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Twisted Eisenstein series, cotangent-zeta sums, and quantum modular forms

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Abstract

We define twisted Eisenstein series $E_s^{\pm}(h,k;\tau)$ for $s \in \mathbb{C}$, and show how their associated period functions, initially defined on the upper half complex plane $\mathbb{H}$, have analytic continuation to all of $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$. We also use this result, as well as properties of various zeta functions, to show that certain cotangent-zeta sums behave like quantum modular forms of (complex) weight s.

1. *Introduction and statement of results*

Let $s \in \mathbb{C}$, and let h and k be integers satisfying $0 \leqslant h < k$ and $\gcd(h,k) = 1$. We define the twisted divisor functions $\sigma_s^{\pm}(h,k;n)$ for $n \in \mathbb{N}$ by

$$\sigma_s^{\pm}(h,k;n) := \sum_{\substack{dd'=n, d>0 \\ d \equiv -h \pmod{k}}} d^s \zeta_k^{\pm hd'}.$$

This function is realized as the usual divisor function $\sigma_s(n)$ in the special case $h = 0, k = 1$; that is, $\sigma_s^{\pm}(0,1;n) = \sigma_s(n)$. We define a generating function for $\sigma_s^{\pm}(h,k;n)$ by

$$\mathscr{S}_s^{\pm}(h,k;\tau) := \sum_{n=1}^{\infty} \sigma_{s-1}^{\pm}(h,k;n) q^{\frac{n}{k}}$$

where $q = e^{2\pi i \tau}$ is the usual modular variable, with $\tau \in \mathbb{H}$, the upper half complex plane, and using $\mathscr{S}_s^{\pm}$, we define the twisted Eisenstein series

$$E_s^{\pm}(h,k;\tau) := c_s \mathscr{S}_s^{\pm}(h,k;\tau) + d_s^{\pm},$$

where

$$c_s := \frac{(-2\pi i)^s}{\Gamma(s) k^{s-1}}, \qquad d_s^{\pm} := e^{-\frac{\pi i s}{2}} \cos\left(\frac{\pi s}{2}\right) Li_s(\zeta_k^{\mp h}).$$

Here and throughout, the polylogarithm function, also called Jonquière's function (see [10] and Section 2),

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

is an analytic function of z where $|z| < 1$, for any fixed $s \in \mathbb{C}$. The series above also converges when $|z| = 1$ when $\Re(s) > 1$. For any fixed $s \in \mathbb{C}$, for $|z| \geqslant 1$, the function is defined by analytic continuation. The Riemann zeta function is given by $Li_s(1) = \zeta(s)$.

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When $h = 0$ and $k = 1$, it is immediate that $E_s^+(0,1;\tau) = E_s^-(0,1;\tau) =: E_s(0,1;\tau)$ (using that $\sigma_s^+(0,1;n) = \sigma_s^-(0,1;n) = \sigma_s(n)$ and $d_s^+ = d_s^- = e^{-\pi is/2}\cos(\pi s/2)\zeta(s) =: d_s$), and in this case,

$$E_s(0,1;\tau)(d_s)^{-1} = 1 + \frac{2}{\zeta(1-s)}\sum_{n=1}^{\infty}\sigma_{s-1}(n)q^n. \tag{1.1}$$

(We have used the functional equations of the Riemann zeta function and the Gamma function, and elementary trigonometric identities, to establish (1.1)). In particular, when s is a positive even integer at least 4, then the function in (1.1) reduces to the usual modular Eisenstein series of even positive weight.

Our first result surrounds a period function for the twisted Eisenstein series $E_s^{\pm}(h,k;\tau)$ (see Theorem 1 for a precise statement). Generally speaking, period functions ψ are real analytic and satisfy a three-term relation

$$\psi(\tau) = \psi(\tau+1) + (\tau+1)^{-2s}\psi\left(\frac{\tau}{\tau+1}\right). \tag{1.2}$$

Extending the classical theory of period polynomials and modular forms, Lewis and Zagier [**11**] defined spaces of period functions for Maass cusp forms and real analytic Eisenstein series. A main result in [**11**] shows that the period functions for Maass cusp forms (with $s = 1/2 + it$ and eigenvalue $s(1-s)$ in the simplest case) are characterized by the three-term relation in (1.2) and certain growth conditions. Moreover, they show in [**11**] that the period functions, initially defined on $\mathbb{H}$, have analytic continuation to $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$, all of the complex plane except for the non-positive real axis. In the later work [**4**], Bettin and Conrey define a period function for the function defined in (1.1) (denoted by $E_s(\tau)$ [**4**, p.219]), and show that it can be analytically continued to $\mathbb{C}'$. See also [**1**], in which the authors give explicit formulas for the period functions appearing in the work of Bettin and Conrey.

Results by Bettin and Conrey on such period functions in [**4**], and later related work by Lewis and Zagier in [**12**] which gives equivalent criteria for the Riemann hypothesis in this context, are additionally motivated by connections to quantum modular forms. Loosely speaking, quantum modular forms are functions $f(x)$ defined for $x \in \mathbb{Q}$, and whose errors to modularity there, namely the functions $h_{\gamma,\kappa}(x) := f(x) - \chi_\gamma(Cx+D)^{-\kappa}f(\gamma x)$, where $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\kappa \in \frac{1}{2}\mathbb{Z}$, and $|\chi_\gamma| = 1$, extend to suitably analytic or continuous functions in $\mathbb{R}$. (See [**16**] for additional details.) There are now myriad examples and applications of quantum modular forms, arising from diverse areas of mathematics including topology, number theory, representation theory, and more (see, for example, [**6**, Chapter 21]).

A prototype example of a quantum modular form originating in [**16**] is the Dedekind sum

$$s(b,a) := \sum_{n=1}^{a-1}\left(\left(\frac{n}{a}\right)\right)\left(\left(\frac{nb}{a}\right)\right),$$

where $((x)) := x - \lfloor x \rfloor - \frac{1}{2}$ for $x \notin \mathbb{Z}$. This well-known function appears in the modular transformation of the Dedekind eta function. In [**16**], Zagier defines a normalized version of this function, namely $S : \mathbb{Q} \to \mathbb{Q}$ defined by $S(b/a) := 12s(b,a)$ (note that $s(-b,a) = -s(b,a)$), and shows that

$$S\left(\frac{a}{b}\right) + S\left(\frac{b}{a}\right) = \frac{a}{b} + \frac{b}{a} - 3\,\mathrm{sgn}\left(\frac{a}{b}\right) + \frac{1}{ab}. \tag{1.3}$$

If one replaces a/b by $x \in \mathbb{Q}$, then equation (1.3) yields the transformation of the function S under $x \mapsto -1/x$, which corresponds to the generator $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$. (Under the other generator $x \mapsto x+1$, the function S is invariant.) Thus, the function S is nearly an example of a quantum modular form of weight 0; the obstruction to calling S a true quantum modular form is the fact that the last term in its error to transformation under $x \mapsto -1/x$ shown in (1.3) depends on the numerator and denominator of the rational input separately, and thus yields a nonsmooth correction term (when viewed as a function in $\mathbb{R}$). While this renders S an *imperfect* quantum modular form (an adjective also used in [**4**]), the example is particularly elegant in that S is a discrete function which may be written down concisely as a simple finite sum, as well as the fact that it is a well-known and widely studied number theoretic function.

The aforementioned example from [**16**] involving the Dedekind sum S has since spawned additional related forms. Namely, as alluded to above, work of Bettin and Conrey in [**4**] on the period function associated to the function in (1.1), a normalized version of $E_s^{\pm}(0,1;\tau)$, led to a family of imperfect quantum modular forms c_s constructed as certain cotangent sums; this family from [**4**] contains the Dedekind sum S as a special case, namely, $12c_{-1} = 2\pi S$. See also [**3**] for additional related work. Our second main result in this paper leads to what may essentially be viewed as an infinite family of (true) quantum modular forms, arising from period functions of our twisted Eisenstein series $E_s^{\pm}(h,k;\tau)$ (see Theorem 2 for a precise statement).

We now precisely state our main results, introduced above, extending work in [**4, 11**], and relating to work in [**12**]. In what follows, we let $0 < h < k$ with $\gcd(h,k) = 1$ (as the case of $(h,k) = (0,1)$ was treated in [**4**]). We define the period function

$$\psi_s(h,k;\tau) := E_s^+(h,k;\tau) - \tau^{-s} E_s^-(h,k;-1/\tau).$$

In Theorem 1 and throughout, $\zeta(s;x)$ denotes the Hurwitz zeta function (see also Section 2), and $\left\langle {n \atop \nu} \right\rangle$ denotes the Eulerian numbers (see [**8**, p. 51 and § 6.5] or [**14**, § 26.14]). We have the following theorem.

Theorem 1. *The period function $\psi_s(h,k;\tau)$ extends to an analytic function on $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$ via the representation*

$$\psi_s(h,k;\tau)$$
$$= ie^{-\frac{\pi i s}{2}} \sin\left(\frac{\pi s}{2}\right) Li_s(\zeta_k^h)\tau^{-s} + g_s(h,k;M) + \frac{(-2\pi i)^{s-1}}{2i\Gamma(s)} \int_{(-\frac{1}{2}-2M)} \frac{\Gamma(w) Z_s(h,k;w)}{\sin(\pi(w-s))(2\pi z)^w} dw,$$

where

$$g_s(h,k;M) := e^{\frac{-\pi i s}{2}} \cos\left(\frac{\pi s}{2}\right) Li_s(\zeta_k^{-h}) + \frac{i(-2\pi i)^s}{2\Gamma(s)} \zeta_{2k}^h \csc\left(\frac{\pi h}{k}\right) \zeta\left(1-s; 1-\frac{h}{k}\right)$$
$$+ \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{2M} \frac{(2\pi i z)^n \zeta\left(1-n-s; 1-\frac{h}{k}\right)}{n!(1-\zeta_k^h)^{n+1}} \sum_{\nu=0}^{n-1} \left\langle {n \atop \nu} \right\rangle \zeta_k^{h(n-\nu)}$$

and

$$Z_s(h,k;w) := \sum_{\pm} e^{\pm\frac{\pi i s}{2}} Li_w(\zeta_k^{\pm h}) \Big(\zeta\left(1+w-s; \tfrac{h}{k}\right) e^{\pm\frac{\pi i (w-s)}{2}} - \zeta\left(1+w-s; 1-\tfrac{h}{k}\right) e^{\mp\frac{\pi i (w-s)}{2}} \Big),$$

and M is any integer greater than $-\frac{1}{2}\min(0, \mathrm{Re}(s-1))$.

Moreover, $\widetilde{\psi}_s^{\pm}(h,k;\tau) := E_s^{\pm}(h,k;k\tau) - \tau^{-s} E_s^-(h,k;-k/\tau)$ satisfies the three-term period relation

$$\widetilde{\psi}_s^{\pm}(h,k;\tau) - \widetilde{\psi}_s^{\pm}(h,k;\tau+1) = (\tau+1)^{-s} \widetilde{\psi}_s^-\Big(h,k;\tfrac{\tau}{\tau+1}\Big).$$

Our next theorem, which is partially proved by using Theorem 1, essentially establishes an infinite family of (true) quantum modular forms, arising from the cotangent-zeta sums defined for $a, b \in \mathbb{N}$ with $\gcd(a,b)=1$, and $a \not\equiv b \pmod{k}$, by

$$\mathfrak{c}_s\Big(h,k;\frac{a}{b}\Big) = \mathfrak{c}_s(h,k;a,b) := b^{s-1}\sum_{\ell=0}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{a}{b}\left(\frac{h}{k}+\ell\right)\right)\right)\zeta\Big(1-s;1-\frac{h+\ell k}{bk}\Big). \tag{1.4}$$

Theorem 2. *The cotangent-zeta sums $\mathfrak{c}_s(h,k;\frac{a}{b})$ satisfy the following weight s reciprocity relationship:*

$$\begin{aligned}
&\mathfrak{c}_s\Big(h,k;\frac{a}{b}\Big) + \Big(\frac{b}{a}\Big)^s \mathfrak{c}_s\Big(h,k;\frac{b}{a}\Big)\\
&\qquad = \frac{ie^{\frac{\pi i s}{2}}}{(2\pi)^{s-1}\Gamma(1-s)\sin(\pi s)}\psi_s\Big(h,k;\frac{a}{b}\Big)\\
&\qquad\quad - \frac{e^{\frac{\pi i s}{2}}}{\sin(\pi s)}\left(1+\Big(\frac{b}{a}\Big)^s e^{-\pi i s}\right)\cos\Big(\frac{\pi s}{2}\Big)\big(\zeta\big(1-s;\tfrac{h}{k}\big)-\zeta\big(1-s;1-\tfrac{h}{k}\big)\big).
\end{aligned}\tag{1.5}$$

Moreover, upon replacing a/b by the variable z, the right-hand side of (1.5) extends to an analytic function (in z) in $\mathbb{C}'$, all of the complex plane except for the nonpositive real axis.

Remark. Theorem 2 essentially shows that the cotangent-zeta sums $\mathfrak{c}_s(h,k;a/b)$ (when viewed as functions in a/b) are quantum modular forms of weight s, which may in general be complex; in particular, the real and imaginary parts of the errors to transformation on the right-hand side of (1.5) extend to smooth functions in $\mathbb{R}_{>0}$. The right-hand side of (1.5) additionally possesses the stronger property that it extends to an analytic function in all of $\mathbb{C}'$ (upon replacing a/b by z, and viewing as a function of z).

Before proving our main results, we illustrate Theorem 2 and the cotangent-zeta sums involved in some special cases.

Example 1. Let $s=2$, and $h=1, k=4$. We plot in Figure 1 the cotangent-zeta function $\mathfrak{c}_2(1,4;a/b)$ for $1 \leqslant a \leqslant 100$ and $1 \leqslant b \leqslant 100$, with $\gcd(a,b)=1$ and $a \not\equiv b \pmod{4}$. In Figure 2, we plot the modular error possessed by this function on $\mathbb{Q}$, and see that it is 'smoothed out'.

Example 2. As a second example, we let $s = 3.3+1.2i$, and $h=5, k=17$. We plot in Figure 3 the real part of the associated cotangent-zeta function, namely $\Re(c_{3.3+1.2i}(5,17;a/b))$, for $1 \leqslant a \leqslant 100$ and $1 \leqslant b \leqslant 100$, with $\gcd(a,b)=1$ and $a \not\equiv b \pmod{17}$. (We omit the plot of the imaginary part, as it is similar in appearance to the plot of the real part shown in Figure 3.) In Figure 4, we plot the real and imaginary parts of the modular error possessed by $\mathfrak{c}_{3.3+1.2i}(5,17;a/b)$ on $\mathbb{Q}$, and see that it is 'smoothed out'.

The remainder of the paper is structured as follows. In Section 2, we state some basic information on certain zeta functions appearing throughout. In Section 3, we prove Theorem 1, and in Section 4, we prove Theorem 2.

2. *Zeta functions*

In this section, we review some basic information on certain zeta functions, and relations between them. We begin with the Hurwitz zeta function

$$\zeta(s;x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s},$$

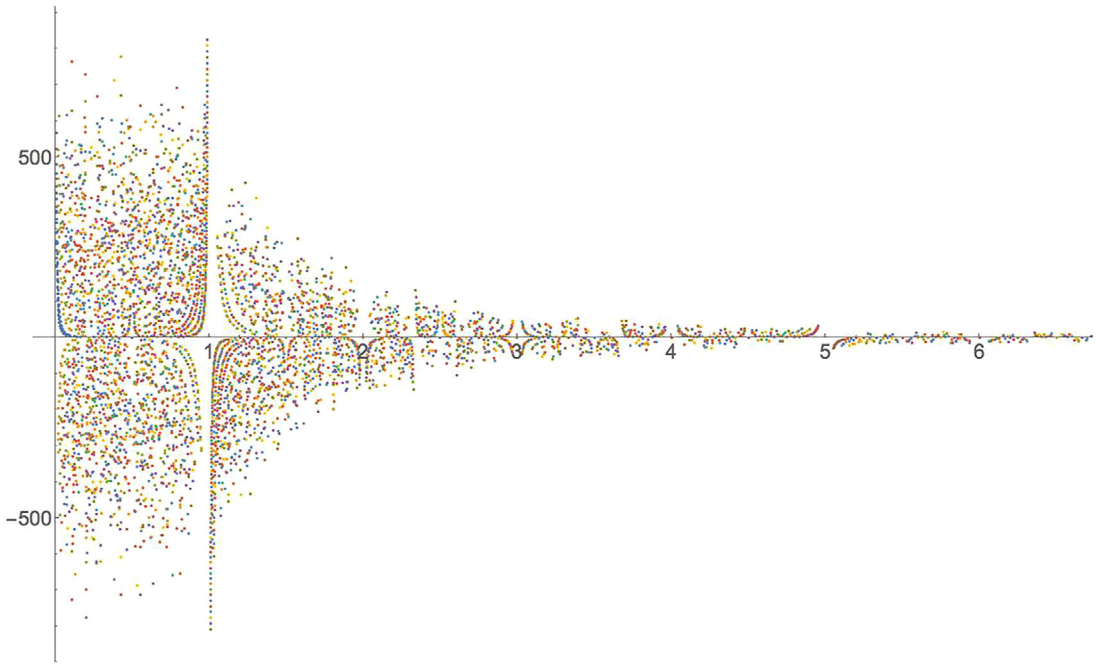

FIGURE 1 (colour online). *The cotangent-zeta function* $\mathfrak{c}_2(1, 4; \frac{a}{b})$.

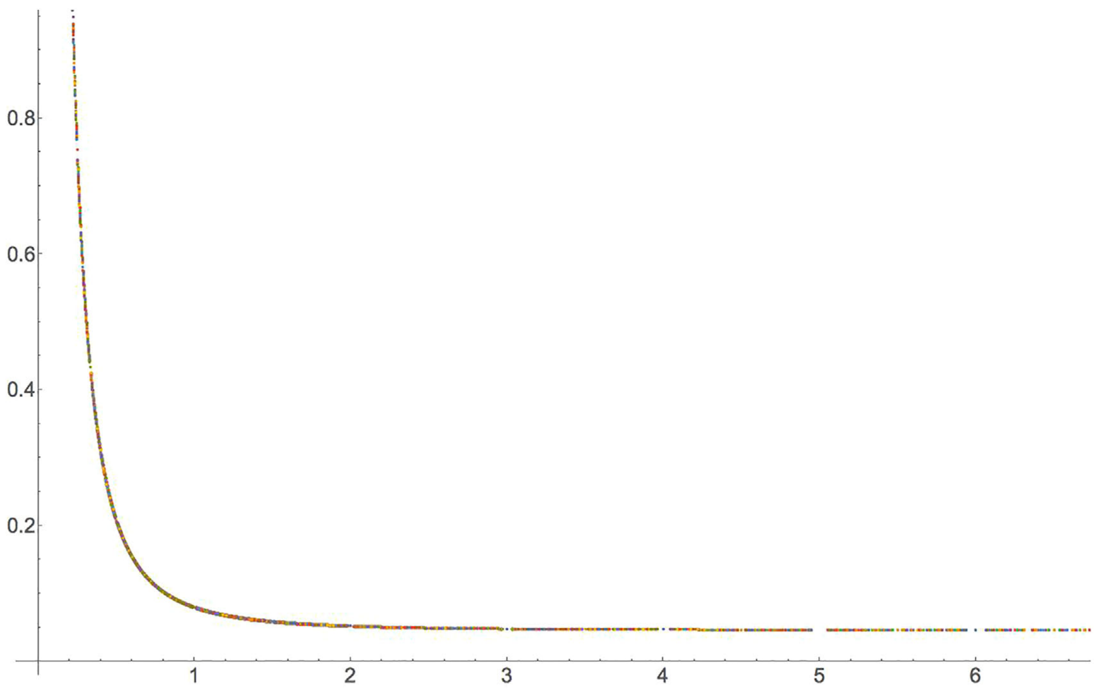

FIGURE 2 (colour online). *The quantum modular error* $\mathfrak{c}_2(1, 4; \frac{a}{b}) + (\frac{b}{a})^2 \mathfrak{c}_2(1, 4; \frac{b}{a})$.

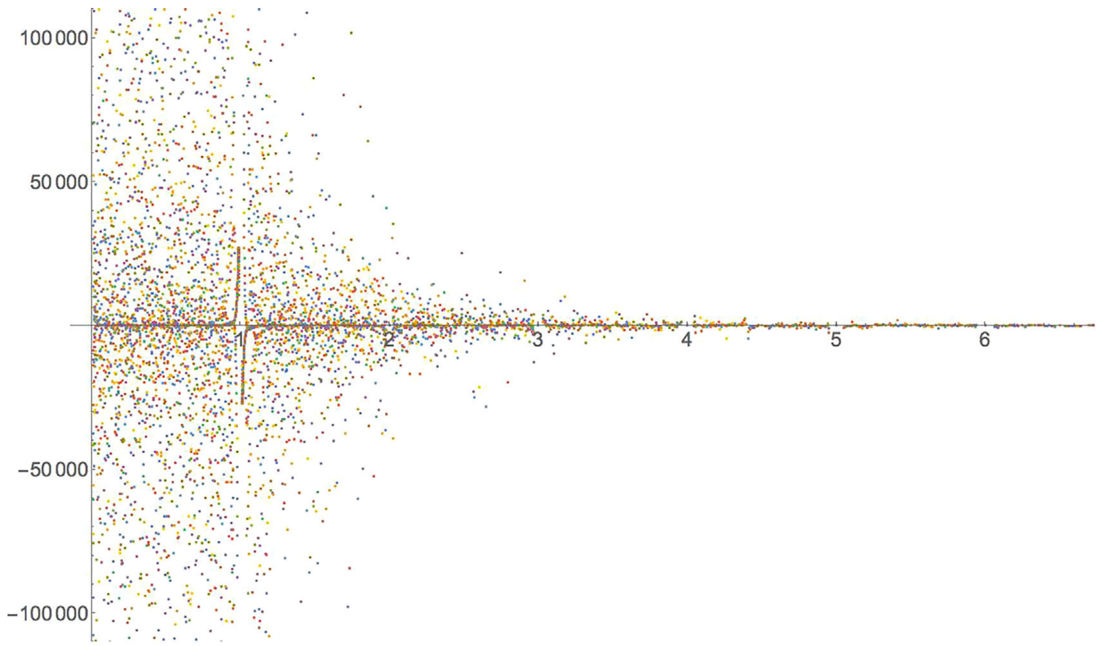

FIGURE 3 (colour online). *The real part of a cotangent-zeta function,* $\Re(\mathfrak{c}_{3.3+1.2i}(5, 17; \frac{a}{b}))$.

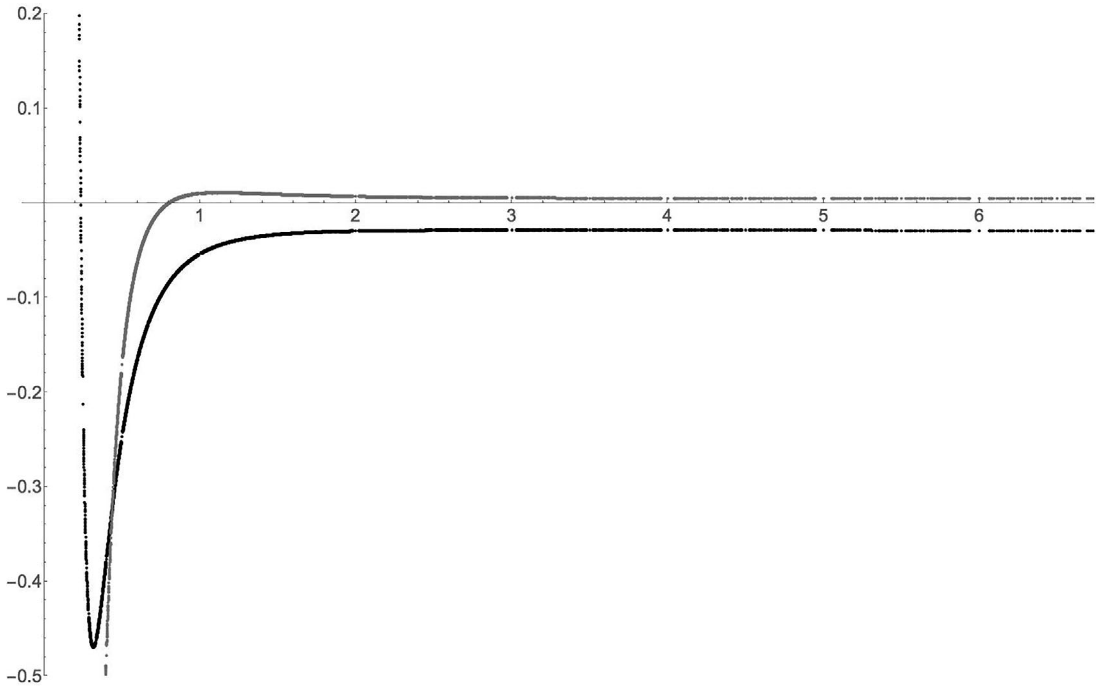

FIGURE 4. *Real and imaginary quantum modular errors,* $\Re(\mathfrak{c}_{3.3+1.2i}(5, 17; \frac{a}{b}) + (\frac{b}{a})^{3.3+1.2i} \mathfrak{c}_{3.3+1.2i}(5, 17; \frac{b}{a}))$ *(shown in black), and* $\Im(\mathfrak{c}_{3.3+1.2i}(5, 17; \frac{a}{b}) + (\frac{b}{a})^{3.3+1.2i}\mathfrak{c}_{3.3+1.2i}(5, 17; \frac{b}{a}))$ *(shown in gray).*

which is defined for $\Re(s) > 1$ and $x \notin -\mathbb{N}_0$, and which can be extended to a meromorphic function for all $s \in \mathbb{C} \setminus \{1\}$, with its only singularity being a simple pole at $s = 1$ with residue 1. The Riemann zeta function is given by $\zeta(s; 1) = \zeta(s)$. As a function of x with $s \neq 1$ fixed, $\zeta(s; x)$ is analytic in $\Re(x) > 0$.

The polylogarithm, also called Jonquière's function [10],

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

is an analytic function of z where $|z| < 1$, for any fixed $s \in \mathbb{C}$. The series above also converges when $|z| = 1$ when $\Re(s) > 1$. For any fixed $s \in \mathbb{C}$, for $|z| \geqslant 1$, the function is defined by analytic continuation. The Riemann zeta function is given by $Li_s(1) = \zeta(s)$.

We will make use of Jonquière's relation between these two functions [9, Section 1.11; 10]

$$\zeta\big(s; \tfrac{h}{k}\big) = \frac{i\Gamma(1-s)}{(2\pi)^{1-s}} \Big(e^{\frac{\pi i s}{2}} Li_{1-s}\big(\zeta_k^h\big) - e^{-\frac{\pi i s}{2}} Li_{1-s}\big(\zeta_k^{-h}\big) \Big), \tag{2.1}$$

as well as its inversion (see also [5, Appendix B] for related identities)

$$Li_s\big(\zeta_k^h\big) = \frac{i\Gamma(1-s)}{(2\pi)^{1-s}} \Big(e^{-\frac{\pi i s}{2}} \zeta\big(1-s; \tfrac{h}{k}\big) - e^{\frac{\pi i s}{2}} \zeta\big(1-s; 1-\tfrac{h}{k}\big) \Big). \tag{2.2}$$

When $s = -n$, $n \in \mathbb{N}$, is a negative integer, the polylogarithm can be evaluated explicitly as

$$Li_{-n}(z) = \frac{\displaystyle\sum_{\nu=0}^{n-1} \left\langle {n \atop \nu} \right\rangle z^{n-\nu}}{(1-z)^{n+1}}, \tag{2.3}$$

where $\left\langle {n \atop \nu} \right\rangle$ are the Eulerian numbers (not to be confused with Euler numbers) (see [8, p. 51 and § 6.5] or [14, § 26.14]). The identity in (2.3) may be verified using that $Li_0(z) = z/(1-z)$, and that for $n \in \mathbb{N}_0$, $z^n \frac{d^n}{dz^n} Li_0(z) = Li_{-n}(z)$. (One reference containing some results related to the case $s = -n$ is [2].)

3. *Proof of Theorem 1*

The three-term relation given for $\widetilde{\psi}_s(h, k; \tau)$ follows after a short calculation using the definitions of the functions $E_s^{\pm}(h, k; \tau)$. We devote the rest of this section to proving the analytic representation of the function $\psi_s(h, k; \tau)$ given in Theorem 1. We begin by establishing the following lemma.

Lemma 3. *Assume the hypotheses above, and let*

$$f(w) = f_{s,h,k}(w) := \Gamma(w) Li_w(\zeta_k^h) \zeta\big(1 + w - s; 1 - \tfrac{h}{k}\big).$$

Then we have that

$$\operatorname*{Res}_{w=0} f(w) = \frac{i}{2} \zeta_{2k}^h \csc\left(\frac{\pi h}{k}\right) \zeta\big(1-s; 1-\tfrac{h}{k}\big),$$

$$\operatorname*{Res}_{w=s} f(w) = \Gamma(s) Li_s\big(\zeta_k^h\big),$$

and for $n \in \mathbb{N}$,

$$\operatorname*{Res}_{w=-n} f(w) = \frac{(-1)^n}{n!} \zeta\big(1-n-s; 1-\tfrac{h}{k}\big) \frac{\displaystyle\sum_{\ell=0}^{n-1} \left\langle {n \atop \ell} \right\rangle \zeta_k^{h(n-\ell)}}{(1-\zeta_k^h)^{n+1}}.$$

Proof. The first result follows from the fact that $\Gamma(w)$ has a simple pole at $w = 0$ with residue 1, and that $Li_0(\zeta_k^h) = \zeta_k^h/(1-\zeta_k^h) = i\zeta_{2k}^h \csc(\pi h/k)/2$. The second result follows from the fact that $\zeta(1+w-s; 1-\frac{h}{k})$ has a simple pole at $w = s$ with residue 1. The last result follows from the fact that $\Gamma(w)$ has a simple pole at $-n$ with residue $(-1)^n/n!$, and (2.3). $\square$

Resuming the proof of Theorem 1, we proceed in a similar manner to the proof of [**4**, Theorem 1]. We have that

$$
\begin{aligned}
\mathscr{S}_s^{\pm}(h,k;\tau) &= \sum_{n=1}^{\infty} \sigma_{s-1}^{\pm}(h,k;n) e^{\frac{2\pi i n\tau}{k}} \\
&= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \sigma_{s-1}^{\pm}(h,k;n) \int_{(2+\max(0,\Re(s-1)))} \Gamma(w) \left(\frac{-2\pi i n\tau}{k}\right)^{-w} dw \\
&= \frac{1}{2\pi i} \int_{(2+\max(0,\Re(s-1)))} \Gamma(w) e^{\frac{\pi i w}{2}} \left(\frac{2\pi\tau}{k}\right)^{-w} \sum_{n=1}^{\infty} \sigma_{s-1}^{\pm}(h,k;n) n^{-w} dw \\
&= \frac{k^{s-1}}{2\pi i} \int_{(2+\max(0,\Re(s-1)))} \Gamma(w) e^{\frac{\pi i w}{2}} (2\pi\tau)^{-w} Li_w\big(\zeta_k^{\pm h}\big) \zeta\big(1+w-s; 1-\tfrac{h}{k}\big) dw,
\end{aligned} \tag{3.1}
$$

where we have used that

$$
\sum_{n=1}^{\infty} \sigma_{s-1}^{\pm}(h,k;n) n^{-w} = k^{s-1-w} Li_w\big(\zeta_k^{\pm h}\big) \zeta\big(1+w-s; 1-\tfrac{h}{k}\big),
$$

a fact that is easily verifiable using the definitions of the functions involved. In particular, we have that

$$
\begin{aligned}
\mathscr{S}_s^{+}(h,k;\tau) = \frac{k^{s-1}}{2\pi i} &\int_{(-\frac{1}{2}-2M)} \Gamma(w) e^{\frac{\pi i w}{2}} (2\pi\tau)^{-w} Li_w\big(\zeta_k^{h}\big) \zeta\big(1+w-s; 1-\tfrac{h}{k}\big) dw \\
&+ k^{s-1} r_{s-1,M}^{h,k}(\tau),
\end{aligned} \tag{3.2}
$$

where $r_{s-1,M}^{h,k}(\tau)$ is the sum of the residues of the integrand (excluding the factor in front of the integral) in the vertical strip between $(-\frac{1}{2}-2M)$ and $(2+\max(0,\Re(s-1)))$, where M is any integer greater than $-\frac{1}{2}\min(0,\Re(s-1))$.

Specifically, using Lemma 3, we have that

$$
\begin{aligned}
r_{s-1,M}^{h,k}(\tau) = \frac{i}{2}\zeta_{2k}^h \csc\left(\frac{\pi h}{k}\right) \zeta\big(1-s; 1-\tfrac{h}{k}\big) &+ \Gamma(s) Li_s(\zeta_k^h) e^{\pi i s/2} (2\pi\tau)^{-s} \\
&+ \sum_{n=1}^{2M} \frac{(2\pi i\tau)^n \zeta(1-n-s; 1-\frac{h}{k})}{n!(1-\zeta_k^h)^{n+1}} \sum_{\nu=0}^{n-1} \left\langle {n \atop \nu} \right\rangle \zeta_k^{h(n-\nu)}.
\end{aligned} \tag{3.3}
$$

From (3.1), we also have that

$$
\frac{1}{\tau^s} \mathscr{S}_s^{-}(h,k;-1/\tau) = \frac{k^{s-1}}{2\pi i \tau^s} \int_{(2+\max(0,\Re(s-1)))} \Gamma(w) e^{-\frac{\pi i w}{2}} \left(\frac{2\pi}{\tau}\right)^{-w} Li_w\big(\zeta_k^{-h}\big) \zeta\big(1+w-s; 1-\tfrac{h}{k}\big) dw.
$$

$$\tag{3.4}$$

We employ (2.1) and (2.2) and find (also using the functional equation for the Γ-function) that (3.4) equals

$$\begin{aligned}
&-\frac{k^{s-1}}{4\pi}\int_{(-1+\min(0,\Re(s-1)))}\frac{\Gamma(w)(2\pi\tau)^{-w}}{\sin(\pi(1+w-s))}e^{\frac{\pi i(1+w-s)}{2}}\\
&\qquad\times\left(i^{1+w-s}\zeta\left(1+w-s;1-\tfrac{h}{k}\right)+i^{-1-w+s}\zeta\left(1+w-s;\tfrac{h}{k}\right)\right)\left(i^{-w}Li_w\left(\zeta_k^{-h}\right)+i^{w}Li_w\left(\zeta_k^{h}\right)\right)dw\\
&=-\frac{k^{s-1}}{4\pi}\int_{(-\frac12-2M)}\frac{\Gamma(w)(2\pi\tau)^{-w}}{\sin(\pi(1+w-s))}e^{\frac{\pi i(1+w-s)}{2}}\\
&\qquad\times\left(i^{1+w-s}\zeta\left(1+w-s;1-\tfrac{h}{k}\right)+i^{-1-w+s}\zeta\left(1+w-s;\tfrac{h}{k}\right)\right)\left(i^{-w}Li_w\left(\zeta_k^{-h}\right)+i^{w}Li_w\left(\zeta_k^{h}\right)\right)dw,
\end{aligned}\tag{3.5}$$

where we have also used that there is no residue of the integrand to the left of $-1+\min(0,\Re(s-1))$.

Subtracting (3.5) from (3.2), we obtain

$$\begin{aligned}
&\mathscr{S}_s^{+}(h,k;\tau)-\frac{1}{\tau^s}\mathscr{S}_s^{-}(h,k;-1/\tau)\\
&\quad=\frac{k^{s-1}}{2\pi i}\int_{(-\frac12-2M)}\Gamma(w)e^{\frac{\pi i w}{2}}(2\pi\tau)^{-w}Li_w\left(\zeta_k^{h}\right)\zeta\left(1+w-s;1-\tfrac{h}{k}\right)dw+k^{s-1}r_{s-1,M}^{h,k}(\tau)\\
&\qquad+\frac{k^{s-1}}{4\pi}\int_{(-\frac12-2M)}\frac{\Gamma(w)(2\pi\tau)^{-w}}{\sin(\pi(1+w-s))}e^{\frac{\pi i(1+w-s)}{2}}\\
&\quad\times\left(i^{1+w-s}\zeta\left(1+w-s;1-\tfrac{h}{k}\right)+i^{-1-w+s}\zeta\left(1+w-s;\tfrac{h}{k}\right)\right)\left(i^{-w}Li_w\left(\zeta_k^{-h}\right)+i^{w}Li_w\left(\zeta_k^{h}\right)\right)dw\\
&\quad=\frac{k^{s-1}}{2\pi i}\int_{(-\frac12-2M)}\Gamma(w)e^{\frac{\pi i w}{2}}(2\pi\tau)^{-w}Li_w\left(\zeta_k^{h}\right)\zeta\left(1+w-s;1-\tfrac{h}{k}\right)\\
&\quad\times\left(1+\frac{ie^{\pi i(1+w-s)}}{2\sin(\pi(1+w-s))}\right)dw+k^{s-1}r_{s-1,M}^{h,k}(\tau)+\frac{k^{s-1}}{4\pi}\int_{(-\frac12-2M)}\frac{\Gamma(w)(2\pi\tau)^{-w}e^{\frac{\pi i w}{2}}}{\sin(\pi(1+w-s))}\\
&\quad\times\Bigg(i^{-2(s-1)}\zeta\left(1+w-s;1-\tfrac{h}{k}\right)Li_w\left(\zeta_k^{-h}\right)\\
&\qquad+i^{-2w}\zeta\left(1+w-s;\tfrac{h}{k}\right)Li_w\left(\zeta_k^{-h}\right)+\zeta\left(1+w-s;\tfrac{h}{k}\right)Li_w\left(\zeta_k^{h}\right)\Bigg)dw.
\end{aligned}\tag{3.6}$$

Now we simplify

$$1+\frac{ie^{\pi i(1+w-s)}}{2\sin(\pi(1+w-s))}=\frac{ie^{-\pi i(1+w-s)}}{2\sin(\pi(1+w-s))}$$

so that (3.6) (after some further simplifications) equals

$$\begin{aligned}
&\frac{k^{s-1}}{4\pi}\int_{(-\frac12-2M)}\frac{\Gamma(w)(2\pi\tau)^{-w}}{\sin(\pi(1+w-s))}\left(\sum_{\pm}e^{\pm\frac{\pi i(s-1)}{2}}Li_w(\zeta_k^{\pm h})\left(\zeta\left(1+w-s;\tfrac{h}{k}\right)e^{\pm\pi i\frac{(1+w-s)}{2}}\right.\right.\\
&\qquad\left.\left.+\zeta\left(1+w-s;1-\tfrac{h}{k}\right)e^{\mp\pi i\frac{(1+w-s)}{2}}\right)\right)dw+k^{s-1}r_{s-1,M}^{h,k}(\tau).
\end{aligned}\tag{3.7}$$

The analytic representation of $\psi_s(h,k;\tau)$ given in Theorem 1 now follows from (3.7), (3.3), and the definitions of $E_s^{\pm}(h,k;\tau)$, after some additional simplifications.

4. *Proof of Theorem* 2

Assuming the truth of (1.5), we apply Theorem 1 to it, which yields the analytic properties claimed in Theorem 2. We devote the rest of this section to proving (1.5). The method of proof we employ below is straightforward, albeit lengthy and technical, and is inspired by older work of Chapman in [**7**]. We have chosen this method due to its explicit and tractable nature. Using Lipschitz summation [**13, 15**], with $\Re(s) > 1$, we find that

$$\begin{aligned}\mathscr{S}_s^{\pm}(h,k;\tau) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(kn-h)^{s-1}\zeta_k^{\pm hm}q^{\frac{m(kn-h)}{k}}\\ &= k^{s-1}\sum_{m=1}^{\infty}\zeta_k^{\pm hm}\sum_{n=1}^{\infty}\left(n-\frac{h}{k}\right)^{s-1}q^{m\left(n-\frac{h}{k}\right)}\\ &= k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\frac{\zeta_k^{h(n\pm m)}}{(m\tau+n)^s}.\end{aligned}$$

Thus, we have that

$$\begin{aligned}&\mathscr{S}_s^{+}(h,k;\tau) - \tau^{-s}\mathscr{S}_s^{-}(h,k;-1/\tau)\\ &= k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\left(\sum_{m\in\mathbb{N}}\frac{\zeta_k^{hm}}{(m\tau)^s} - \frac{\zeta_k^{-hm}}{(-m)^s} + \sum_{m,n\in\mathbb{N}}\left(\sum_{\pm}\frac{\zeta_k^{h(\pm n+m)}}{(m\tau\pm n)^s} - \sum_{\pm}\frac{\zeta_k^{h(\pm n-m)}}{(\pm n\tau - m)^s}\right)\right)\\ &= k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\left(\tau^{-s}Li_s\left(\zeta_k^{h}\right) - e^{-\pi i s}Li_s\left(\zeta_k^{-h}\right) + G_s(h,k;\tau)\right),\end{aligned}\tag{4.1}$$

where

$$G_s(h,k;\tau) := \sum_{m,n\in\mathbb{N}}\frac{\zeta_k^{h(n+m)} - e^{-\pi i s}\zeta_k^{-h(n+m)}}{(m\tau+n)^s}.$$

Using (4.1) and the definition of $E_s^{\pm}$, we have that

$$\psi_s(h,k;\tau) = G_s(h,k;\tau) + ie^{-\frac{\pi i s}{2}}\sin\left(\frac{\pi s}{2}\right)\left(Li_s\left(\zeta_k^{-h}\right) + \tau^{-s}Li_s\left(\zeta_k^{h}\right)\right).\tag{4.2}$$

We let $\tau \to \frac{a}{b}$, where $a,b\in\mathbb{N}$, and $\gcd(a,b) = 1$. Let

$$G_s(h,k;a,b) = b^{-s}\lim_{\tau\to\frac{a}{b}}G_s(h,k;\tau) = \sum_{m,n\in\mathbb{N}}\frac{\zeta_k^{h(n+m)} - e^{-\pi i s}\zeta_k^{-h(n+m)}}{(ma+nb)^s}.$$

Then $G_s(h,k;a,b)$ equals

$$\sum_{j,\ell=1}^{k}(\zeta_k^{h(\ell+j)} - e^{-\pi i s}\zeta_k^{-h(\ell+j)})\sum_{m,n\in\mathbb{N}}\frac{1}{((j+k(m-1))a + (\ell+k(n-1))b)^s} = \sum_{n=1}^{\infty}c_n n^{-s},$$

where c_n is the coefficient of X^n in

$$\sum_{j,\ell=1}^{k}(\zeta_k^{h(\ell+j)}-e^{-\pi is}\zeta_k^{-h(\ell+j)})\sum_{m,n\in\mathbb{N}}X^{(j+k(m-1))a+(\ell+k(n-1))b}.$$

This factors as

$$\begin{aligned}&X^{-ka-kb}\left(\sum_{j,\ell=1}^{k}(\zeta_k^{h(\ell+j)}-e^{-\pi is}\zeta_k^{-h(\ell+j)})X^{ja+\ell b}\right)\left(\sum_{m,n\in\mathbb{N}}X^{k(ma+nb)}\right)\\&=X^{b+a}\left(\frac{\zeta_k^{2h}}{(1-\zeta_k^hX^b)(1-\zeta_k^hX^a)}-e^{-\pi is}\frac{\zeta_k^{-2h}}{(1-\zeta_k^{-h}X^a)(1-\zeta_k^{-h}X^b)}\right)\\&=\Phi_+(X)+\Phi_-(X),\end{aligned}$$

where

$$\Phi_\pm(X)=\Phi_\pm(a,b,h,k,s;X):=\frac{X^{a+b}\zeta_k^{\pm2h}\widetilde{\kappa}_\pm(s)}{(1-\zeta_k^{\pm h}X^a)(1-\zeta_k^{\pm h}X^b)}\tag{4.3}$$

with

$$\widetilde{\kappa}_\pm(s):=\begin{cases}1, & +,\\ -e^{-\pi is}, & -.\end{cases}$$

Because $a\not\equiv b \pmod k$, a short calculation reveals that $\Phi_\pm$ may be decomposed as

$$\Phi_\pm(X)=\sum_{\ell=0}^{a-1}\frac{T_\ell^\pm}{1-\zeta_{ka}^{\pm h}\zeta_a^\ell X}+\sum_{r=0}^{b-1}\frac{U_r^\pm}{1-\zeta_{kb}^{\pm h}\zeta_b^rX}\tag{4.4}$$

for some constants $T_\ell^\pm$ and $U_r^\pm$ (which additionally depend on h,k,a,b and s, parameters which we suppress from notation for ease of notation); we will now explicitly determine these constants.

Fix an integer ℓ satisfying $0\leqslant\ell\leqslant a-1$. Then we have from (4.4) that

$$\lim_{X\to\zeta_{ka}^{\mp h}\zeta_a^{-\ell}}(1-\zeta_{ka}^{\pm h}\zeta_a^\ell X)\Phi_\pm(X)=T_\ell^\pm.$$

On the other hand, we compute from (4.3) that

$$\lim_{X\to\zeta_{ka}^{\mp h}\zeta_a^{-\ell}}(1-\zeta_{ka}^{\pm h}\zeta_a^\ell X)\Phi_\pm(X)=\frac{1}{a}\frac{\widetilde{\kappa}_\pm(s)\zeta_{ka}^{\mp hb}\zeta_a^{-b\ell}\zeta_k^{\pm h}}{1-\zeta_k^{\pm h}\zeta_{ka}^{\mp hb}\zeta_a^{-\ell b}}.$$

Thus, after using the definition of csc, that it is an odd function, the fact that $e^{i\theta}\csc(\theta)-\cot(\theta)=i$, and that cot is an odd function, we have that

$$\begin{aligned}T_\ell^\pm&=\frac{\widetilde{\kappa}_\pm(s)}{2ia}\zeta_{2k}^{\pm h}\zeta_{2ka}^{\mp hb}\zeta_{2a}^{-\ell b}\csc\left(\pi\left(\mp\frac{h}{k}\pm\frac{hb}{ka}+\frac{\ell b}{a}\right)\right)\\&=-\frac{\widetilde{\kappa}_\pm(s)}{2ia}\left(\cot\left(\pi\left(\pm\frac{h}{k}\mp\frac{hb}{ka}-\frac{\ell b}{a}\right)\right)+i\right)\\&=\widetilde{T}_\ell^\pm\mp\frac{\kappa_\pm(s)}{2a},\end{aligned}$$

where

$$\widetilde{T}_\ell^\pm:=\frac{\kappa_\pm(s)}{2ia}\cot\left(\pi\left(-\frac{h}{k}+\frac{hb}{ka}\pm\frac{\ell b}{a}\right)\right)$$

and

$$\kappa_{\pm}(s) := \begin{cases} 1, & +, \\ e^{-\pi i s}, & -. \end{cases}$$

A nearly identical calculation shows that

$$U_r^{\pm} = \widetilde{U}_r^{\pm} \mp \frac{\kappa_{\pm}(s)}{2b}$$

where

$$\widetilde{U}_r^{\pm} := \frac{\kappa_{\pm}(s)}{2ib} \cot\left(\pi\left(-\frac{h}{k} + \frac{ha}{kb} \pm \frac{ra}{b}\right)\right).$$

Thus, we have that

$$\begin{aligned}
\Phi_+(X) + \Phi_-(X) &= \sum_{\ell=0}^{a-1}\sum_{\pm} \frac{T_\ell^{\pm}}{1-\zeta_{ka}^{\pm h}\zeta_a^{\ell}X} + \sum_{r=0}^{b-1}\sum_{\pm} \frac{U_r^{\pm}}{1-\zeta_{kb}^{\pm h}\zeta_b^{r}X} \\
&= \sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{a-1}\sum_{\pm} \widetilde{T}_\ell^{\pm}\left(\zeta_{ka}^{\pm h}\zeta_a^{\ell}\right)^n + \sum_{r=0}^{b-1}\sum_{\pm}\widetilde{U}_r^{\pm}\left(\zeta_{kb}^{\pm h}\zeta_b^{r}\right)^n\right)X^n \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{1}{2a}\sum_{\ell=0}^{a-1}\sum_{\pm} \mp\kappa_{\pm}(s)\left(\zeta_{ka}^{\pm h}\zeta_a^{\ell}\right)^n + \frac{1}{2b}\sum_{r=0}^{b-1}\sum_{\pm}\mp\kappa_{\pm}(s)\left(\zeta_{kb}^{\pm h}\zeta_b^{r}\right)^n\right)X^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{a-1}\sum_{\pm} \widetilde{T}_\ell^{\pm}\left(\zeta_{ka}^{\pm h}\zeta_a^{\ell}\right)^n + \sum_{r=0}^{b-1}\sum_{\pm}\widetilde{U}_r^{\pm}\left(\zeta_{kb}^{\pm h}\zeta_b^{r}\right)^n\right)X^n \\
&\quad + \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{\pm}\mp\kappa_{\pm}(s)\zeta_k^{\pm hn}\right)\left(X^{na}+X^{nb}\right).
\end{aligned}$$

Hence, we have shown that

$$\begin{aligned}
G_s(h,k,a,b) = &\sum_{\ell=0}^{a-1}\sum_{\pm}\widetilde{T}_\ell^{\pm} Li_s\left(\zeta_{ka}^{\pm h}\zeta_a^{\ell}\right) + \sum_{r=0}^{b-1}\sum_{\pm}\widetilde{U}_r^{\pm} Li_s\left(\zeta_{kb}^{\pm h}\zeta_b^{r}\right) \\
&+ \frac{(a^{-s}+b^{-s})}{2}\sum_{\pm}\mp\kappa_{\pm}(s) Li_s\left(\zeta_k^{\pm h}\right). \qquad (4.5)
\end{aligned}$$

To this expression in (4.5), we wish to apply (2.2). The identity in (2.2) is directly applicable to $Li_s(\zeta_k^h)$, and also to $Li_s(\zeta_{ak}^{h+\ell k})$ and $Li_s(\zeta_{bk}^{h+rk})$ for all $0 \leqslant \ell \leqslant a-1$ and $0 \leqslant r \leqslant b-1$, respectively; it is also applicable to $Li_s(\zeta_{ak}^{-h+\ell k})$ and $Li_s(\zeta_{bk}^{-h+rk})$ for all $1 \leqslant \ell \leqslant a-1$ and $1 \leqslant r \leqslant b-1$, respectively. For the other polylogarithm functions that appear, we re-write the functions as follows:

$$Li_s\left(\zeta_k^{-h}\right) = Li_s\left(\zeta_k^{k-h}\right),\ Li_s\left(\zeta_{ak}^{-h}\right) = Li_s\left(\zeta_{ak}^{ak-h}\right),\ \text{and } Li_s\left(\zeta_{bk}^{-h}\right) = Li_s\left(\zeta_{bk}^{bk-h}\right),$$

so that (2.2) is applicable. Precisely, we define

$$\alpha_\ell^{\pm} = \alpha_{\ell,a,k}^{\pm} := \begin{cases} ak, & \text{if } - \text{ and } \ell = 0, \\ 0, & \text{else,} \end{cases} \qquad \beta_r^{\pm} = \beta_{r,b,k}^{\pm} := \begin{cases} bk, & \text{if } - \text{ and } r = 0, \\ 0, & \text{else,} \end{cases}$$

$$\delta^{\pm} = \delta_k^{\pm} := \begin{cases} k, & \text{if } -, \\ 0, & \text{if } +. \end{cases}$$

and apply (2.2) accordingly to obtain for the expression in (4.5)

$$\begin{aligned}
&\frac{i\Gamma(1-s)}{(2\pi)^{1-s}}\sum_{\pm}\sum_{\ell=0}^{a-1}\widetilde{T}_\ell^{\pm}\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\alpha_\ell^{\pm}\pm h+\ell k}{ka}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\alpha_\ell^{\pm}\pm h+\ell k}{ka}\right)\right)\\
&+\frac{i\Gamma(1-s)}{(2\pi)^{1-s}}\sum_{\pm}\sum_{r=0}^{b-1}\widetilde{U}_r^{\pm}\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\beta_r^{\pm}\pm h+rk}{kb}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\beta_r^{\pm}\pm h+rk}{kb}\right)\right)\\
&+\frac{i\Gamma(1-s)(a^{-s}+b^{-s})}{2(2\pi)^{1-s}}\sum_{\pm}\mp\kappa_{\pm}(s)\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\delta^{\pm}\pm h}{k}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\delta^{\pm}\pm h}{k}\right)\right).
\end{aligned}\tag{4.6}$$

Now we separate the $\ell=0$ and $r=0$ terms and insert the definitions of $\widetilde{T}_\ell^{\pm}$ and $\widetilde{U}_r^{\pm}$ so that (4.6) becomes

$$\begin{aligned}
&\frac{\Gamma(1-s)}{2a(2\pi)^{1-s}}\sum_{\pm}\kappa_{\pm}(s)\sum_{\ell=1}^{a-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{hb}{ka}\pm\frac{\ell b}{a}\right)\right)\\
&\quad\times\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\alpha_\ell^{\pm}\pm h+\ell k}{ka}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\alpha_\ell^{\pm}\pm h+\ell k}{ka}\right)\right)
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
&+\frac{\Gamma(1-s)}{2b(2\pi)^{1-s}}\sum_{\pm}\kappa_{\pm}(s)\sum_{r=1}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}\pm\frac{ra}{b}\right)\right)\\
&\quad\times\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\beta_r^{\pm}\pm h+rk}{kb}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\beta_r^{\pm}\pm h+rk}{kb}\right)\right)
\end{aligned}\tag{4.8}$$

$$\begin{aligned}
&+\frac{\Gamma(1-s)}{2a(2\pi)^{1-s}}\sum_{\pm}\kappa_{\pm}(s)\cot\left(\pi\left(-\frac{h}{k}+\frac{hb}{ka}\right)\right)\\
&\quad\times\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\alpha_0^{\pm}\pm h}{ka}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\alpha_0^{\pm}\pm h}{ka}\right)\right)
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
&+\frac{\Gamma(1-s)}{2b(2\pi)^{1-s}}\sum_{\pm}\kappa_{\pm}(s)\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}\right)\right)\\
&\quad\times\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\beta_0^{\pm}\pm h}{kb}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\beta_0^{\pm}\pm h}{kb}\right)\right)
\end{aligned}\tag{4.10}$$

$$+\frac{i\Gamma(1-s)(a^{-s}+b^{-s})}{2(2\pi)^{1-s}}\sum_{\pm}\mp\kappa_{\pm}(s)\left(e^{\frac{-\pi is}{2}}\zeta\left(1-s;\frac{\delta^{\pm}\pm h}{k}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\frac{\delta^{\pm}\pm h}{k}\right)\right).\tag{4.11}$$

Expanding the sum in (4.11) we see that it simplifies to

$$\frac{e^{\frac{\pi i}{2}(1-s)}\Gamma(1-s)}{(2\pi)^{1-s}}(a^{-s}+b^{-s})\left(\cos(\pi s)\zeta\left(1-s;1-\frac{h}{k}\right)-\zeta\left(1-s;\frac{h}{k}\right)\right).$$

Similarly, (4.9) equals

$$\frac{-i\Gamma(1-s)e^{-\frac{\pi i s}{2}}}{(2\pi)^{1-s}a}\zeta\left(1-s;1-\frac{h}{ka}\right)\cot\left(\pi\left(-\frac{h}{k}+\frac{hb}{ka}\right)\right)\sin\left(\pi s\right)$$

and (4.10) equals

$$\frac{-i\Gamma(1-s)e^{-\frac{\pi i s}{2}}}{(2\pi)^{1-s}b}\zeta\left(1-s;1-\frac{h}{kb}\right)\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}\right)\right)\sin\left(\pi s\right).$$

For the sums over ℓ and r that remain in (4.7) and (4.8), we take the $-$ part of the sum and change indices of summation from ℓ to $b-\ell$, and r to $b-r$. In the case of the sum over r we obtain, also using that cot is π-periodic, that (4.8) equals

$$\begin{aligned}
&\frac{\Gamma(1-s)}{2b(2\pi)^{1-s}}\\
&\quad\times\left[\sum_{r=1}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}+\frac{ra}{b}\right)\right)\left(e^{-\frac{\pi i s}{2}}\zeta\left(1-s;\frac{h+rk}{bk}\right)-e^{\frac{\pi i s}{2}}\zeta\left(1-s;1-\frac{h+rk}{bk}\right)\right)\right.\\
&\qquad+e^{-\pi i s}\sum_{r=1}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}-\frac{(b-r)a}{b}\right)\right)\\
&\quad\left.\times\left(e^{-\frac{\pi i s}{2}}\zeta\left(1-s;\frac{-h+(b-r)k}{bk}\right)-e^{\frac{\pi i s}{2}}\zeta\left(1-s;1-\frac{-h+(b-r)k}{bk}\right)\right)\right]\\
&\quad=\frac{\Gamma(1-s)}{2b(2\pi)^{1-s}}\\
&\quad\times\left[\sum_{r=1}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}+\frac{ra}{b}\right)\right)\left(e^{-\frac{\pi i s}{2}}\zeta\left(1-s;\frac{h+rk}{bk}\right)-e^{\frac{\pi i s}{2}}\zeta\left(1-s;1-\frac{h+rk}{bk}\right)\right)\right.\\
&\qquad+\sum_{r=1}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}+\frac{ra}{b}\right)-\pi a\right)\left(e^{-\frac{3\pi i s}{2}}\zeta\left(1-s;1-\frac{h+rk}{bk}\right)\right.\\
&\qquad\left.\left.-e^{-\frac{\pi i s}{2}}\zeta\left(1-s;\frac{h+rk}{bk}\right)\right)\right]\\
&\quad=\frac{\Gamma(1-s)}{2b(2\pi)^{1-s}}\left[-2ie^{-\frac{\pi i s}{2}}\sin(\pi s)\sum_{r=1}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}+\frac{ra}{b}\right)\right)\zeta\left(1-s;1-\frac{h+rk}{bk}\right)\right].
\end{aligned}$$

The ℓ-sum in (4.7) is similar. We obtain for (4.7)

$$\frac{\Gamma(1-s)}{2a(2\pi)^{1-s}}\left[-2ie^{-\frac{\pi i s}{2}}\sin(\pi s)\sum_{\ell=1}^{a-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{hb}{ka}+\frac{\ell b}{a}\right)\right)\zeta\left(1-s;1-\frac{h+\ell k}{ak}\right)\right].$$

Overall, we obtain that (4.6) equals

$$\frac{e^{-\frac{\pi i}{2}(s+1)}\Gamma(1-s)\sin(\pi s)}{a(2\pi)^{1-s}}\sum_{\ell=0}^{a-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{hb}{ka}+\frac{\ell b}{a}\right)\right)\zeta\left(1-s;1-\frac{h+\ell k}{ak}\right)$$

$$+\frac{e^{-\frac{\pi i}{2}(s+1)}\Gamma(1-s)\sin(\pi s)}{b(2\pi)^{1-s}}\sum_{r=0}^{b-1}\cot\left(\pi\left(-\frac{h}{k}+\frac{ha}{kb}+\frac{ra}{b}\right)\right)\zeta\left(1-s;1-\frac{h+rk}{bk}\right)$$
$$+\frac{e^{\frac{\pi i}{2}(1-s)}\Gamma(1-s)}{(2\pi)^{1-s}}(a^{-s}+b^{-s})\left(\cos(\pi s)\zeta\left(1-s;1-\frac{h}{k}\right)-\zeta\left(1-s;\frac{h}{k}\right)\right).$$

Now we multiply back through by b^s, and find that

$$\lim_{\tau\to\frac{a}{b}} G_s(h,k;\tau)$$
$$=\frac{b^{s-1}}{(2\pi)^{1-s}}e^{-\frac{\pi i}{2}(s+1)}\Gamma(1-s)\sin(\pi s)\left(\frac{b}{a}a^{1-s}\mathfrak{c}_s\left(\frac{b}{a}\right)+b^{1-s}\mathfrak{c}_s\left(\frac{a}{b}\right)\right)$$
$$+\frac{e^{-\frac{\pi i}{2}(s-1)}\Gamma(1-s)}{(2\pi)^{1-s}}\left(\left(\frac{b}{a}\right)^s+1\right)\left(\cos(\pi s)\zeta\left(1-s;1-\frac{h}{k}\right)-\zeta\left(1-s;\frac{h}{k}\right)\right)$$
$$=\frac{1}{(2\pi)^{1-s}}e^{-\frac{\pi i}{2}(s+1)}\Gamma(1-s)\sin(\pi s)\left(\left(\frac{b}{a}\right)^s\mathfrak{c}_s\left(\frac{b}{a}\right)+\mathfrak{c}_s\left(\frac{a}{b}\right)\right) \tag{4.12}$$
$$+\frac{e^{-\frac{\pi i}{2}(s-1)}\Gamma(1-s)}{(2\pi)^{1-s}}\left(\left(\frac{b}{a}\right)^s+1\right)\left(\cos(\pi s)\zeta\left(1-s;1-\frac{h}{k}\right)-\zeta\left(1-s;\frac{h}{k}\right)\right), \tag{4.13}$$

where the cotangent-zeta sums $\mathfrak{c}_s(\frac{a}{b})=\mathfrak{c}_s(h,k;\frac{a}{b})$ are as defined in (1.4). (Here, we have suppressed the dependence on h and k for ease of notation.)

We now must add the terms in (4.2) coming from the polylogarithm functions. We find, using (2.2) with $Li_s(\zeta_k^{-h})=Li_s(\zeta_k^{k-h})$, that

$$ie^{-\frac{\pi is}{2}}\sin\left(\frac{\pi s}{2}\right)\left(Li_s(\zeta_k^{-h})+\tau^{-s}Li_s(\zeta_k^{h})\right)$$
$$=-e^{-\frac{\pi is}{2}}\frac{\sin(\frac{\pi s}{2})\Gamma(1-s)}{(2\pi)^{1-s}}\left(\zeta\left(1-s;\frac{h}{k}\right)\left(\tau^{-s}e^{-\frac{\pi is}{2}}-e^{\frac{\pi is}{2}}\right)\right.$$
$$\left.+\zeta\left(1-s;1-\frac{h}{k}\right)\left(e^{-\frac{\pi is}{2}}-\tau^{-s}e^{\frac{\pi is}{2}}\right)\right).$$

We let $\tau=a/b$ and add this to (4.13) to obtain

$$-\frac{i\Gamma(1-s)}{(2\pi)^{1-s}}\left(1+\left(\frac{b}{a}\right)^s e^{-\pi is}\right)\cos\left(\frac{\pi s}{2}\right)\left(\zeta(1-s;\tfrac{h}{k})-\zeta(1-s;1-\tfrac{h}{k})\right). \tag{4.14}$$

Adding (4.12) and (4.14), we obtain

$$\psi_s\left(h,k;\frac{a}{b}\right)=-\frac{i\Gamma(1-s)}{(2\pi)^{1-s}}\left(1+\left(\frac{b}{a}\right)^s e^{-\pi is}\right)\cos\left(\frac{\pi s}{2}\right)\left(\zeta(1-s;\tfrac{h}{k})-\zeta(1-s;1-\tfrac{h}{k})\right)$$
$$-\frac{i}{(2\pi)^{1-s}}e^{-\frac{\pi is}{2}}\Gamma(1-s)\sin(\pi s)\left(\left(\frac{b}{a}\right)^{1+\mathfrak{a}}\mathfrak{c}_s\left(\frac{b}{a}\right)+\mathfrak{c}_s\left(\frac{a}{b}\right)\right).$$

Equivalently,

$$\mathfrak{c}_s\left(\frac{a}{b}\right)+\left(\frac{b}{a}\right)^s\mathfrak{c}_s\left(\frac{b}{a}\right)=\frac{ie^{\frac{\pi is}{2}}}{(2\pi)^{s-1}\Gamma(1-s)\sin(\pi s)}\psi_s\left(h,k;\frac{a}{b}\right)$$
$$-\frac{e^{\frac{\pi is}{2}}}{\sin(\pi s)}\left(1+\left(\frac{b}{a}\right)^s e^{-\pi is}\right)\cos\left(\frac{\pi s}{2}\right)\left(\zeta\left(1-s;\frac{h}{k}\right)-\zeta\left(1-s;1-\frac{h}{k}\right)\right),$$

which is the identity given in (1.5) of Theorem 2 for $\Re(s) > 1$, and we extend to $\mathbb{C}$ by analytic continuation.

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