}$, is its "generating function", which is of the form $P(x) := 1 + \sum_{n\geq 1} p(n)x^n$, where x is a variable. It is not difficult to see by a counting argument that $P(x) = \prod_{n\geq 1}(1-x^n)^{-1}$. Returning to modular forms, it is also well known that $\eta(\tau)$ has an infinite q-product expansion, and upon replacing x with $q := e^{2\pi i \tau}$ in $P(x)$, one finds the following relationship between the modular form $\eta(\tau)$ and the partition generating function $P(x)$: $P(q) = q^{1/24}/\eta(\tau)$.

Many combinatorial generating functions like $P(x)$ are related to modular forms with q-infinite product expansions in similar ways, and having such relationships often allows one to use the theory of modular forms to further explain various aspects of the combinatorial functions. A famous example of this is the work of Hardy, Ramanujan, and H. Rademacher, who first used the theory of modular forms to describe the asymptotic behavior of $p(n)$ as $n \to \infty$. The mock theta functions stood out in that they too seemed to be related to combinatorial functions. We now know that the mock theta function $f(q)$, for example, is related to F. Dyson's rank-generating function, where the rank of a partition is equal to its largest part minus the number of its parts.

By the time of the Ramanujan Centenary Conference in 1987, it had become clear that "the mock theta functions give us tantalizing hints of a grand synthesis still to be discovered," as Dyson said. "Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of the modular forms…This," he said, "remains a challenge for the future." Despite the volumes of literature produced by many famous mathematicians on the subject since the Hardy-Ramanujan era, the "what *is*" remained unanswered for eighty-two years.

Enter S. Zwegers, a 2002 doctoral student under D. Zagier, whose thesis finally provided long awaited explanations, one being: while the mock theta functions were indeed not modular, they could be "completed" (after multiplying by a suitable power of q) by adding a certain nonholomorphic component, and then packaged together to produce real analytic vector valued functions that exhibit appropriate modular behavior. This one-sentence description does not do justice to the scope of Zwegers's results, which are much broader and also realize the mock theta functions within other contexts.

Zwegers's breakthrough didn't simply put an end to the mystery surrounding the mock theta functions; it (fortunately) opened the door to many more unanswered questions! Notably, K. Bringmann, K. Ono, and collaborators developed an overarching theory of "weak Maass forms" (see the work of J. Bruinier and J. Funke for a debut appearance in the literature), a space of functions to which we now understand the mock theta functions belong. Weak Maass forms are nonholomorphic modular forms that are also eigenfunctions of a certain differential (Laplacian) operator. (For the reader familiar with usual Maass forms or the more modern "Langlands program", while there is a small intersection with the theory of weak Maass forms, the theories are distinct in that the required growth condition on weak Maass forms is relaxed, for example, hence the descriptor "weak".)

How do the mock theta functions fit into this framework of weak Maass forms and lead to the answer to the question "what *is* a mock modular form"? As alluded to above, to associate modular behavior to a given mock theta function $m(q)$, one first needs to 1) define a suitable multiple $h(q) := q^M m(q)$ for some $M \in \mathbb{Q}$, and 2) add to $h(q)$ an appropriate nonholomorphic function $g^*(\tau)$, constructed from a modular theta series $g(\tau)$, dubbed *the shadow* of h (after Zagier). The final object $\widehat{h}(\tau) := h(q) + g^*(\tau)$ is then a nonholomorphic modular form. (Implicit here is that one must also replace q by $e^{2\pi i\tau}, \tau \in \mathbb{H}$, in the function $h(q)$.) Thus, one gains modularity at the expense of holomorphicity: $h(q)$ is holomorphic but not modular, while $\widehat{h}(\tau)$ is modular but not holomorphic. In particular, the function $\widehat{h}(\tau)$ is a weak Maass form, whose holomorphic part is essentially the mock theta function $m(q)$.

Loosely speaking then, a "mock modular form" is a holomorphic part of a weak Maass form. One particularly beautiful example of a mock modular form due to Zagier is the generating function for Hurwitz class numbers of algebraic number theory, whose completion (associated weak Maass form) is the so-called Zagier-Eisenstein series, and whose shadow is given by the classical modular theta function $\sum_{n\in\mathbb{Z}} q^{n^2}$.

The ability to realize the mock theta functions within the theory of weak Maass forms has led to many important discoveries. One notable example towards Dyson's "challenge for the future" within partition theory is due to Bringmann and Ono, who show that generalized rank-generating functions (which include the mock theta function $f(q)$ as a special case) are mock modular forms. Their work not only led to deeper results in the theory of partitions by making use of the theory of weak Maass forms but also exhibited a new perspective on the roles played by modular forms in both theories.

To grasp a more precise formulation of mock modular forms, note that functions exhibiting modular behavior (said to satisfy modular "transformations") come equipped with an integer or half-integer "weight" k. One has two clear formulations of the space $\mathbb{M}_k$ of weight k mock modular forms due to Zagier. First, consider the space $\mathfrak{M}_k$ of real analytic functions that exhibit a modular transformation of weight k and satisfy suitable growth conditions. Understanding the roles played by the shadows g, and also the differential Laplacian operator, leads to the fact that the space $\widehat{\mathbb{M}}_k := \{F \in \mathfrak{M}_k \mid \partial_\tau(y^k \partial_{\overline{\tau}} F) = 0\}$ not only consists of weight k weak Maass forms (with special eigenvalue $\frac{k}{2}(1-\frac{k}{2})$) but is also isomorphic to the space of mock modular forms $\mathbb{M}_k$ of weight k by mapping $h \in \mathbb{M}_k$ with shadow g to its completion $\widehat{h}! = h + g^* \in \widehat{\mathbb{M}}_k$. Second, one may also realize $\mathbb{M}_k$ via the exact sequence $0 \to M_k^! \to \mathbb{M}_k \xrightarrow{S} M_{2-k} \to 0$, where M_{2-k} is the space of holomorphic modular forms of weight $2-k$, $M_k^!$ is the space of weakly holomorphic modular forms of weight k (allowing additional poles at points called cusps) and $S(h) := g$ for the mock modular form h with shadow g. While this definition as stated is arguably the most natural, it may be of interest to consider other generalizations, some of which are currently being explored.

The tale of mock theta functions, mock modular forms, and weak Maass forms, while rooted in analytic number theory, has bled into many other areas of mathematics: sometimes mock modular forms are combinatorial generating functions, sometimes they answer questions about the nonvanishing of L-functions, sometimes they are related to class numbers, sometimes they are characters for affine Lie superalgebras, and sometimes they tell us about topological invariants—to name just a few of their many roles. Watson was right: "Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder…."

What *is* next?

References

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