ROGERS-RAMANUJAN MOMENT BIAS

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Abstract. The celebrated Rogers-Ramanujan identities are a pair of combinatorial identities stating that number of integer partitions of n ($n \in \mathbb{N}_0$) with parts congruent to $\pm 1 \pmod 5$ (respectively $\pm 2 \pmod 5$) equals the number of partitions of n with super-distinct parts (respectively super-distinct parts with no 1s). In this paper, we consider related combinatorial moment functions, and establish bias results, generalizing both the Rogers-Ramanujan identities and results of Ballantine and the author. We also interpret these results in terms of k-marked partitions, and pose related open problems of interest.

Dedicated to George Andrews and Bruce Berndt in honor of their 85th birthdays.

1. Introduction, results, and open problems

The celebrated Rogers-Ramanujan identities are a pair of combinatorial identities stating that number of integer partitions of n ($n \in \mathbb{N}_0$) with parts congruent to $\pm 1 \pmod 5$ (respectively $\pm 2 \pmod 5$) equals the number of partitions of n with super-distinct parts (respectively super-distinct parts with no 1s). Super-distinct parts are also referred to as 2-distinct parts, and must differ by at least 2. For example, among the eleven integer partitions of $n = 6$, there are three with parts $\pm 1 \pmod 5$, namely 6, $4+1+1$, and $1+1+1+1+1+1$, and three with super-distinct parts, namely 6, $5+1$, and $4+2$. In analytic form, the Rogers-Ramanujan identities are expressed as

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{2}$$

noting that the products and series appearing are the corresponding partition generating functions. Here and throughout, the q-Pochhammer symbol is defined for $n \in \mathbb{N}_0 \cup \{\infty\}$ by

$$(a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$

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and we assume $|q| < 1$ for absolute convergence.

Rogers and Ramanujan independently discovered the identities in the late 19th century/early 20th century, and they have proved to be of great mathematical interest and importance. Rogers established the first known proof [25], and later published a joint proof with Ramanujan [26]. Around the same time, Schur independently rediscovered and proved the identities [27]. In the decades following, the identities have played signficiant roles in and have made connections to diverse areas, including combinatorics, q-hypergeometric series, Lie algebras, modular forms, statistical mechanics, and more (see, e.g., [1, 5, 9, 15, 17, 22–24, 28, 29], for more). In particular (up to minor normalizations) when viewed as functions of $\tau \in \mathbb{H}$ with $q = e^{2\pi i \tau}$, the functions defining the Rogers-Ramanujan identities (1)-(2) are modular forms, and thus are such that their Fourier coefficients are given by the relevant restricted partition numbers.

In joint work of Ballantine and the author [8], we further dissect the restricted partitions from the Rogers-Ramanujan identities by examining their numbers of parts. Namely, we study in [8] the following natural question: given that the Rogers-Ramanujan identities hold, is it also true that the number of parts in all partitions of n with parts congruent to $\pm 1 \pmod 5$ (respectively $\pm 2 \pmod 5$) equals the number of parts in all super-distinct partitions of n (respectively super-distinct partitions of n with no 1s)? This type of question originated for other partitions in a well-known conjecture of Beck, now a theorem of Andrews [2]. While in general the aforementioned natural question on parts in the Rogers-Ramanujan partition functions is not true, Ballantine and the author established biases in the following two theorems from [8].

Theorem BF1 (Ballantine-Folsom, [8])**.** *The excess of the number of parts in all partitions of n with parts congruent to $\pm 1 \pmod 5$ over the number of parts in all super-distinct partitions of n equals the number of pairs of partitions $(\lambda, (a^b))$ satisfying all of the following conditions: λ is a super-distinct partition of $n - ab$, $a \equiv \pm 1 \pmod 5$, $b \geq 1$, and if $a = 1$, then at least one of $b-1, b, b+1$ is a part of λ.*

We use the notation (a^b) to stand for the partition $a + a + \cdots + a$ consisting of b copies of the part a.

Theorem BF2 (Ballantine-Folsom, [8])**.** *The excess of the number of parts in all partitions of n with parts congruent to $\pm 2 \pmod 5$ over the number of parts in all partitions of n with super-distinct parts and no 1s equals the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ such that λ has super-distinct parts and no 1s, $a \equiv \pm 2 \pmod 5$, $b \geq 1$, and satisfying conditions prescribed by $\mathcal{S}(n)$.*

We refer the reader to [8] for the explicit description of the set $\mathcal{S}(n)$.

To illustrate Theorem BF1 from [8] with $n = 6$, we see from our earlier example that there are 10 total parts in the three partitions of 6 with parts $\pm 1 \pmod 5$, and 5 total parts in the three partitions of 6 with super-distinct parts. The $10 - 5 = 5$ excess pairs of partitions satisfying the conditions stated in the theorem are $(4+1, (1^1)), (3+1, (1^2)), (3, (1^3)), (2, (4^1)), (\emptyset, (6^1))$.

In this paper, we further extend our study of Rogers-Ramanujan partitions from [8] to their moments. More broadly, partition moment functions have been of significance. For example, work of Atkin and Garvan [7] connecting partition ranks and cranks introduces

kth moments of partition ranks

$$N_k(n) := \sum_{m=-\infty}^{\infty} m^k N(m,n),$$

where $N(m,n) = p(n\colon \text{rank } m)$. We briefly recall that the rank of a partition is the difference between its largest part and its number of parts, introduced by Dyson [16] in order to combinatorially explain two of Ramanujan's famous partition congruences (mod 5 and 7); the existence of its counterpart the crank was an important discovery of Andrews and Garvan, who showed that it could be used to explain all three Ramanujan congruences (mod $5, 7$ and 11) [6, 21].

Later in the influential work [3], Andrews provided a combinatorial interpretation of the Atkin-Garvan rank moments involving symmetrized kth rank moment functions

$$\eta_k(n) := \sum_{m=-\infty}^{\infty} \binom{m + \left\lfloor \frac{k-1}{2} \right\rfloor}{k} N(m,n), \tag{3}$$

and provided an interpretation in terms of k-marked Durfee symbols of partitions. Further important work of Bringmann [10], followed by work of Bringmann-Garvan-Mahlburg [11], Kimport and the author [20], and also others including [18, 19], established associated mock and quantum modularity (broadly speaking) of related generating functions. Others have studied and proved significant combinatorial inequalities among rank and crank moments as well (see, e.g., [12, 13]).

Here we define a generalization of the Rogers-Ramanujan partition functions, namely the following shifted kth moment functions ($k \in \mathbb{N}_0$) for partitions enumerated by the Rogers-Ramanujan identities:

$$\alpha_k^{(j)}(n) := \sum_{m=0}^{\infty} \binom{m}{k} a_j(n,m), \tag{4}$$

$$\beta_k^{(j)}(n) := \sum_{m=0}^{\infty} \binom{m}{k} b_j(n,m), \tag{5}$$

for $j \in \{1, 2\}$, where

$$\begin{aligned}
a_1(n,m) &:= p(n\colon \text{ parts } \equiv \pm 1 \pmod 5 \text{ and exactly } m \text{ parts}),\\
a_2(n,m) &:= p(n\colon \text{ parts } \equiv \pm 2 \pmod 5 \text{ and exactly } m \text{ parts}),\\
b_1(n,m) &:= p(n\colon \text{ super-distinct parts and exactly } m \text{ parts}),\\
b_2(n,m) &:= p(n\colon \text{ super-distinct parts, no 1s, and exactly } m \text{ parts}).
\end{aligned}$$

(We point out that the ordering (n,m) of dependence on n and m in these coefficients and as in [8] is opposite that of $N(m,n)$ discussed above.) These shifted rank moments defined in (4)-(5) encode both the Rogers-Ramanujan identity partition functions (discussed above) as well as those studied in terms of parts in [8], in the special cases $k = 0$ and $k = 1$, respectively.

That is, for $k = 0$, we have that

$$\alpha_0^{(1)}(n) = p(n\colon \text{parts } \equiv \pm 1 \pmod 5),$$
$$\beta_0^{(1)}(n) = p(n\colon \text{super-distinct parts}),$$

and

$$\alpha_0^{(2)}(n) = p(n\colon \text{parts } \equiv \pm 2 \pmod 5),$$
$$\beta_0^{(2)}(n) = p(n\colon \text{super-distinct parts, no 1s}),$$

and hence by the Rogers-Ramanujan identities, the 0th shifted Rogers-Ramanujan moments agree for all $n \in \mathbb{N}_0$, i.e.,

$$\alpha_0^{(1)}(n) = \beta_0^{(1)}(n), \tag{6}$$
$$\alpha_0^{(2)}(n) = \beta_0^{(2)}(n). \tag{7}$$

For $k = 1$, we have that

$$\alpha_1^{(1)}(n) = \#\{\text{parts among the partitions of } n \text{ with parts} \equiv \pm 1 \pmod 5\},$$
$$\beta_1^{(1)}(n) = \#\{\text{parts among the partitions of } n \text{ with super-distinct parts}\},$$

and

$$\alpha_1^{(2)}(n) = \#\{\text{parts among the partitions of } n \text{ with parts} \equiv \pm 2 \pmod 5\},$$
$$\beta_1^{(2)}(n) = \#\{\text{parts among the partitions of } n \text{ with super-distinct parts, no 1s}\},$$

By Theorems BF1 and BF2 stated above from [8], it follows that

$$\alpha_1^{(1)}(n) \geq \beta_1^{(1)}(n) \tag{8}$$

and

$$\alpha_1^{(2)}(n) \geq \beta_1^{(2)}(n) \tag{9}$$

for all $n \in \mathbb{N}_0$, exhibiting a bias for these first shifted Rogers-Ramanujan moments with parts $\pm 1 \pmod 5$ over those with super-distinct parts, and with parts $\pm 2 \pmod 5$ over those with super-distinct parts and no 1s.

Given these results (i.e., (6)-(9)) corresponding to the cases $k = 0$ and $k = 1$, it is natural to ask about potential shifted Rogers-Ramanujan moment biases for $k \geq 2$. For example, with $k = 2$ and $n = 6$, we compute as in the examples preceding that

$$\alpha_2^{(1)}(6) = \binom{1}{2} a(6,1) + \binom{3}{2} a(6,3) + \binom{6}{2} a(6,6) = 0 + 3 + 15 = 18,$$

while

$$\beta_2^{(1)}(6) = \binom{1}{2} b(6,1) + \binom{2}{2} b(6,2) = 0 + 2 = 2.$$

To further motivate our study of these shifted Rogers-Ramanujan moments (4)-(5), we offer an additional combinatorial interpretation in terms of k-marked partitions. We do so in the spirit of Andrews' symmetrized rank moment functions $\eta_k(n)$ in (3) above which he showed may be interpreted in terms of k-marked Durfee symbols. To explain this, we first recall that a marked partition is one with a single part marked (e.g., by an asterisk). Note that, for example, $4 + 1^* + 1$ and $4 + 1 + 1^*$ are different marked partitions. Thus, the number of parts in all partitions of n satisfying certain conditions is equal to the number of marked partitions of n satisfying the conditions. Hence, we may realize the first shifted Rogers-Ramanujan moments $\alpha_1^{(1)}(n)$ (respectively $\alpha_1^{(2)}(n)$) discussed above as also equaling the number of marked partitions of n with parts $\equiv \pm 1 \pmod 5$ (respectively $\equiv \pm 2 \pmod 5$); similarly $\beta_1^{(1)}(n)$ (respectively $\beta_1^{(2)}(n)$) may also be interpreted as equaling the number of marked partitions of n with super-distinct parts (respectively super-distinct parts and no 1s). More generally we may combinatorially interpret the Rogers-Ramanujan shifted kth moments (4)-(5) ($k \in \mathbb{N}_0$) as:

$$\alpha_k^{(1)}(n) = \#\{k\text{-marked partitions of } n \text{ with parts} \equiv \pm 1 \pmod 5\},$$
$$\beta_k^{(1)}(n) = \#\{k\text{-marked partitions of } n \text{ with super-distinct parts}\}, \tag{10}$$
$$\alpha_k^{(2)}(n) = \#\{k\text{-marked partitions of } n \text{ with parts} \equiv \pm 2 \pmod 5\},$$
$$\beta_k^{(2)}(n) = \#\{k\text{-marked partitions of } n \text{ with super-distinct parts, no 1s}\}, \tag{11}$$

where we say a partition is k-marked if k of its parts are marked. (Thus, a 1-marked partition is the same as a marked partition as defined above.) For example, $4^* + 1^* + 1$, $4^* + 1 + 1^*$, and $4 + 1^* + 1^*$ are all (different) 2-marked partitions of $n = 6$ into parts $\equiv \pm 1 \pmod 5$. This interpretation of the kth shifted Rogers-Ramanujan moments $\alpha_k^{(j)}(n)$ and $\beta_k^{(j)}(n)$ in (4)-(5) is especially relevant since a main goal is to study the number of parts in partitions. Moreover, we use this combinatorial interpretation of the moments in a portion of our proofs of our main theorems, Theorem 1.1 and Theorem 1.2 below.

Our main results in this paper establish biases for the kth shifted Rogers-Ramanujan moment functions, generalizing both the Rogers-Ramanujan identities (6) and (7) ($k = 0$) and the parts biases (8) and (9) ($k = 1$) from [8].

Theorem 1.1. *For all $k, n \in \mathbb{N}_0$, we have that*

$$\alpha_k^{(1)}(n) \geq \beta_k^{(1)}(n).$$

Theorem 1.2. *For all $k, n \in \mathbb{N}_0$, we have that*

$$\alpha_k^{(2)}(n) \geq \beta_k^{(2)}(n).$$

The proof of Theorem 1.2 is more difficult than the proof of Theorem 1.1. We offer the following related open problems of interest.

Open Problems.

Problem 1. As stated above, a simple combinatorial interpretation of the excess $\alpha_1^{(1)}(n) - \beta_1^{(1)}(n)$ corresponding to the case of $k = 1$ (and $j = 1$, with respect to the first Rogers-Ramanujan identity partition functions) was established in [8] (see Theorem BF1). More generally for all $k \geq 0$, a combinatorial interpretation of the excesses $\alpha_k^{(1)}(n) - \beta_k^{(1)}(n)$ (which are nonnegative due to Theorem 1.1) may be deduced from proof of Theorem 1.1. However, a simpler, and manifestly positive, combinatorial description of this excess for $k \geq 2$ is of interest. We leave this as an open problem.

Problem 2. As remarked above, somewhat surprisingly, it is more difficult to establish Theorem 1.2 corresponding to the second Rogers-Ramanujan identity (than Theorem 1.1). (A similar remark is made in [8] regarding Theorem BF2.) A simpler proof of Theorem

1.2, in particular a simpler combinatorial injection than the one given at the end of the proof of Theorem 1.2, is of interest, and we leave this as an open problem. From the proof of Theorem 1.2 and [8] it is also possible to deduce a combinatorial interpretation of the excesses $\alpha_k^{(2)}(n) - \beta_k^{(2)}(n)$ (which are non-negative in general for $k \geq 0$ due to Theorem 1.2); this was done explicitly for $k = 1$ in [8]. However, a simpler, and manifestly positive, combinatorial description of this excess for all k (different from the results and methods in this paper and [8]) is of interest, and we leave this as a related open problem.

2. Proof of Theorem 1.1

Our proof follows by both analytic and combinatorial methods. We remark that the theorem holds in the case of $k = 0$ by the first Rogers-Ramanujan identity (and with strict equality) (1), and results implying the truth of the theorem in the case of $k = 1$ are proved in [8] (see also Section 1). It thus suffices to prove the theorem for integers $k \geq 2$, however our proof below establishes the result for $k \geq 1$. The two-variable Rogers-Ramanujan parts generating functions for $a_1(n,m)$ and $b_1(n,m)$ are given by

$$P_1(z;q) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_1(n,m)z^m q^n = \frac{1}{(zq;q^5)_\infty (zq^4;q^5)_\infty}$$

and

$$R_1(z;q) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_1(n,m)z^m q^n = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n}, \tag{12}$$

respectively (using notation from [8]). From this it follows that generating functions for the kth shifted Rogers-Ramanujan moment functions (4)-(5) with $j = 1$ may be given in closed form as

$$\sum_{n=0}^{\infty} \alpha_k^{(1)}(n) q^n = \frac{1}{k!}\frac{\partial^k}{\partial z^k} P_1(z;q)\Big|_{z=1} \tag{13}$$

and

$$\sum_{n=0}^{\infty} \beta_k^{(1)}(n) q^n = \frac{1}{k!}\frac{\partial^k}{\partial z^k} R_1(z;q)\Big|_{z=1}.$$

To prove the theorem, it thus suffices to show that $P_1^{(k)}(1;q) - R_1^{(k)}(1;q) \succeq 0$ for $k \geq 2$; as implied by remarks made above we will establish this for $k \geq 1$. Here and below we use the notation $F^{(j)}(1;q) := \frac{\partial^j}{\partial z^j} F(z;q)\big|_{z=1}$, and say that a q-series $F(q) := \sum_{n=0}^{\infty} c(n)q^n$ satisfies $F(q) \succeq 0$ if $c(n) \geq 0$ for all $n \geq 0$. To prove the result for $k \geq 1$, noting that the result holds for $k = 0$ by the first Rogers-Ramanujan identity (1), we proceed by induction, concluding with a combinatorial argument. To this end, we suppose the result is true for some $k \geq 0$. We compute using the definition of $P_1(z;q)$ that

$$P_1^{(1)}(z;q) = P_1(z;q)L_1(z;q),$$

where

$$L_1(z;q) := \sum_{n=1}^{\infty} \left(\frac{q^{5n-4}}{1 - zq^{5n-4}} + \frac{q^{5n-1}}{1 - zq^{5n-1}} \right),$$

and we use the notation $F^{(j)}(z;q) := \frac{\partial^j}{\partial z^j} F(z;q)$. Then

$$P_1^{(k+1)}(z;q) = \sum_{j=0}^{k} \binom{k}{j} P_1^{(j)}(z;q) L_1^{(k-j)}(z;q) \tag{14}$$

(where $F^{(0)} := F$). We compute for $j \in \mathbb{N}_0$ that

$$L_1^{(j)}(z;q) = j! \sum_{n=1}^{\infty} \left(\frac{q^{(j+1)(5n-4)}}{(1-zq^{5n-4})^{j+1}} + \frac{q^{(j+1)(5n-1)}}{(1-zq^{5n-1})^{j+1}} \right),$$

and hence $L_1^{(j)}(1;q) \succeq 0$. From this and (13) it follows that each summand in (14) satisfies $\binom{k}{j} P_1^{(j)}(1;q) L_1^{(k-j)}(1;q) \succeq 0$. Thus, to prove the desired result for $k+1$, it now suffices to show

$$P_1^{(k)}(1;q) L_1(1;q) - R_1^{(k+1)}(1;q) \succeq 0. \tag{15}$$

By inductive hypotheses, and hence $\frac{q}{1-q}(P_1^{(k)}(1;q) - R_1^{(k)}(1;q)) \succeq 0$, as well as the fact that $P_1^{(k)}(1;q)\big(L_1(1;q) - \frac{q}{1-q}\big) \succeq 0$, we have that (15) would follow from the truth of

$$\frac{q}{1-q} R_1^{(k)}(1;q) - R_1^{(k+1)}(1;q) \succeq 0 \tag{16}$$

which we now establish combinatorially.

We compute using (10) and (12) that (for $j \in \mathbb{N}_0$)

$$\frac{1}{j!} R_1^{(j)}(1;q) = \sum_{n=j}^{\infty} \frac{\binom{n}{j} q^{n^2}}{(q;q)_n} = \sum_{n=0}^{\infty} p\,(n\colon \text{super-distinct parts, } j \text{ marked parts})\, q^n.$$

To prove (16), it thus suffices to show that

$$\begin{aligned} &\frac{q}{1-q} \sum_{n=0}^{\infty} p\,(n\colon \text{super-distinct parts, } k \text{ marked parts})\, q^n \\ &\qquad - (k+1) \sum_{n=0}^{\infty} p\,(n\colon \text{super-distinct parts, } k+1 \text{ marked parts})\, q^n \succeq 0. \end{aligned} \tag{17}$$

To prove (17) and hence Theorem 1.1, we establish a combinatorial injection. Namely, we map $k+1$ copies (recalling that $k+1 \geq 1$) of a partition of n into super-distinct parts with $k+1$ marked parts into pairs $(\pi, (1^b))$ where $b \geq 1$ and π is a partition of $n-b$ with super-distinct parts and k marked parts, as follows. Suppose the $k+1$ marked parts of the starting partition are $p_1 > p_2 > \cdots > p_{k+1} \geq 1$. For each $1 \leq j \leq k+1$, we map the jth copy of this partition to $(\pi \setminus p_j, (1^{p_j}))$. Then each $\pi \setminus p_j$ has super-distinct parts with exactly k marked, and is a partition of $n - p_j$. Moreover, the pairs in the set $\{(\pi_i \setminus p_{i_j}, (1^{p_{i_j}})) \mid 1 \leq j \leq k+1\}$ are distinct, where $\{\pi_i\}$ is the set of super-distinct partitions of n with $k+1$ marked parts $p_{i_1} > \cdots > p_{i_{k+1}}$. This establishes the desired injection, proving (17) and hence Theorem 1.1 by the arguments preceding. $\square$

3. Proof of Theorem 1.2

The proof is similar to our proof of Theorem 1.1 above, however it is (somewhat surprisingly) more complicated. The truth of the identity in the case of $k = 0$ follows from the second Rogers-Ramanujan identity (and with strict equality) (2), and results implying the truth of the theorem in the case of $k = 1$ are proved in [8] (see also Section 1). It thus suffices to prove the theorem for integers $k \geq 2$, however our proof below establishes the result for $k \geq 1$. The two-variable Rogers-Ramanujan parts generating functions for $a_2(n, m)$ and $b_2(n, m)$ are given by

$$P_2(z; q) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_2(n, m) z^m q^n = \frac{1}{(zq^2; q^5)_\infty (zq^3; q^5)_\infty}$$

and

$$R_2(z; q) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_2(n, m) z^m q^n = \sum_{n=0}^{\infty} \frac{z^n q^{n^2+n}}{(q; q)_n}, \tag{18}$$

respectively (using notation from [8]). From this it follows that generating functions for the kth shifted Rogers-Ramanujan moment functions (4)-(5) with $j = 2$ may be given in closed form as

$$\sum_{n=0}^{\infty} \alpha_k^{(2)}(n) q^n = \frac{1}{k!} \frac{\partial^k}{\partial z^k} P_2(z; q)\Big|_{z=1} \tag{19}$$

and

$$\sum_{n=0}^{\infty} \beta_k^{(2)}(n) q^n = \frac{1}{k!} \frac{\partial^k}{\partial z^k} R_2(z; q)\Big|_{z=1}.$$

To prove the theorem, it thus suffices to show that $P_2^{(k)}(1; q) - R_2^{(k)}(1; q) \succeq 0$ for $k \geq 2$; as implied by remarks made above we will establish this for $k \geq 1$. To prove the result for $k \geq 1$, noting that the result holds for $k = 0$ by the second Rogers-Ramanujan identity (2), we proceed by induction, concluding with a combinatorial argument (which is more complicated than the one given in the proof of Theorem 1.1). To this end, we suppose the result is true for some $k \geq 0$. We compute using the definition of $P_2(z; q)$ that

$$P_2^{(1)}(z; q) = P_2(z; q) L_2(z; q),$$

where

$$L_2(z; q) := \sum_{n=1}^{\infty} \left(\frac{q^{5n-3}}{1 - zq^{5n-3}} + \frac{q^{5n-2}}{1 - zq^{5n-2}} \right).$$

Then

$$P_2^{(k+1)}(z; q) = \sum_{j=0}^{k} \binom{k}{j} P_2^{(j)}(z; q) L_2^{(k-j)}(z; q) \tag{20}$$

(where $F^{(0)} := F$). We compute for $j \in \mathbb{N}_0$ that

$$L_2^{(j)}(z; q) = j! \sum_{n=1}^{\infty} \left(\frac{q^{(j+1)(5n-3)}}{(1 - zq^{5n-3})^{j+1}} + \frac{q^{(j+1)(5n-2)}}{(1 - zq^{5n-2})^{j+1}} \right),$$

and hence $L_2^{(j)}(1;q) \succeq 0$. From this and (19) it follows that each summand in (20) satisfies $\binom{k}{j}P_2^{(j)}(1;q)L_2^{(k-j)}(1;q) \succeq 0$. Thus, to prove the desired result for $k+1$, it now suffices to show $P_2^{(k)}(1;q)L_2(1;q) - R_2^{(k+1)}(1;q) \succeq 0$. By inductive hypotheses, this would follow from

$$R_2^{(k)}(1;q)L_2(1;q) - R_2^{(k+1)}(1;q) \succeq 0, \tag{21}$$

which we now prove combinatorially.

We compute using (11) and (18) that (for $j \in \mathbb{N}_0$)

$$\frac{1}{j!}R_2^{(j)}(1;q) = \sum_{n=j}^{\infty} \frac{\binom{n}{j} q^{n^2+n}}{(q;q)_n} = \sum_{n=0}^{\infty} p\,(n\colon \text{super-distinct parts, no 1s, } j \text{ marked parts})\, q^n.$$

To prove (21), it thus suffices to show that

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} p\,(n\colon \text{super-distinct parts, no 1s, } k \text{ marked parts})\, q^n\right)\left(\sum_{n=1}^{\infty}\left(\frac{q^{5n-3}}{1-zq^{5n-3}}+\frac{q^{5n-2}}{1-zq^{5n-2}}\right)\right)\\ &\quad - (k+1)\sum_{n=0}^{\infty} p\,(n\colon \text{super-distinct parts, no 1s, } k+1 \text{ marked parts})\, q^n \succeq 0.\end{aligned} \tag{22}$$

To prove (22) and hence Theorem 1.2, we establish a combinatorial injection. This injection generalizes the injection defined in the proof of [8, Theorem 4.1] by Ballantine and the author (which corresponds to the case $k=0$ and $k+1=1$ in the current proof). Namely, we map $k+1$ copies (recalling that $k+1 \geq 1$) of a partition μ of n into super-distinct parts with no 1s and $k+1$ marked parts into pairs $(\pi, (a^b))$ where $a \equiv \pm 2 \pmod 5, b \geq 1$, and π is a partition of $n-ab$ with super-distinct parts, no 1s, and k marked parts. Before exactly defining this injection which involves several cases, we summarize for clarity. See also Example 3.1 below. Suppose the $k+1$ marked parts of the starting partition μ are $p_1 > p_2 > \cdots > p_{k+1} > 1$. For each $1 \leq j \leq k+1$, we map the jth copy of this partition μ to a pair of partitions $(\pi, (a^b))$ with the prescribed conditions on π, a and b by first removing the marked part p_j from μ; depending on the parity of p_j and also its residue class (mod 5), we add and subtract other parts to construct π, as well as the second element (a^b) in the target pair.

For the remainder of the proof we will express partitions as ordered sets of parts, e.g., $\mu = \{\mu_1, \mu_2, \dots, \mu_\ell\}, \ell \geq 1$, where $\mu_1 > \mu_2 > \cdots > \mu_\ell > 1$. To define our map on the $k+1$ copies of μ, we apply the map defined below in terms of an individual marked part of μ $k+1$ times, once for each marked part of μ. That is, denote the $k+1$ marked parts of μ by $p_1 > p_2 > \cdots > p_{k+1} > 1$. We will omit dependence on the index j of the marked part p_j in the notation defining the map that follows for ease of notation, and begin by denoting p_j by c. Then $c = \mu_i$ for some $1 \leq i \leq \ell$. Our map in terms of a marked part p_j which we apply $k+1$ times, once for each marked part of μ in order to obtain a suitable pair $(\pi, (a^b))$, is defined as follows. (We again refer the reader to Example 3.1.)

Define

$$x := \begin{cases} \mu_{i+1} & \text{if } i < \ell \\ 0 & \text{if } i = \ell, \end{cases}$$

and let $y = c - x$. Thus, if the marked part is not the last part of μ, y is the difference between the marked part and the next part. Otherwise, y is equal to the marked part.

Hence, $y \geq 2$. We denote by $\widetilde{\mu}$ the partition obtained from μ by removing the (jth) marked part, i.e, $\widetilde{\mu} := \mu \setminus \{c\}$ and with any marked parts in μ (other than c) remaining marked in $\widetilde{\mu}$.

<u>Case 1:</u> $c = 2t$, $t \geq 1$. Map μ to the pair $(\widetilde{\mu}, (2^t))$.

<u>Case 2:</u> $c = 2t + 1$, $t \geq 1$.

(A) If $c \equiv 2$ or $3 \pmod 5$, map μ to $(\widetilde{\mu}, (c))$.

(B) If $c \not\equiv 2$ or $3 \pmod 5$, then $c = 2t + 1 \geq 5$, i.e., $t \geq 2$, and we consider several subcases according to the size of y.

(i) If $y = 2$ or 3, then $x \neq 0$. If x is a marked part of μ, map μ to $(\widetilde{\mu} \setminus \{x\} \cup \{x+1\}, (2^t))$ and with $x+1$ marked. If x is unmarked, similarly map μ to $(\widetilde{\mu} \setminus \{x\} \cup \{x+1\}, (2^t))$ and with $x+1$ unmarked.

(ii) If $y \geq 4$ and $c \equiv 0 \pmod 5$, since c is odd, we write $c = 10j + 5 = 2(5j+2) + 1$ with $j \geq 0$. Notice that if $j = 0$, then $x = 0$. We map μ to

$$\begin{cases} (\widetilde{\mu} \setminus \{x\} \cup \{x+1\}, ((5j+2)^2)) \text{ and with } x+1 \text{ marked if } x \text{ is marked} & \text{if } x \neq 0, \\ (\widetilde{\mu} \cup (5j+3), (5j+2)) & \text{if } x = 0. \end{cases}$$

(iii) If $y \geq 4$ and $c \equiv 4 \pmod 5$, since c is odd, we write $c = 10j + 9 = 2(5j+3) + 3$ with $j \geq 0$. Map μ to

$$\begin{cases} (\widetilde{\mu} \setminus \{x\} \cup \{x+3\}, ((5j+3)^2)) \text{ and with } x+3 \text{ marked if } x \text{ is marked} & \text{if } x \neq 0, \\ (\widetilde{\mu} \cup \{3\}, ((5j+3)^2)) & \text{if } x = 0. \end{cases}$$

(iv) If $y \geq 4$ and $c \equiv 1 \pmod 5$, since c is odd, we write $c = 10j + 1$ with $j \geq 1$.
If $c = 20h + 11 = 4(5h+2) + 3$ for some $h \geq 0$, map μ to

$$\begin{cases} (\widetilde{\mu} \setminus \{x\} \cup \{x+3)\}, ((5h+2)^4)) \text{ and with } x+3 \text{ marked if } x \text{ is marked} & \text{if } h > 0, x \neq 0, \\ (\widetilde{\mu} \cup \{3\}, ((5h+2)^4)) & \text{if } h > 0, x = 0, \\ (\widetilde{\mu} \setminus \{x\} \cup \{x+2\}, (3^3)) \text{ and with } x+2 \text{ marked if } x \text{ is marked} & \text{if } h = 0, x \neq 0, \\ (\widetilde{\mu} \cup \{2\}, (3^3)) & \text{if } h = 0, x = 0. \end{cases}$$

If $c = 20h + 1$ for some $h \geq 1$, write $c = 3m + r$ with $0 \leq r \leq 2$. Note that $m \geq 7$. Moreover, if $r = 0$, then $m \equiv 7 \pmod{20}$; if $r = 1$, then $m \equiv 0 \pmod{20}$; and if $r = 2$, then $m \equiv 13 \pmod{20}$. Map μ to

$$\begin{cases} (\widetilde{\mu} \setminus \{x\} \cup \{x+r\}, (3^m)) \text{ and with } x+r \text{ marked if } x \text{ is marked} & \text{if } x \neq 0, \\ (\widetilde{\mu} \cup \{r\}, (3^m)) & \text{if } x = 0, r \neq 1, \\ (\widetilde{\mu} \cup \{5(h-1)+8, 5(h-1)+6, 5(h-1)+4\}, (5(h-1)+3)) & \text{if } x = 0, r = 1. \end{cases}$$

Remark. The anonymous referee suggested redefining the injection in this case (Case 2 (B) (iv) $c = 20h+1$), by writing $c = 3m + r$ for some $2 \leq r \leq 4$ and $m \geq 6$, and omitting the third case $(x = 0, r = 1)$ from the injection just defined above.

The cases above define the desired injection; the collection of image sets $(1 \leq j \leq k+1)$ are disjoint by inspection. Thus we have proved (22) and hence Theorem 1.2 by the arguments preceding. □

Example 3.1. We illustrate the injection defined towards the end of the proof of Theorem 1.2 in the case of $k = 1$ and $n = 14$. There are 11 partitions of $n = 14$ into super-distinct parts with no 1s, with $k + 1 = 2$ parts marked as follows:

$12^* + 2^*$	$8^* + 4^* + 2$	$7^* + 5^* + 2$
$11^* + 3^*$	$8^* + 4 + 2^*$	$7^* + 5 + 2^*$
$10^* + 4^*$	$8 + 4^* + 2^*$	$7 + 5^* + 2^*$
$9^* + 5^*$		
$8^* + 6^*$		

The injection defined in the proof of Theorem 1.2 maps $k + 1 = 2$ copies of each of these partitions to the following 22 distinct pairs of partitions $(\pi, (a^b))$, where π is a partition of $14 - ab$ into super-distinct parts with no 1s and $k = 1$ part marked, and $a \equiv \pm 2 \pmod 5$, $b \geq 1$. That is, 2 copies of the 2-marked partitions in each grid box above map to the 2 pairs of partitions in the corresponding box in the following grid:

$(2^*, (2^6)), (12^*, (2^1))$	$(4^* + 2, (2^4)), (8^* + 2, (2^2))$	$(5^* + 2, (7)), (7^* + 3, (2^2))$
$(5^*, (3^3)), (11^*, (3^1))$	$(4 + 2^*, (2^4)), (8^* + 4, (2^1))$	$(5 + 2^*, (7)), (7^* + 5, (2^1))$
$(4^*, (2^5)), (10^*, (2^2))$	$(8 + 2^*, (2^2)), (8 + 4^*, (2^1))$	$(7 + 3^*, (2^2)), (7 + 5^*, (2^1))$
$(8^*, (3^2)), (9^* + 3, (2^1))$		
$(6^*, (2^4)), (8^*, (2^3))$		

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