

A SHORT PROOF OF THE MOCK THETA CONJECTURES USING MAASS FORMS

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ABSTRACT. A celebrated work of D. Hickerson gives a proof of the Mock Theta Conjectures using Hecke-type identities discovered by G. Andrews. Here, we respond to a remark by K. Bringmann, K. Ono and R. Rhoades and provide a short proof of the Mock Theta Conjectures by realizing each side of the identities as the holomorphic projection of a harmonic weak Maass form.

1. INTRODUCTION

Throughout we assume $\tau \in \mathbb{H}$, and let $q = q(\tau) := e^{2\pi i\tau}$, $e(\alpha) := e^{2\pi i\alpha}$. Ramanujan's "Lost" Notebook, discovered by George Andrews in 1976, contains a number of q -series identities involving Ramanujan's 17 original "mock theta" functions, which are classified by "order" 3, 5 or 7. In [2], the authors show that the ten 5th order identities naturally divide into two classes and prove that any given identity is true if and only if all identities are true for every member of its class. Thus, the truth of all 5th order identities reduces to the truth of two identities, which are called in [2] the "Mock Theta Conjectures". Before stating the Mock Theta Conjectures, we introduce the functions involved. We do so using the original notation of G. N. Watson ([15], [16]), who proved claims stated about these mock theta functions in Ramanujan's last letter to G. H. Hardy [13]. As in [16] we define

$$\chi_0(q) := \sum_{n \geq 0} \frac{q^{n^2+n}}{(q^{n+1}; q)_n},$$
$$\chi_1(q) := \sum_{n \geq 0} \frac{q^n}{(q^{n+1}; q)_{n+1}},$$

where $(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$. The Mock Theta Conjectures relate the functions $\chi_0(q)$ and $\chi_1(q)$ to differences of "rank generating functions"

$$R_{b,c}(d; q) := \sum_{n \geq 0} (N(b, 5, 5n + d) - N(c, 5, 5n + d))q^n,$$

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where $N(b, t, r)$ denotes the number of partitions of r with rank congruent to b mod t . The rank of a given partition is a statistic defined by F.J. Dyson [9] as the number of parts of the partition subtracted from the largest part of the partition. (For example, the partition $1 + 1 + 1 + 3 + 4$ of 10 has rank equal to $4 - 5 = -1$.)

Conjecture (The Mock Theta Conjectures¹).

$$\begin{aligned}\chi_0(q) - 1 &= R_{1,0}(0; q), \\ \chi_1(q) &= R_{2,1}(3; q) + R_{2,0}(3; q).\end{aligned}$$

2. PROOF OF THE MOCK THETA CONJECTURES USING MAASS FORMS

D. Hickerson, in the celebrated work [10], proves the Mock Theta Conjectures using Hecke-type identities discovered by G. Andrews [1]. More recently, in [6], the authors explicitly construct families of modular forms after offering a systematic treatment of more general mock theta-type identities. As a side remark to their results, [6, §1 Remark 2], the authors state:

...the results of Zwegers' thesis ... [and] the proof of [6, Thm. 1.1] ... reduces the proof of the Mock Theta Conjectures to the verification of two simple identities for classical weakly holomorphic modular forms.

Our purpose here in this paper is to respond to this remark and provide a proof of the Mock Theta Conjectures by realizing each side of the identities as the holomorphic projection of a harmonic weak Maass form. Until recently (as exposed in [17] and [4]), not much was known regarding the role of Ramanujan's mock theta functions within the context of modular forms. S. Zwegers "completed" many of Ramanujan's mock theta functions to obtain weight $1/2$ weak Maass forms, and in [4] K. Bringmann and K. Ono further generalize the results in [17]. (For the definition of weak Maass forms, see [8] or [4]-[6].) We make use of these results to prove the Mock Theta Conjectures.

For a vector valued function $V(\tau) := (v_1(\tau), v_2(\tau), \dots, v_n(\tau))^T$, we let $V^{(i)}(\tau) := v_i(\tau)$ denote the i^{th} component function. We use the following identities of Watson [16]:

$$\begin{aligned}\chi_0(q) &= 2F_0(q) - \phi_0(-q), \\ \chi_1(q) &= 2F_1(q) + q^{-1}\phi_1(-q),\end{aligned}$$

where $F_j(q)$ and $\phi_j(q)$, $j = 0, 1$, are additional fifth order mock theta functions whose definitions (q -series expansions) may be found in [2], for example.

Using the vectors $F_{5,1}(\tau)$ and $F_{5,2}(\tau)$ defined in [17, p. 74 and p. 79], we have

$$\begin{aligned}q^{\frac{1}{240}} F_{5,1}^{(3)}(\tau) + 1 &= F_0(q^{\frac{1}{2}}), & q^{\frac{1}{240}} F_{5,2}^{(3)}(\tau) &= \phi_0(-q^{\frac{1}{2}}), \\ q^{-\frac{71}{240}} F_{5,1}^{(4)}(\tau) &= F_1(q^{\frac{1}{2}}), & -q^{\frac{49}{240}} F_{5,2}^{(4)}(\tau) &= \phi_1(-q^{\frac{1}{2}}),\end{aligned}$$

¹We point out that the second Mock Theta Conjecture stated here is corrected from [2, Eq. (4.10)], where an extra " -1 " appears. This extraneous " -1 " first appears in [2, Eq. (4.7)], which one confirms is merely miscopied from [3, Eq. (6.18)].

so that

$$\begin{aligned}\chi_0(q^{\frac{1}{2}}) - 1 &= 1 + q^{\frac{1}{240}} \left(2F_{5,1}^{(3)}(\tau) - F_{5,2}^{(3)}(\tau) \right), \\ \chi_1(q^{\frac{1}{2}}) &= q^{-\frac{71}{240}} \left(2F_{5,1}^{(4)}(\tau) - F_{5,2}^{(4)}(\tau) \right).\end{aligned}$$

We define the differences

$$\begin{aligned}K_1(\tau) &:= q^{-\frac{1}{240}}(\chi_0(q^{\frac{1}{2}}) - 1) - 2G_{5,1}^{(3)}(\tau) + G_{5,2}^{(3)}(\tau) - q^{-\frac{1}{240}}, \\ N_1(\tau) &:= q^{\frac{71}{240}}(\chi_1(q^{\frac{1}{2}}) - 2G_{5,1}^{(4)}(\tau) + G_{5,2}^{(4)}(\tau))\end{aligned}$$

and find

$$(2.1) \quad K_1(\tau) = 2H_{5,1}^{(3)}(\tau) - H_{5,2}^{(3)}(\tau),$$

$$(2.2) \quad N_1(\tau) = 2H_{5,1}^{(4)}(\tau) - H_{5,2}^{(4)}(\tau),$$

where the vectors $G_{5,j}(\tau)$ and $H_{5,j}(\tau)$, $j = 1, 2$, are defined in [17]. Using (2.1), (2.2), and [17, Prop. 4.10, Prop. 4.13, Prop. 4.2 (2)], we deduce the following proposition.

Proposition 2.1. *The functions $K_1(\tau)$ and $N_1(\tau)$ are vector valued harmonic weak Maass forms of weight $1/2$, whose non-holomorphic parts $K_1^{\text{nh}}(\tau)$ and $N_1^{\text{nh}}(\tau)$ of $K_1(\tau)$ and $N_1(\tau)$ (respectively) are given by*

$$\begin{aligned}K_1^{\text{nh}}(\tau) &= -\frac{3i}{4\sqrt{30}} \int_{-\bar{\tau}}^{i\infty} \frac{\sum_{m \equiv 19, 29 \pmod{60}} me^{\pi im^2 z/120} - \sum_{m \equiv 49, 59 \pmod{60}} me^{\pi im^2 z/120}}{\sqrt{-i(z + \tau)}} dz, \\ N_1^{\text{nh}}(\tau) &= -\frac{3i}{4\sqrt{30}} \int_{-\bar{\tau}}^{i\infty} \frac{\sum_{m \equiv 13, 23 \pmod{60}} me^{\pi im^2 z/120} - \sum_{m \equiv 43, 53 \pmod{60}} me^{\pi im^2 z/120}}{\sqrt{-i(z + \tau)}} dz.\end{aligned}$$

Next we define the functions²

$$\begin{aligned}\tilde{K}_2(\tau) &:= q^{-\tilde{\ell}_5} R_{1,0}(0, q^{5\ell_5}) - q^{-\tilde{\ell}_5} \\ &= \sum_{n \geq 0} (N(1, 5, 5n) - N(0, 5, 5n)) q^{5\ell_5 n - \tilde{\ell}_5} - q^{-\tilde{\ell}_5}, \\ \tilde{N}_2(\tau) &:= q^{3\ell_5 - \tilde{\ell}_5} (R_{2,1}(3; q^{5\ell_5}) + R_{2,0}(3; q^{5\ell_5})) \\ &= \sum_{n \geq 0} (2N(2, 5, 5n + 3) - N(1, 5, 5n + 3) - N(0, 5, 5n + 3)) q^{\ell_5(5n+3) - \tilde{\ell}_5},\end{aligned}$$

where throughout we assume the notation as in [4] and let $\ell_5 := 25 \cdot 24$, and $\tilde{\ell}_5 := 25$. We deduce from [4, Theorem 1.3] that $\tilde{K}_2(\tau)$ and $\tilde{N}_2(\tau)$ are restricted sums of the holomorphic parts of harmonic weak Maass forms. To make this statement more precise, as introduced in [4], we define $S_1\left(\frac{\alpha}{5}, \tau\right)$ by

$$S_1\left(\frac{\alpha}{5}, \tau\right) := -i \sin\left(\frac{\pi\alpha}{5}\right) 10\sqrt{2} \int_{-\bar{\tau}}^{i\infty} \frac{\Theta\left(\frac{\alpha}{5}; 5^2 \cdot 24 \cdot z\right)}{\sqrt{-i(\tau + z)}} dz,$$

²Different definitions for $N(0, t, 0)$ exist in the literature, e.g. $N(0, t, 0) := 0$ in [2], [3], while in other places (e.g. [4], [6]), $N(0, t, 0) := 1$. We adopt the former convention, therefore requiring the extra term $-q^{-\tilde{\ell}_5}$ in the definition of $\tilde{K}_2(\tau)$. (See also Proposition 2.2.)

where

$$\Theta\left(\frac{\alpha}{5}; \tau\right) := \sum_{m \pmod{10}} (-1)^m \sin\left(\frac{\alpha\pi(6m+1)}{5}\right) \sum_{n \equiv 6m+1 \pmod{60}} ne(n^2\tau/24).$$

After a brief argument (see also [4, Prop. 4.1]), one finds that the function $S_1\left(\frac{\alpha}{5}, \tau\right)$ has Fourier development of the form

$$S_1\left(\frac{\alpha}{5}, \tau\right) = \sum_{\substack{m \pmod{10} \\ n \equiv 6m+1 \pmod{60}}} \beta_{n,m} q^{-(5n)^2}.$$

We define $S_1^*\left(\frac{\alpha}{5}; \tau\right)$ (resp. $S_1^{**}\left(\frac{\alpha}{5}; \tau\right)$) to be the restriction of (the Fourier development of) $S_1\left(\frac{\alpha}{5}; \tau\right)$ to those n such that $n^2 \equiv 1 \pmod{120}$ (respectively $n^2 \equiv 49 \pmod{120}$). Finally, we define

$$K_2(\tau) := \tilde{K}_2(\tau) - \frac{1}{5} \sum_{j=1}^4 (\zeta_5^{-j} - 1) S_1^*\left(\frac{j}{5}; \tau\right),$$

$$N_2(\tau) := \tilde{N}_2(\tau) - \frac{1}{5} \sum_{j=1}^4 (2\zeta_5^{-2j} - \zeta_5^{-j} - 1) S_1^{**}\left(\frac{j}{5}; \tau\right),$$

where $\zeta_5 := e^{2\pi i/5}$. We deduce the following proposition.

Proposition 2.2. *The functions $K_2(\tau)$ and $N_2(\tau)$ are harmonic weak Maass forms of weight $\frac{1}{2}$ on $\Gamma_1(144 \cdot 10^2 \cdot 5^4)$.*

Proof. By Theorem 1.3 in [4], for $0 \leq r < 5$, the difference³

$$\sum_{n \geq 0} \left(N(r, 5; n) - \frac{p(n)}{5} \right) q^{24 \cdot 5^2 \cdot n - 5^2} + \lambda_r$$

is the holomorphic part of a weak Maass form of weight $\frac{1}{2}$ on $\Gamma_1(144 \cdot 10^2 \cdot 5^2)$, where $p(n)$ denotes the number of partitions of n , and $\lambda_r := 0$ for $r \neq 0$, and $\lambda_0 := q^{-25}$. Turning to the proof of this statement [4, §3.3, p. 24], we find that the non-holomorphic part is given by $-\frac{1}{5} \sum_{j=1}^4 \zeta_5^{-rj} S_1\left(\frac{j}{t}; \tau\right)$. For $0 \leq j \leq 4$ we define

$$\mathfrak{D}_j(\tau) := \sum_{n \geq 0} \left(N(r, 5; n) - \frac{p(n)}{5} \right) q^{24 \cdot 5^2 \cdot n - 5^2} - \frac{1}{5} \sum_{j=1}^4 \zeta_5^{-rj} S_1\left(\frac{j}{t}; \tau\right) + \lambda_j,$$

$$\mathfrak{D}_j^*(\tau) := \sum_{n \geq 0} \left(N(r, 5; 5n) - \frac{p(5n)}{5} \right) q^{24 \cdot 5^2 \cdot 5n - 5^2} - \frac{1}{5} \sum_{j=1}^4 \zeta_5^{-rj} S_1^*\left(\frac{j}{t}; \tau\right) + \lambda_j,$$

$$\mathfrak{D}_j^{**}(\tau) := \sum_{n \geq 0} \left(N(r, 5; 5n+3) - \frac{p(5n+3)}{5} \right) q^{24 \cdot 5^2 \cdot (5n+3) - 5^2}$$

$$- \frac{1}{5} \sum_{j=1}^4 \zeta_5^{-rj} S_1^{**}\left(\frac{j}{t}; \tau\right).$$

³The factor q^{-25} must be introduced when $r = 0$, as we have adopted the convention $N(0, t, 0) := 0$ as in [2], [3], whereas $N(0, t, 0) := 1$ in [4].

Then one finds

$$(2.3) \quad K_2(\tau) = \mathfrak{D}_1^*(\tau) - \mathfrak{D}_0^*(\tau),$$

$$(2.4) \quad N_2(\tau) = 2\mathfrak{D}_2^{**}(\tau) - \mathfrak{D}_1^{**}(\tau) - \mathfrak{D}_0^{**}(\tau).$$

By [4], the forms $\mathfrak{D}_j(\tau)$ are weight $\frac{1}{2}$ harmonic weak Maass forms on $\Gamma_1(144 \cdot 10^2 \cdot 5^2)$. The forms $\mathfrak{D}_j^*(\tau)$ and $\mathfrak{D}_j^{**}(\tau)$ are restrictions of $\mathfrak{D}_j(\tau)$ to the arithmetic progressions $n \equiv 0 \pmod{5}$ and $n \equiv 3 \pmod{5}$ and thus, too, are harmonic weak Maass forms by a classical argument using the orthogonality of Dirichlet characters ([11], Proposition 17).⁴ After restriction, the group $\Gamma_1(144 \cdot 10^2 \cdot 5^2)$ becomes $\Gamma_1(144 \cdot 10^2 \cdot 5^4)$. Proposition 2.2 now follows from (2.3) and (2.4). \square

Next we explicitly compute the non-holomorphic part $K_2^{\text{nh}}(\tau)$ (resp. $N_2^{\text{nh}}(\tau)$) of $K_2(\tau)$ (resp. $N_2(\tau)$). We first consider $K_2(\tau)$. By definition, the Fourier expansion of $S_1^*(\alpha, \tau)$ reduces to a sum over those integers n for which $n^2 \equiv 1 \pmod{120}$. The integers n satisfying this condition are

$$n \in \Omega := \{1, 11, 19, 29, 31, 41, 49, 51, 61, 71, 79, 89, 91, 101, 109, 119\}(\text{mod } 120).$$

Further, of all residue classes $m \pmod{10}$, we find there are only 8 values for $n \in \Omega$ that satisfy $n \equiv 6m + 1 \pmod{60}$, and these are

$$n \in \Omega' := \{1, 19, 31, 49, 61, 79, 91, 109\}(\text{mod } 120),$$

corresponding to, in order,

$$m \in \Psi := \{0, 3, 5, 8, 10, 13, 15, 18\}.$$

We observe that

$$\Omega'(\text{mod } 60) = \{19, -29, 49, -59\}(\text{mod } 60)$$

and find

$$K_2^{\text{nh}}(\tau) = -\frac{15i}{\sqrt{2}} \left(\sum_{n \equiv 19, 29 \pmod{60}} \int_{-\bar{\tau}}^{i\infty} \frac{ne^{2\pi i(5n)^2 z}}{\sqrt{-i(\tau+z)}} dz - \sum_{n \equiv 49, 59 \pmod{60}} \int_{-\bar{\tau}}^{i\infty} \frac{ne^{2\pi i(5n)^2 z}}{\sqrt{-i(\tau+z)}} dz \right).$$

Letting $\tau \mapsto 10\ell_5\tau$ in Proposition 2.1, we find

$$(2.5) \quad \begin{aligned} K_1^{\text{nh}}(10\ell_5\tau) &= -\frac{15i}{\sqrt{2}} \int_{-\bar{\tau}}^{i\infty} \frac{\sum_{m \equiv 19, 29 \pmod{60}} me^{2\pi i(5m)^2 z} - \sum_{m \equiv 49, 59 \pmod{60}} me^{2\pi i(5m)^2 z}}{\sqrt{-i(z+\tau)}} dz \\ &= K_2^{\text{nh}}(\tau). \end{aligned}$$

⁴To be more precise, for those readers who wish to carry out the details of this assertion, one obtains such restricted forms by applying the Hecke operators U and V , and twisting. For example, if $g(\tau) := \sum_n a_n q^{5^2(24n-1)}$, then

$$g^*(\tau) := \sum_{n \equiv 0 \pmod{5}} a_n q^{5^2(24n-1)} = \frac{1}{4} \sum_{\chi \pmod{5}} (\chi(4)(g|U_{25}) \otimes \bar{\chi}|V_{25}).$$

For the functions $N_2(\tau)$, we proceed similarly, using the sets

$$\begin{aligned} \Psi_2 &:= \{1, 2, 6, 7\}(\bmod 10), \\ \Omega'_2(\bmod 60) &:= \{13, -23, 43, -53\}(\bmod 60), \end{aligned}$$

and conclude $N_1^{\text{nh}}(10\ell_5\tau) = N_2^{\text{nh}}(\tau)$.

To conclude the proof of the Mock Theta Conjectures, by Proposition 2.2 we have that $K_2(\tau)$ and $N_2(\tau)$ are harmonic weak Maass forms of weight $\frac{1}{2}$ on $\Gamma_1(144 \cdot 10^2 \cdot 5^4)$. The forms $K_1(10\ell_5\tau)$ and $N_1(10\ell_5\tau)$ are also harmonic weak Maass forms of weight $\frac{1}{2}$ by Proposition 2.1. We remark that the groups on which all forms $K_1(\tau)$, $K_2(\tau)$, $N_1(\tau)$ and $N_2(\tau)$ live are dictated by the groups associated to the theta functions appearing in their respective period integrals.⁵ By a lifting argument as in [8] and as used in [4], one finds an optimal (larger) subgroup associated to $K_1(10\ell_5\tau)$ and $N_1(10\ell_5\tau)$, namely $\Gamma_1(144 \cdot 10^2 \cdot 5^4)$. For brevity, we proceed by arguing the proof of the first Mock Theta Conjecture, as the second follows similarly. By (2.5), we see that the non-holomorphic parts of the forms $K_1(10\ell_5\tau)$ and $K_2(\tau)$ are identical, and thus $\mathfrak{K}(\tau) := K_1(10\ell_5\tau) - K_2(\tau)$ is a weakly holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_1(144 \cdot 10^2 \cdot 5^4)$. We argue that $\mathfrak{K}(\tau) = 0$ and provide two arguments.

Argument 1. We first claim that the weakly holomorphic modular form $\mathfrak{K}(\tau)$ is in fact holomorphic. From [17] and [4], one has all transformation laws for $\mathfrak{K}(\tau)$. Moreover, one has vector-valued forms containing $\mathfrak{K}(\tau)$ (from the vectors $H_{5,1}(\tau)$ and $H_{5,2}(\tau)$ of [17], and [4, Lemma 3.1]). With this, and the fact that the principal parts and constant terms are equal, an application of the circle method implies that the asymptotic growth of the coefficients of $\mathfrak{K}(\tau)$ is $\mathcal{O}(n^\alpha)$ for some constant α . Indeed, one may see this result carried out in explicit detail in [7, Thm. 1.1 and Thm. 1.2] for the forms $R_{b,c}(d; q)$. (See also [5, Cor 4.2 and p. 22].) Thus, $\mathfrak{K}(\tau)$ is a holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_1(144 \cdot 10^2 \cdot 5^4)$. By the Serre-Stark Basis Theorem [12, Thm. 1.45] and [12, Eq. (1.7)], one has as a basis for the space of weight $\frac{1}{2}$ modular forms a finite number of theta functions, and one verifies that $\mathfrak{K}(\tau)$ is identically zero.

Argument 2. Arguing using complex analysis, one has the well-known valence formula⁶

$$(2.6) \quad \sum_{\tau \in \mathbb{H}} \text{ord}_\tau \mathfrak{K} + \sum_{\text{cusps } \kappa} \text{ord}_\kappa \mathfrak{K} + \text{ord}_\infty \mathfrak{K} = \frac{1}{24} [\text{SL}_2(\mathbb{Z}) : \Gamma_1(144 \cdot 10^2 \cdot 5^4)].$$

A priori, $\mathfrak{K}(\tau)$ is holomorphic on \mathbb{H} , so the first summand in (2.6) is non-negative. Further, we are able to determine a bound $-M \leq 0$ such that $\text{ord}_\kappa \mathfrak{K} \geq -M$ for any cusp κ . This integer $-M$ is at worst the largest (in magnitude) negative order at ∞ of any other component in the vector-valued form to which \mathfrak{K} belongs. Thus, by computing sufficiently many coefficients in the Fourier expansion of \mathfrak{K} at ∞ , one can contradict equation (2.6). Although the number of coefficients that must be checked is large due to the large index $[\text{SL}_2(\mathbb{Z}) : \Gamma_1(144 \cdot 10^2 \cdot 5^4)]$, this number is indeed finite.

⁵See for example [14] for the transformation laws of the theta functions.

⁶The notation “ord” denotes appropriately weighted orders.

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