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Asymptotic expansions, partial theta functions, and radial limit differences of mock modular and modular forms

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Dedicated to Bruce Berndt, in honor of his 80*th birthday.*

In 1920, Ramanujan studied the asymptotic differences between his mock theta functions and modular theta functions, as q tends towards roots of unity singularities radially from within the unit disk. In 2013, the bounded asymptotic differences predicted by Ramanujan with respect to his mock theta function $f(q)$ were established by Ono, Rhoades, and the author, as a special case of a more general result, in which they were realized as special values of a quantum modular form. Our results here are threefold: we realize these radial limit differences as special values of a partial theta function, provide full asymptotic expansions for the partial theta function as q tends towards roots of unity radially, and explicitly evaluate the partial theta function at roots of unity as simple finite sums of roots of unity.

Keywords: Mock theta functions; mock modular forms; Ramanujan; partial theta functions; asymptotic expansions; radial limits.

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1. Introduction and Statement of Results

Ramanujan's last letter to Hardy, written in 1920 [2, 22], explores the asymptotic properties of his mock theta functions. Specifically, in the letter, Ramanujan states without proof that as q approaches an even order 2κ root of unity radially from within the unit disk, that his mock theta function

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$

(defined using the q-Pochhammer symbol $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ $(n \in \mathbb{N} \cup \{\infty\})$) satisfies

$$f(q) - (-1)^{\kappa}(1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots) = O(1). \tag{1.1}$$

It is not difficult to see that the even order 2κ roots of unity are singularities of the mock theta function $f(q)$, and that the function $b(q)$ appearing in (1.1), defined by $b(q) := (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots)$, is a modular form (up to multiplication by a suitable power of q and under the usual modular change of variable $q = e^{2\pi i\tau}, \tau \in \mathbb{H} := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$). Thus, Ramanujan's claim in (1.1) may be interpreted as stating that the modular forms $\pm b(q)$ carve out the exponential singularities of the mock theta function $f(q)$.

Decades later, the truth of (1.1) was proved, first in [15] as a special case of a more general result, in which the implied $O(1)$ constants are realized as special values of quantum modular forms, and the proof of which uses the more recent theory of mock modular forms. (See [23; 4, Chap. 21] for more on quantum modular forms, and [4, Part 2] for more on mock modular forms.) This more general result, namely, [15, Theorem 1.2], which proves (1.1) as a special case, replaces $f(q)$ (respectively, $b(q)$) by the more general mock modular (respectively, modular) partition rank (respectively, crank) generating function $R(w;q)$ (respectively, $C(w;q)$). Note that $f(q) = R(-1;q)$ (respectively, $b(q) = C(-1;q)$). The functions R and C are given by

$$R(w;q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}, \quad C(w;q) = \frac{(q;q)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty}.$$

With the w-variable fixed to be suitable roots of unity ζ_b^a (with $\zeta_N := e^{2\pi i/N}$), the $O(1)$ constants implied in the generalization of (1.1) in [15, Theorem 1.2] as q radially tends towards roots of unity ζ_k^h are realized (up to an explicit constant multiple) as specializations (at $(w;q) = (\zeta_b^a;\zeta_k^h)$) of the quantum modular strongly unimodal sequence rank generating function $U(w;q)$,[a] given by

$$U(w;q) = \sum_{n=0}^{\infty} (wq;q)_n(w^{-1}q;q)_n q^{n+1}.$$

Precisely, we have the following theorem.

Theorem 1.1 ([15, Theorem 1.2]). *Let $1 \le a < b$ and $1 \le h < k$ be integers with $\gcd(a,b) = \gcd(h,k) = 1$ and $b \,|\, k$. If $h' \in \mathbb{Z}$ satisfies $hh' \equiv -1 \pmod{k}$ then we have that*

$$\lim_{t\to 0^+} \left(R(\zeta_b^a;\zeta_k^h e^{-2\pi t}) - \zeta_{b^2}^{-a^2h'k} C(\zeta_b^a;\zeta_k^h e^{-2\pi t})\right) = -(1-\zeta_b^a)(1-\zeta_b^{-a})U(\zeta_b^a;\zeta_k^h).$$

Ramanujan's claim (1.1) (with an added explicit implied $O(1)$ constant) is deduced from [15, Theorem 1.2] (stated above) by setting $(a,b) = (1,2)$. See also [8, 10, 12–14, 17, 20, 24] for more recent related work.

[a] We caution the reader that the function U is defined with slightly different normalizations (using the same notation for the function) in different sources. Here, we have used the definition from [15].

Remark 1.2. The above theorem is stated slightly differently than in [15] but is equivalent: namely, we have replaced the variable q by $\zeta_k^h e^{-2\pi t}$, and have replaced the limit as $q \to \zeta_k^h$ radially from within the unit disk by the limit as $t \to 0^+$.

As alluded to above, the functions R, C, and U are now well known to possess different types of modular properties, namely, they are mock modular, modular, and quantum modular, respectively, when appropriately normalized and specialized in q and w. When expanded as two-variable series in q and w, these functions are also well-known to be combinatorial generating functions for partition ranks, partition cranks, and ranks of strongly unimodal sequences, respectively. (See, for example, [4, 15] for more.)

At the June 2019 conference "Analytic and Combinatorial Number Theory: The Legacy of Ramanujan" in honor of Bruce Berndt at the University of Illinois Urbana-Champaign, Peter Paule (RISC, Johannes Kepler University, Linz, Austria) asked the author after her lecture on this subject about the existence of full asymptotic expansions for implied constant values in the $O(1)$ expressions in (1.1) as q tends to roots of unity radially within the unit disk [19]. One of the main results of this paper is to establish these asymptotic expansions; in fact, we do so as a special case of a more general result given in Theorem 1.3, which builds from and extends results in [15].

Our Theorem 1.3 is stated in terms of the two-variable partial theta function ϑ, (with $q = e^{2\pi i\tau}$ and $w = e^{2\pi iz}$), defined by

$$\widetilde{\vartheta}(z;\tau) := \sum_{n=0}^{\infty} \chi_{12}(n) q^{\frac{n^2}{24}} w^{\frac{n}{2}},$$

where $\chi_{12}(n) := \left(\frac{12}{n}\right)$ is defined by the Kronecker symbol. Although the case $z=0$ (which corollaryresponds to $w = 1$) is irrelevant in Theorem 1.3, we remark for context that $\widetilde{\vartheta}(0;\tau) = \eta(24\tau)$, the weight $1/2$ modular η-function. For more general z, the function $\widetilde{\vartheta}$ may be described as a (two-variable) partial, or false, theta function, and is related to certain holomorphic Eichler integrals, the latter of which have been of historical importance in the theory of modular forms [11, 21]. More recently, such functions have been shown to play important roles in the theory of quantum modular forms, mathematical physics, representation theory, and the intersections of these areas (for example, see [3–7, 9, 15, 23]).

In the three parts of Theorem 1.3 and Corollary 1.4, we realize the radial limit differences studied by Ramanujan and later, more generally, in [15], as special values of the partial theta function $\widetilde{\vartheta}$ defined above; we provide full asymptotic expansions for the partial theta function $\widetilde{\vartheta}$ as q tends towards roots of unity radially; and we explicitly evaluate the partial theta function $\widetilde{\vartheta}$ at roots of unity as simple finite sums of roots of unity, thereby producing new simple expressions for the radial limit differences in question.

Throughout, $B_k(x)$ denotes the kth Bernoulli polynomial.

Theorem 1.3. *Let* $1 \leq a < b$ *and* $1 \leq h < k$ *be integers with* $\gcd(a,b) = \gcd(h,k) = 1$ *and* $b \,|\, k$, *so that* $bb' = k$ *for some integer* b'. *Let* $h' \in \mathbb{Z}$ *satisfy* $hh' \equiv -1 \pmod{k}$. *The following are true*:

(i) *We have that*

$$\lim_{t \to 0^+} (R(\zeta_b^a; \zeta_k^h e^{-2\pi t}) - \zeta_{b^2}^{-a^2 h' k} C(\zeta_b^a; \zeta_k^h e^{-2\pi t}))$$
$$= -2i\zeta_{24k}^h \sin\left(\frac{\pi a}{b}\right) \widetilde{\vartheta}\left(\frac{a}{b}; -\frac{h}{k}\right). \tag{1.2}$$

(ii) *The partial theta function* $\widetilde{\vartheta}$ *appearing on the right-hand side of* (1.2) *has the asymptotic expansion as* $t \to 0^+$

$$\widetilde{\vartheta}\left(\frac{a}{b}; -\frac{h}{k} + \frac{12it}{\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r, d) \frac{(-t)^r}{r!},$$

where the L-values are given by

$$L(-r, d) = -\frac{(12k)^r}{r+1} \sum_{n=1}^{12k} d(n) B_{r+1}\left(\frac{n}{12k}\right), \quad (r = 0, 1, 2, \ldots)$$

with

$$d(n) = d_{a,b,h,k}(n) := \chi_{12}(n) \zeta_{24k}^{-hn^2} \zeta_{2b}^{an}.$$

(iii) *In particular, the value* $\widetilde{\vartheta}(\frac{a}{b}; -\frac{h}{k})$ *appearing on the right-hand side of* (1.2) *may be explicitly computed as*

$$\widetilde{\vartheta}\left(\frac{a}{b}; -\frac{h}{k}\right) = -\frac{1}{12k} \sum_{n=1}^{12k} nd(n),$$

with $d(n)$ *as in part* (ii). *Combining this with part* (i), *we have that*

$$\lim_{t \to 0^+} (R(\zeta_b^a; \zeta_k^h e^{-2\pi t}) - \zeta_{b^2}^{-a^2 h' k} C(\zeta_b^a; \zeta_k^h e^{-2\pi t}))$$
$$= (6k)^{-1} i\zeta_{24k}^h \sin\left(\frac{\pi a}{b}\right) \sum_{n=1}^{12k} nd(n).$$

From Theorem 1.3 with $(a,b) = (1,2)$ and $k = 2\kappa$, we obtain results directly pertaining to Ramanujan's original radial limit (1.1) and his mock theta function $f(q)$.

Corollary 1.4. *Let* $\kappa \in \mathbb{N}$. *The following are true*:

(i) *We have that*

$$\lim_{t \to 0^+} (f(\zeta_{2\kappa}^h e^{-2\pi t}) - (-1)^\kappa b(\zeta_{2\kappa}^h e^{-2\pi t})) = -2i\zeta_{48\kappa}^h \widetilde{\vartheta}\left(\frac{1}{2}; -\frac{h}{2\kappa}\right).$$

(ii) *The partial theta function* $\widetilde{\vartheta}$ *appearing on the right-hand side of* (1.2) *has the asymptotic expansion as* $t \to 0^+$

$$\widetilde{\vartheta}\left(\frac{1}{2}; -\frac{h}{2\kappa} + \frac{12it}{\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r, d)\frac{(-t)^r}{r!},$$

where the L*-values are given by*

$$L(-r, d) = -\frac{(24\kappa)^r}{r+1} \sum_{n=1}^{24\kappa} d(n) B_{r+1}\left(\frac{n}{24\kappa}\right), \quad (r = 0, 1, 2, \dots)$$

with

$$d(n) = d_{1,2,h,2\kappa} = d_{h,\kappa}(n) := \chi_{12}(n)\zeta_{48\kappa}^{-hn^2} i^n.$$

(iii) *In particular, the value* $\widetilde{\vartheta}(\frac{1}{2}; -\frac{h}{2\kappa})$ *appearing on the right-hand side of* (1.2) *may be explicitly computed as*

$$\widetilde{\vartheta}\left(\frac{1}{2}; -\frac{h}{2\kappa}\right) = -\frac{1}{24\kappa} \sum_{n=1}^{24\kappa} nd(n),$$

with $d(n)$ *as in part* (ii). *Combining this with part* (i), *we have that*

$$\lim_{t \to 0^+} (f(\zeta_{2\kappa}^h e^{-2\pi t}) - (-1)^\kappa b(\zeta_{2\kappa}^h e^{-2\pi t})) = (12\kappa)^{-1} i \zeta_{48\kappa}^h \sum_{n=1}^{24\kappa} nd(n).$$

Remark 1.5. It is interesting to compare the explicit expressions given on the right-hand sides of the radial limits in Theorem 1.3 (and Corollary 1.4) with those given in [15, Theorem 1.2 (and Theorem 1.1)]. From Theorem 1.3, we have that

$$\lim_{t \to 0^+} (R(\zeta_b^a; \zeta_k^h e^{-2\pi t}) - \zeta_{b^2}^{-a^2 h' k} C(\zeta_b^a; \zeta_k^h e^{-2\pi t}))$$
$$= (6k)^{-1} i \zeta_{24k}^h \sin\left(\frac{\pi a}{b}\right) \sum_{n=1}^{12k} n\chi_{12}(n)\zeta_{24k}^{-hn^2} \zeta_{2b}^{an}. \tag{1.3}$$

On the other hand, from [15, Theorem 1.2], we have that

$$\lim_{t \to 0^+} (R(\zeta_b^a; \zeta_k^h e^{-2\pi t}) - \zeta_{b^2}^{-a^2 h' k} C(\zeta_b^a; \zeta_k^h e^{-2\pi t}))$$
$$= -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) \sum_{n=0}^{c(a,b,h,k)} (\zeta_b^a \zeta_k^h; \zeta_k^h)_n (\zeta_b^{-a} \zeta_k^h; \zeta_k^h)_n \zeta_k^{h(n+1)}, \tag{1.4}$$

where $c(a, b, h, k)$ is a non-negative integer (which can be explicitly computed) depending on a, b, h and k.

Without Theorem 1.3 and [15, Theorem 1.2], it is not obvious that the finite sum of roots of unity on the right-hand sides of (1.3) and the finite q-hypergeometric sum at roots of unity on the right-hand side of (1.4) are equal. We state this as an open problem.

Open Problem. Give a direct proof of the fact that the finite sum on the right-hand side of (1.3) equals the finite sum on the right-hand side of (1.4), under the hypotheses given, avoiding theorems such as Theorem 1.3 and [15, Theorem 1.2] (and their proofs and corollaries), and instead using, for example, elements from the theory of q-hypergeometric series, partial theta functions, or other elementary or direct methods.

Example. Let $(a, b) = (1, 2)$ and $(h, k) = (5, 6)$. Then by part (ii) of Theorem 1.3 (or part (ii) of Corollary 1.4), as $t \to 0^+$,

$$\widetilde{\vartheta}\left(\frac{1}{2}; -\frac{5}{6} + \frac{12it}{\pi}\right) \sim -\sum_{r=0}^{\infty} \frac{(72)^{2r}}{2r+1} \sum_{n=1}^{72} \chi_{12}(n) \zeta_{144}^{-5n^2} i^n B_{2r+1}\left(\frac{n}{72}\right) \frac{(-t)^r}{r!}.$$

By part (iii) of Theorem 1.3 (or part (iii) of Corollary 1.4),

$$\vartheta\left(\frac{1}{2}, -\frac{5}{6}\right) = -\frac{1}{72} \sum_{n=1}^{72} n\chi_{12}(n) \zeta_{144}^{-5n^2} i^n.$$

Combining this with part (i) of Theorem 1.3 (or part (i) of Corollary 1.4), we have that Ramanujan's radial limit satisfies

$$\begin{aligned}\lim_{t \to 0^+} (f(\zeta_6^5 e^{-2\pi t}) - (-1)^\kappa b(\zeta_6^5 e^{-2\pi t})) &= (36)^{-1} i \zeta_{144}^5 \sum_{n=1}^{72} n\chi_{12}(n) \zeta_{144}^{-5n^2} i^n \\ &\approx -2 + 3.4641i.\end{aligned}$$

On the other hand, using [15, Theorem 1.2] (see (1.4)), we have that this same radial limit satisfies

$$\begin{aligned}\lim_{t \to 0^+} (f(\zeta_6^5 e^{-2\pi t}) - (-1)^\kappa b(\zeta_6^5 e^{-2\pi t})) &= -4 \sum_{n=0}^{2} (-\zeta_6^5; \zeta_6^5)_n^2 \zeta_6^{5(n+1)} \\ &\approx -2 + 3.4641i.\end{aligned}$$

2. Proofs

Proof of Theorem 1.3. We begin by establishing (i). From [14, Corollary 1.4], an earlier result of Ki *et al.*, and the author, we see that the right-hand side of claimed radial limit may be expressed as the value $-(1 - \zeta_b^a)(1 - \zeta_b^{-a}) F(\zeta_b^a; \zeta_k^{-h})$, where

$$F(w; q) := \sum_{n=0}^{\infty} w^{n+1} (wq; q)_n.$$

We next invoke a result of Hikami on certain difference equations, namely [16, Theorem 8], which shows after some simplification that $(1 - w) w^{-\frac{1}{2}} q^{\frac{1}{24}} F(w; q) = \widetilde{\vartheta}(z; \tau)$, with $w = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. Replacing $w = \zeta_b^a$ (hence $z = a/b$) and $q = \zeta_k^{-h}$ (hence $\tau = -h/k$), we have that the right-hand side of the claimed radial limit is

$$-\zeta_{24k}^{h}\zeta_{2b}^{a}(1-\zeta_b^{-a})\widetilde{\vartheta}\left(\frac{a}{b};-\frac{h}{k}\right) = -2i\zeta_{24k}^{h}\sin\left(\frac{\pi a}{b}\right)\widetilde{\vartheta}\left(\frac{a}{b};-\frac{h}{k}\right)$$

as claimed.

Now, we establish (ii) Using the definition of $\widetilde{\vartheta}$, for any $t>0$, with hypotheses as given on a,b,h,k, we have that

$$\widetilde{\vartheta}\left(\frac{a}{b};-\frac{h}{k}+\frac{12it}{\pi}\right) = \sum_{n=0}^{\infty} d(n)e^{-n^2t},$$

where the coefficients $d(n)$ are as defined in Theorem 1.3. Towards the proof of (ii), we will show (below) that the coefficients $d(n)$ are periodic with period $12k$ and have mean value zero, in order to apply the following result of Lawrence and Zagier.

Proposition 2.1 ([18, p. 98]). *Let $C:\mathbb{Z}\to\mathbb{C}$ be a periodic function with mean value* 0. *Then the associated L-series $L(s,C)=\sum_{n=1}^{\infty}C(n)n^{-s}$ $(\mathrm{Re}(s)>1)$ extends holomorphically to all of $\mathbb{C}$ and the function $\sum_{n=1}^{\infty}C(n)e^{-n^2t}$ $(t>0)$ has the asymptotic expansion*

$$\sum_{n=1}^{\infty}C(n)e^{-n^2t}\sim\sum_{r=0}^{\infty}L(-2r,C)\cdot\frac{(-t)^r}{r!}$$

as $t\to 0^+$. The L-values $L(-r,C)$ are given explicitly by

$$L(-r,C)=-\frac{M^r}{r+1}\sum_{n=1}^{M}C(n)B_{r+1}\left(\frac{n}{M}\right)\quad (r=0,1,\ldots)$$

where M is any period of the function $C(n)$.

First, it is clear by definition, and using the fact that $b\,|\,k$, that the $d(n)$ are periodic mod $12k$. To prove that the $d(n)$ have mean value 0, we begin by using the definition of χ_{12} and re-write

$$\begin{aligned}\sum_{n=0}^{12k-1}d(n) &= \sum_{\pm}\left(\sum_{\substack{n=0\\ n\equiv\pm1\,(\mathrm{mod}\,12)}}^{12k-1}\zeta_{24k}^{-hn^2}\zeta_{2b}^{an} - \sum_{\substack{n=0\\ n\equiv\pm5\,(\mathrm{mod}\,12)}}^{12k-1}\zeta_{24k}^{-hn^2}\zeta_{2b}^{an}\right)\\ &= \sum_{\pm}\left(\sum_{n=\frac{1\mp1}{2}}^{k-\frac{1\pm1}{2}}\zeta_{24k}^{-h(12n\pm1)^2}\zeta_{2b}^{a(12n\pm1)} - \sum_{n=\frac{1\mp1}{2}}^{k-\frac{1\pm1}{2}}\zeta_{24k}^{-h(12n\pm5)^2}\zeta_{2b}^{a(12n\pm5)}\right)\\ &= \sum_{\pm}\left(\sum_{n\,(\mathrm{mod}\,k)}\zeta_{24k}^{-h(12n\pm1)^2}\zeta_{2b}^{a(12n\pm1)}\right.\\ &\qquad\left.-\sum_{n\,(\mathrm{mod}\,k)}\zeta_{24k}^{-h(12n\pm5)^2}\zeta_{2b}^{a(12n\pm5)}\right)\end{aligned}\tag{2.1}$$

where we have also used that the summands $\zeta_{24k}^{h(12n\pm\nu)^2}\zeta_{2b}^{a(12n\pm\nu)}$, where $\nu \in \{1,5\}$, are periodic mod k (again using that $b \mid k$).

Case 1. $\gcd(k,6) > 1$. In this case, we will show that each of the four sums on n in (2.1) are identically zero. Up to a constant multiple, each of the sums on n, where $\nu \in \{1,5\}$, and b' is such that $bb' = k$, may be re-written as $G(-6h, 6ab' \mp h\nu, k)$, where $G(A,B,C)$ denotes the Gauss sum

$$G(A,B,C) := \sum_{n \bmod C} \zeta_C^{An^2+Bn}.$$

It is well known that if $\gcd(A,C) = g > 1$ and $g \nmid B$, that $G(A,B,C) = 0$ [1]. We will use this fact here, to show that each of the four sums on n are identically zero. Here, we have that $\gcd(A,C) = \gcd(6h,k) = \gcd(6,k) = g > 0$ by hypothesis. Suppose $g \mid B = 6ab' \mp h\nu$. Then since $g \mid 6$, and $\gcd(g,\nu) = 1$, we have that $g \mid h$. But $g \mid k$ as well, so this is a contradiction, because $\gcd(h,k) = 1$ and $g > 1$. Hence, $g \nmid 6ab' \mp h\nu$ and $G(6h, 6ab' \mp h, k) = 0$.

Case 2. $\gcd(k,6) = 1$. In this case, we write, using that the sum (at the start of (2.2)) may be taken over any set of representatives mod k,

$$\sum_{n \pmod k} \zeta_{24k}^{-h(12n\pm5)^2}\zeta_{2b}^{a(12n\pm5)} = \sum_{n \pmod k} \zeta_{24k}^{-h(12(n\mp\alpha)\pm5)^2}\zeta_{2b}^{a(12(n\mp\alpha)\pm5)}, \tag{2.2}$$

where α is any integer satisfying $3\alpha \equiv 1 \pmod k$ – a number we know exists in this case since $\gcd(3,k) = 1$. After some expanding and simplifying in the exponents of the roots of unity in the summands, and writing $b = k/b'$, we rewrite (2.2) as

$$\begin{aligned}
&\zeta_k^{\pm2ab'(1-3\alpha)}\zeta_{3k}^{-2h(1-3\alpha)^2} \sum_{n \pmod k} \zeta_{24k}^{-h(12n\pm1)^2}\zeta_{2k}^{ab'(12n\pm1)}\zeta_{3k}^{\mp h(12n\pm1)(1-3\alpha)} \\
&\quad = \zeta_k^{\pm2ab'(1-3\alpha)}\zeta_k^{-h(3\alpha-1)(2\alpha-1)} \sum_{n \pmod k} \zeta_{24k}^{-h(12n\pm1)^2}\zeta_{2k}^{ab'(12n\pm1)}\zeta_k^{\mp4hn(1-3\alpha)} \\
&\quad = \sum_{n \pmod k} \zeta_{24k}^{-h(12n\pm1)^2}\zeta_{2k}^{ab'(12n\pm1)},
\end{aligned} \tag{2.3}$$

where we have used that $3\alpha \equiv 1 \pmod k$ to obtain the last line in (2.3). Substituting (2.2) for (2.3) (and replacing $b' = k/b$), (2.1) becomes $\sum_\pm 0 = 0$.

Having established the periodic and mean value 0 nature of the $d(n)$, we invoke the proposition [18, Proposition, p. 98] stated above, which yields, as $t \to 0^+$,

$$\widetilde{\vartheta}\left(\frac{a}{b}; -\frac{h}{k} + \frac{12it}{\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r,d)\frac{(-t)^r}{r!},$$

with L-values $L(-r,d)$ as defined in Theorem 1.3, as claimed.

Part (iii) of Theorem 1.3 follows from the asymptotic expansion established in part (ii), by letting $t \to 0^+$. To see this, using the definition of $B_1(x)$ and the previously established facts that the $d(n)$ have period $12k$ and mean value 0, we have that that

$$\widetilde{\vartheta}\left(\frac{a}{b};-\frac{h}{k}\right) = -\sum_{n=1}^{12k} d(n)B_1\left(\frac{n}{12k}\right) = -\sum_{n=1}^{12k} d(n)\left(\frac{n}{12k}-\frac{1}{2}\right) = -\frac{1}{12k}\sum_{n=1}^{12k} nd(n)$$

as claimed. $\square$

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