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Modularity and the distinct rank function

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Dedicated to George Andrews on his 70th birthday

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Abstract If $R(\omega, q)$ denotes Dyson's partition rank generating function, due to work of Bringmann and Ono, it is known that for roots of unity $\omega \neq 1$, $R(\omega, q)$ is the "holomorphic part" of a harmonic weak Maass form. Dating back to Ramanujan, it is also known that $\widehat{R}(\omega, q) := R(\omega, q^{-1})$ is given by Eichler integrals and modular forms. In analogy to these results, more recently Monks and Ono have shown that modular forms arise in a natural way from $G(\omega, q)$, the generating function for ranks of partitions into distinct parts. Moreover, Monks and Ono pose the following problem: determine whether the function $\widehat{G}(\omega, q) := G(\omega, q^{-1})$ appears naturally in the theory of modular forms. Here we answer this question of Monks and Ono, and show that $\widehat{G}(\omega, q)$, when combined with $\widehat{G}(\omega^{-1}, q)$ and a twisted third-order mock theta of Ramanujan, form a weight 1 modular form. We provide a more general result on the modularity of certain expressions involving basic hypergeometric series and then show that our result on $\widehat{G}(\omega, q)$ may be deduced from this as a special case.

Keywords Modular forms · Integer partitions · Basic hypergeometric series

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1 Introduction and statement of results

Of great interest in number theory are identities that relate modular forms to arithmetic or combinatorial generating functions. One of the most celebrated examples of

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such an identity is the Rogers–Ramanujan identity

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \prod_{n\geq 0} \left(1-q^{5n+1}\right)^{-1}\left(1-q^{5n+4}\right)^{-1}, \tag{1.1}$$

where $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), n \geq 1$, and $(a;q)_0 = 1$. The infinite product on the right-hand side of (1.1) is essentially a weight 0 modular form. Further, the infinite series on the left-hand side of (1.1) may be interpreted combinatorially as the generating function for the number of partitions of an integer n with minimal difference equal to 2. Another highly lauded result involves Dyson's rank generating function

$$R(\omega,q) := \sum_{n\geq 0} N(m,n)\omega^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(\omega q;q)_n(\omega^{-1}q;q)_n}.$$

The rank of an integer partition is defined to be the largest part of the partition minus the number of parts, and $N(m,n)$ denotes the number of partitions of n with rank m. In [2], Bringmann and Ono prove for roots of unity ω, that $R(\omega,q)$ is the "holomorphic part" of a weight $1/2$ harmonic weak Maass form, a form that satisfies an appropriate modular transformation under a subgroup of the modular group $\mathrm{SL}_2(\mathbb{Z})$, is annihilated by a Laplacian operator, and satisfies a certain growth condition in the cusps.

More classically understood, yet in analogy to the modularity associated to $R(\omega,q)$ proved by Bringmann–Ono in [2], is a result dating back to Ramanujan regarding the function $\widehat{R}(\omega,q) := R(\omega,q^{-1})$. Namely, one now understands how $\widehat{R}(\omega,q)$ fits into the theory of modular forms: it can be expressed using a weight 0 modular form and two false theta series. (See [1, 5], and (1.3).)

Also in analogy to Dyson's rank generating function $R(\omega,q)$, more recently in [6], Monks and Ono study the generating function $G(\omega,q)$ for ranks of partitions into distinct parts. That is,

$$G(\omega,q) := \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} Q(m,n)\omega^m q^n = \sum_{n\geq 0} \frac{q^{\frac{n^2+n}{2}}}{(\omega q;q)_n},$$

and $Q(m,n)$ counts the number of partitions of n into distinct parts with rank m. As is true with $R(\omega,q)$, the authors show that modular forms arise from $G(\omega,q)$ in a natural way. Moreover, the authors pose a problem in analogy to the known modularity associated to $\widehat{R}(\omega,q)$ as described above. That is, if $\widehat{G}(\omega,q) := G(\omega,q^{-1})$, Monks and Ono naturally ask the following:

Problem 1 (Monks–Ono [6]) *Determine whether the series $\widehat{G}(\omega,q)$ appears naturally within the theory of automorphic forms.*

In this paper we solve this problem and show that $\widehat{G}(\omega,q)$, combined with $\widehat{G}(\omega^{-1},q)$ and a twisted third-order mock theta function of Ramanujan, form a weight 1 modular form. To describe our results, we begin with the following expression for $\widehat{G}(\omega, q)$, revealed in [6, (1.8)]:

$$\widehat{G}(\omega, q) = \sum_{n\geq 0} \frac{(-\omega^{-1})^n}{(\omega^{-1}q; q)_n}.$$

As Monks and Ono point out, the series is not well defined, yet one of their main theorems [6, Theorem 1.2] shows that, for each integer r such that $0 \leq r < m$ and any mth root of unity $-\omega^{-1} \neq 1$, the series $\lim_{n\to\infty} \widehat{G}_{mn+r}(\omega, q)$ *is* well defined, where $\widehat{G}_t(\omega, q)$ denotes the tth partial sum of $\widehat{G}_t(\omega, q)$. Moreover, they show that this well-defined series $\lim_{n\to\infty} \widehat{G}_{mn+r}(\omega, q)$ differs from $\widehat{G}(\omega, q) = \lim_{n\to\infty} \widehat{G}_{mn}(\omega, q)$ by a constant multiple of the infinite product $q^{\frac{1}{48}}\eta_0(\omega^{-1}, q)^{-1}$, where

$$\eta_0(\omega, q) := q^{\frac{1}{48}} \prod_{n\geq 1} \bigl(1 - \omega q^n\bigr). \tag{1.2}$$

The products $\eta_0(\omega, q)$ may be used to define weight 0 modular forms

$$\eta(\omega, q) := q^{\frac{1}{24}} \eta_0(\omega, q)\eta_0\bigl(\omega^{-1}, q\bigr), \tag{1.3}$$

where $\omega \neq 1$. Note that when $\omega = 1$, we have

$$\eta(1, q) = \eta^2(q),$$

where $\eta(q) := q^{\frac{1}{24}} \prod_{n\geq 1}(1 - q^n)$ is the Dedekind η-function, a weight $\frac{1}{2}$ modular form. To regard these functions as modular forms, one lets $q = e^{2\pi i z}$, where $z \in \mathbb{H}$, the upper-half complex plane. (See, for example, [4].) Similar to what holds for the function $G(\omega, q)$ [6, Theorem 1.1], we will show that $\widehat{G}(\omega, q)$ and $\widehat{G}(\omega^{-1}, q)$ become "factors" of a weight 1 modular form, along with a twisted third-order mock theta function of Ramanujan

$$\psi(\omega, q) := \sum_{n\geq 0} \frac{q^{n^2}\omega^n}{(q; q^2)_n}.$$

(The function $\psi(1, q)$ is the original 3rd-order Ramanujan mock theta function.) As we will see in Theorem 1.1, this weight one modular form is described in terms of the weight 0 modular forms $\eta(\omega, q)$, and the weight $1/2$ modular form $\eta(q)$. More precisely, we define

$$\widehat{D}(\omega, q) := \bigl(1 + \omega^{-1}\bigr)\widehat{G}(\omega, q) + \bigl(1 - \omega^{-2}\bigr)\bigl(\psi\bigl(-\omega^2, q\bigr) - 1\bigr).$$

Replacing q by $e^{2\pi i z}$, $\widehat{D}(\omega, q)$ becomes a function on $\mathbb{H}$. Our main theorem is as follows.

Theorem 1.1 *Let $-\omega^{-1} \neq 1$ be a primitive mth root of unity. Then $q^{-\frac{1}{12}}\widehat{D}(\omega, q) \times \widehat{D}(\omega^{-1}, q)$ is the weight 1 modular form*

$$q^{-\frac{1}{12}}\widehat{D}(\omega, q)\widehat{D}\bigl(\omega^{-1}, q\bigr) = \frac{\eta^4(q^2)\eta^2(\omega^2, q)}{\eta^2(q)\eta^3(\omega^2, q^2)}.$$

Example As an example of Theorem 1.1, we consider the case $\omega = i$. In this case, by (3.4), we see that

$$\widehat{G}(i,q) = (1+i)F\big(q^{-1},0;-1,q^2\big),$$
$$\widehat{G}(-i,q) = (1-i)F\big(q^{-1},0;-1,q^2\big),$$

which shows that $\widehat{D}(i,q) = \widehat{D}(-i,q)$. Thus, we find the weight $1/2$ modular form

$$q^{-\frac{1}{24}}\widehat{D}(i,q) = \frac{\eta^7(q^2)}{\eta^3(q)\eta^3(q^4)} = \frac{q^{\frac{1}{12}}}{4} \cdot \frac{\eta(q^2)}{\eta(q)\eta(q^4)} \cdot D^2(i,q), \tag{1.4}$$

where $D(\omega,q) := (1+w)G(\omega,q) + (1-\omega)G(-\omega,q)$ is defined by Monks and Ono in [6], where it is shown that $q^{\frac{1}{24}}D(i,q)$ is a weight $1/2$ modular form given by a specific η-quotient (agreeing with (1.4)). (See the example following Theorem 1.1 in [6].)

The remainder of the paper is structured as follows. In Sect. 2, beginning with previously known results due to Fine [3], we deduce a more general statement (Proposition 2.2) regarding the modularity of a certain expression involving basic hypergeometric series. In Sect. 3, we prove Theorem 1.1. To do this, we perform initial manipulations on $\widehat{G}(\omega;q)$ and then explain how one can deduce Theorem 1.1 from the results in Sect. 2.

2 Modularity and basic hypergeometric series

In this section, beginning with results due to Fine [3], we deduce a more general statement regarding the modularity of a certain expression involving basic hypergeometric series. The basic hypergeometric series $F(a,b;t,q)$ are defined by

$$F(a,b;t,q) := \sum_{n\geq 0} \frac{(aq;q)_n}{(bq;q)_n} t^n$$

for specified parameters a, b and t. It is well known that the series $F(a,b;t,q)$ are well defined for any choice of parameters a, b, and t with the exception of certain instances when $b = q^{-n}$ or $t = q^{-n}$, $n \in \mathbb{N}$. For a more explicit description, see [3], Chap. 1, Sect. 3. Here, we will consider the following difference of basic hypergeometric series, $D_F(\alpha,\beta,q)$, defined by

$$D_F(\alpha,\beta,q) := F(\alpha,0;\beta,q) - \alpha^{-1}\beta^{-1}F\big(\beta^{-1},0;\alpha^{-1},q\big).$$

Before stating the main result of this section, we must first introduce some notation. First, we define a function on $\frac{1}{2}\mathbb{Z}$ as follows:

$$h(r) := \begin{cases} 0, & r \in \mathbb{Z}, \\ 1, & r \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

Next, we set the following notation.

Definition 2.1 For $r, s \in \frac{1}{2}\mathbb{Z}$ and parameters $\zeta, \xi, \alpha, \beta$ such that the expressions below are defined, we let

$$P(\alpha, \beta) := \frac{(1-\alpha^{-1}\beta^{-1})(1-\alpha\beta)}{(1-\alpha)(1-\alpha^{-1})(1-\beta)(1-\beta^{-1})}, \tag{2.1}$$

$$c_1^0(\xi, r, q) := \left(-\xi^{-1}\right)^{r-1} q^{-r(r-1)/2} \cdot \frac{(1-\xi^{-1})}{(1-\xi q^r)}, \tag{2.2}$$

$$c_{-1}^0(\xi, r, q) := (-\xi)^{-r-1} q^{-r(r+1)/2} \cdot \frac{(1-\xi)}{(1-\xi^{-1}q^{-r})}, \tag{2.3}$$

$$c_1^1(\xi, r, q) := \left(-\xi^{-1}\right)^{r-\frac{1}{2}} q^{-\frac{1}{16}-\frac{r(r-1)}{2}} \cdot \left(1-\xi q^r\right)^{-1}, \tag{2.4}$$

$$c_{-1}^1(\xi, r, q) := (-\xi)^{-r-\frac{1}{2}} q^{-\frac{1}{16}-\frac{r(r+1)}{2}} \cdot \left(1-\xi^{-1}q^{-r}\right)^{-1}, \tag{2.5}$$

$$c_0^0(\xi, r, q) := 1, \tag{2.6}$$

$$\kappa \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \begin{pmatrix} \alpha, \beta & \\ \zeta, \xi \,; & q \\ r, s & \end{pmatrix} := P(\alpha, \beta) \cdot \frac{\left(c_{\delta_1}^{h(r+s)}(\zeta\xi, r+s, q)\right)^2}{c_{\delta_2}^{h(r)}(\zeta, r, q) \cdot c_{\delta_3}^{h(s)}(\xi, s, q)}, \tag{2.7}$$

where $\delta_i \in \{1, -1, 0\}$, $1 \le i \le 3$.

Next, we define the functions $N_1(\zeta, \xi, r, s; q)$ and $N_2(\zeta, r, s; q)$ by

$$N_1(\zeta, \xi, r, s; q) := \begin{cases} \eta(\zeta, q)\eta(\xi, q), & r, s \in \mathbb{Z}, \\ \eta(\zeta, q)\frac{\eta(\xi, q^{\frac{1}{2}})}{\eta(\xi, q)}, & r \in \mathbb{Z}, s \in \frac{1}{2} + \mathbb{Z}, \\ \eta(\xi, q)\frac{\eta(\zeta, q^{\frac{1}{2}})}{\eta(\zeta, q)}, & r \in \frac{1}{2} + \mathbb{Z}, s \in \mathbb{Z}, \\ \frac{\eta(\zeta, q^{\frac{1}{2}})\eta(\xi, q^{\frac{1}{2}})}{\eta(\zeta, q)\eta(\xi, q)}, & r, s \in \frac{1}{2} + \mathbb{Z}, \end{cases} \tag{2.8}$$

$$N_2(\zeta, r, s; q) := \begin{cases} \eta^2(\zeta, q), & r + s \in \mathbb{Z}, \\ \frac{\eta^2(\zeta, q^{\frac{1}{2}})}{\eta^2(\zeta, q)}, & r + s \in \frac{1}{2} + \mathbb{Z}. \end{cases} \tag{2.9}$$

The functions N_1 and N_2 are modular forms for roots of unity ζ and ξ. We define one more function depending on $r, s \in \frac{1}{2}\mathbb{Z}$:

$$p(r, s) := \begin{cases} 0, & r, s \in \mathbb{Z}, \\ \frac{1}{16}, & r + s \in \frac{1}{2} + \mathbb{Z}, \\ -\frac{1}{8}, & r, s \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

The following result shows that up to a finite product, the product $D_F(\zeta_m q^S, \zeta_n q^T, q) \cdot D_F(\zeta_m^{-1} q^{-S}, \zeta_n^{-1} q^{-T}, q)$ is a modular form (with easily computable weight) under certain hypotheses on the parameters defining the arguments, ζ_m, ζ_n, S, and T.

Proposition 2.2 *Let $m, n \in \mathbb{N}$, and let ζ_m be an mth root of unity, and ζ_n an nth root of unity. Next, we define*

$$K := K(S, T, \zeta_m, \zeta_n; q) = \kappa^{-1} \begin{bmatrix} \mathrm{sgn}(S+T) \\ \mathrm{sgn}(S) \\ \mathrm{sgn}(T) \end{bmatrix} \begin{pmatrix} \zeta_m q^S, \zeta_n q^T & \\ \zeta_m, \zeta_n & ; \; q \\ S, T & \end{pmatrix}. \tag{2.10}$$

If ζ_m, ζ_n, S, T satisfy one of the following four conditions

(i) $\zeta_m \neq 1$, $\zeta_n \neq 1$, *and* $S, T \in \frac{1}{2}\mathbb{Z}$,
(ii) $\zeta_m \neq 1$, $\zeta_n = 1$, $S \in \frac{1}{2}\mathbb{Z}$, *and either* $T \in \mathbb{Z}^{>0}$ *or* $T \in \frac{1}{2} + \mathbb{Z}$,
(iii) $\zeta_m = 1$, $\zeta_n \neq 1$, $T \in \frac{1}{2}\mathbb{Z}$, *and either* $S \in \mathbb{Z}^{<0}$ *or* $S \in \frac{1}{2} + \mathbb{Z}$,
(iv) $\zeta_m = \zeta_n = 1$, *and either* $S \in \mathbb{Z}^{<0}$, $T \in \mathbb{Z}^{>0}$, *or* $S, T \in \frac{1}{2} + \mathbb{Z}$,

then the following is true:

$$\begin{aligned} &K \cdot q^{\frac{1}{12} - p(S,T)} \cdot D_F\big(\zeta_m q^S, \zeta_n q^T, q\big) \cdot D_F\big(\zeta_m^{-1} q^{-S}, \zeta_n^{-1} q^{-T}, q\big) \\ &\quad = \frac{\eta^2(q) N_2(\zeta_m \zeta_n, S, T; q)}{N_1(\zeta_m, \zeta_n, S, T; q)}. \end{aligned} \tag{2.11}$$

Moreover, the expression in (2.11) *is a modular form whose weight is determined using* (2.8) *and* (2.9).

We recall the definition of the signum function appearing in Proposition 2.2,

$$\mathrm{sgn}(x) := \begin{cases} 1, & x > 0, \\ -1, & x < 0, \\ 0, & x = 0. \end{cases}$$

We also remark that conditions (i)–(iv) in Proposition 2.2 are to ensure the expressions appearing for D_F are well defined. To prove Proposition 2.2, we require some known results regarding basic hypergeometric series. By observing that the $F(a, b; t, q)$ satisfy difference equations of the form $f_n = L_n + M_n f_{n+1}$, $n \geq 0$, iterative methods allow one to solve for f_0. This procedure is described by Fine in [3], Chap. 1, Sects. 5–7. One result given by these methods that will be useful here is the following.

Lemma 2.3 *Assuming all series are well defined, one has the equality*

$$\begin{aligned} &F(a, 0; t, q) \cdot \frac{(ba^{-1}; q)_\infty}{(bq; q)_\infty (ba^{-1}t^{-1}; q)_\infty} \\ &\quad = F(a, b; t, q) + \frac{ba^{-1}t^{-1}}{1 - ba^{-1}t^{-1}} \sum_{n \geq 0} \frac{(ba^{-1}; q)_n q^n}{(bq; q)_n (bqa^{-1}t^{-1}; q)_n}. \end{aligned} \tag{2.12}$$

Here and throughout the expression $(a; q)_\infty := \prod_{n \geq 0}(1 - aq^n)$. By the iterative procedures mentioned above, one may also deduce the following (see [3]).

Lemma 2.4 *Assuming the series below are well defined, the following identities hold*:

$$\frac{(atq;q)_\infty}{(t;q)_\infty} = F(a,1;t,q), \tag{2.13}$$

$$\sum_{n\geq 0} \frac{(t;q)_n q^n}{(bq;q)_n (q;q)_n} = F\left(bt^{-1},0;t,q\right) \frac{(t;q)_\infty}{(bq;q)_\infty (q;q)_\infty}. \tag{2.14}$$

Armed with Lemma 2.3 and Lemma 2.4, we first apply (2.14) to the series appearing in the right-hand side of (2.12), where we take $b=1$. We find

$$\sum_{n\geq 0} \frac{(a^{-1};q)_n q^n}{(q;q)_n (qa^{-1}t^{-1};q)_n} = F\left(t^{-1},0;a^{-1},q\right) \cdot \frac{(a^{-1};q)_\infty}{(a^{-1}t^{-1}q;q)_\infty (q;q)_\infty}. \tag{2.15}$$

Next we substitute (2.15) into (2.12) (with $b=1$) and replace the $F(a,1;t,q)$ appearing in (2.12) by the appropriate infinite product given by (2.13). Simplifying, we find

$$\begin{aligned} D_F(a,t,q) &= \frac{(q;q)_\infty (a^{-1}t^{-1};q)_\infty (atq;q)_\infty}{(a^{-1};q)_\infty (t;q)_\infty} \\ &= \frac{(1-a^{-1}t^{-1})}{(1-a^{-1})(1-t)} \cdot q^{-\frac{1}{12}} \frac{\eta(q)\eta(at,q)}{\eta_0(a^{-1},q)\eta_0(t,q)} \end{aligned} \tag{2.16}$$

for parameters a,t,q such that this expression is well defined. We will need to prove one additional lemma.

Lemma 2.5 *Assuming notation as above, the following are true.*

(i) *Let $R \in \mathbb{Z}$. Then we have*

$$\eta_0\left(\zeta q^R,q\right)\eta_0\left(\zeta^{-1}q^{-R},q\right) = c^0_{\mathrm{sgn}(R)}(\zeta,R,q) \cdot \eta_0(\zeta,q)\eta_0\left(\zeta^{-1},q\right).$$

(ii) *Let $R \in \frac{1}{2}+\mathbb{Z}$. Then we have*

$$\eta_0\left(\zeta q^R,q\right)\eta_0\left(\zeta^{-1}q^{-R},q\right) = c^1_{\mathrm{sgn}(R)}(\zeta,R,q) \cdot \frac{\eta_0(\zeta,q^{\frac{1}{2}})\eta_0(\zeta^{-1},q^{\frac{1}{2}})}{\eta_0(\zeta,q)\eta_0(\zeta^{-1},q)}.$$

Proof To prove (i), we first note that the case $R=0$ follows trivially by the definition of $c^0_0(\zeta,0,q)$. Next, we note that one need only consider the case $R \in \mathbb{Z}^{\geq 1}$. The result for the case $R \in \mathbb{Z}^{\leq -1}$ follows from the result for $R \geq 1$ after replacing R by $-R$, and ζ by ζ^{-1}. Thus, assuming $R \in \mathbb{Z}^{\geq 1}$, we have

$$\begin{aligned} &\eta_0\left(\zeta q^R,q\right)\eta_0\left(\zeta^{-1}q^{-R},q\right) \\ &= q^{\frac{1}{24}} \prod_{k\geq 1}\left(1-\zeta q^{R+k}\right)\left(1-\zeta^{-1}q^{-R+k}\right) \end{aligned}$$

$$= \eta_0(\zeta,q)\eta_0\big(\zeta^{-1},q\big)\cdot \prod_{k=-R+1}^{0}\big(1-\zeta^{-1}q^k\big)\prod_{k=1}^{R}\big(1-\zeta q^k\big)^{-1}$$

$$= \eta_0(\zeta,q)\eta_0\big(\zeta^{-1},q\big)\cdot \prod_{k=0}^{R-1}\big(1-\zeta^{-1}q^{-k}\big)\prod_{k=1}^{R}\big(1-\zeta q^k\big)^{-1}$$

$$= \eta_0(\zeta,q)\eta_0\big(\zeta^{-1},q\big)\cdot \frac{(1-\zeta^{-1})}{(1-\zeta q^R)}\prod_{k=1}^{R-1}\frac{(1-\zeta^{-1}q^{-k})}{(1-\zeta q^k)}$$

$$= \eta_0(\zeta,q)\eta_0\big(\zeta^{-1},q\big)\cdot \frac{(1-\zeta^{-1})}{(1-\zeta q^R)}\cdot\big(-\zeta^{-1}\big)^{R-1}q^{-R(R-1)/2}$$

$$= c_1^0(\zeta,R,q)\cdot\eta_0(\zeta,q)\eta_0\big(\zeta^{-1},q\big),$$

where we understand that any empty product appearing above (which occurs only in the case $R=1$) equals 1. The proof of (ii) follows similarly. Again we need only consider the case $R\in\frac{1}{2}+\mathbb{Z}^{\geq 0}$, as the case $R\in\frac{1}{2}+\mathbb{Z}^{<0}$ follows from the established result for $R\in\frac{1}{2}+\mathbb{Z}^{\geq 0}$ after replacing R by $-R$, and ζ by ζ^{-1}. For $R\in\frac{1}{2}+\mathbb{Z}^{\geq 0}$, we compute

$$\begin{aligned}
&\eta_0\big(\zeta q^R,q\big)\eta_0\big(\zeta^{-1}q^{-R},q\big)\\
&\quad= q^{\frac{1}{24}}\prod_{k\geq 1}\big(1-\zeta q^{R+k}\big)\big(1-\zeta^{-1}q^{-R+k}\big)\\
&\quad= q^{\frac{1}{24}}\prod_{k\geq 1}\big(1-\zeta q^{\frac{1}{2}(2(R+k-\frac{1}{2})+1)}\big)\big(1-\zeta^{-1}q^{\frac{1}{2}(2(-R+k+\frac{1}{2})-1)}\big)\\
&\quad= q^{\frac{1}{24}}\prod_{k\geq R+\frac{1}{2}}\big(1-\zeta q^{\frac{1}{2}(2k+1)}\big)\prod_{k\geq -R+\frac{3}{2}}\big(1-\zeta^{-1}q^{\frac{1}{2}(2k-1)}\big)\\
&\quad= q^{\frac{1}{24}}\prod_{k=0}^{R-\frac{1}{2}}\big(1-\zeta q^{\frac{1}{2}(2k+1)}\big)^{-1}\prod_{k\geq 0}\big(1-\zeta q^{\frac{1}{2}(2k+1)}\big)\\
&\qquad\times\prod_{-R+\frac{3}{2}}^{1}\big(1-\zeta^{-1}q^{\frac{1}{2}(2k-1)}\big)\prod_{k\geq 2}\big(1-\zeta^{-1}q^{\frac{1}{2}(2k-1)}\big)\\
&\quad= \frac{q^{\frac{1}{24}}}{(1-\zeta q^R)}\prod_{k\geq 0}\big(1-\zeta q^{\frac{1}{2}(2k+1)}\big)\big(1-\zeta^{-1}q^{\frac{1}{2}(2k+1)}\big)\\
&\qquad\times\prod_{k=0}^{R-\frac{3}{2}}\frac{(1-\zeta^{-1}q^{\frac{1}{2}(-2k-1)})}{(1-\zeta q^{\frac{1}{2}(2k+1)})},
\end{aligned}\tag{2.17}$$

where we understand the empty product in (2.17) above (which only occurs when $R = \frac{1}{2}$) to equal 1. Continuing, we simplify and see that (2.17) becomes

$$\frac{q^{\frac{1}{24}}}{(1-\zeta q^R)} \prod_{k\geq 1} \frac{(1-\zeta q^{k/2})(1-\zeta^{-1}q^{k/2})}{(1-\zeta q^k)(1-\zeta^{-1}q^k)} \big(-\zeta^{-1}q^{-\frac{1}{2}}\big)^{R-\frac{1}{2}} q^{-(R-\frac{3}{2})(R-\frac{1}{2})/2}$$

$$= c_1^1(\zeta, r, q) \cdot \frac{\eta_0(\zeta, q^{\frac{1}{2}})\eta_0(\zeta^{-1}, q^{\frac{1}{2}})}{\eta_0(\zeta, q)\eta_0(\zeta^{-1}, q)}. \qquad \square$$

To prove Proposition 2.2, one uses (2.16) to rewrite the product $D_F(\zeta_m q^S, \zeta_n q^T, q) \cdot D_F(\zeta_m^{-1} q^{-S}, \zeta_n^{-1} q^{-T}, q)$ in terms of the functions $\eta(q)$, $\eta(\omega, q)$, and $P(\alpha, \beta)$ (defined in (2.1)). Next, one applies Lemma 2.5 to the expression obtained. Proposition 2.2 follows after using the definition of the finite product κ in (2.7), and the functions N_1 and N_2 in (2.8) and (2.9).

3 Proof of Theorem 1.1

Before we are able to apply Proposition 2.2 to the function $\widehat{G}(\omega, q)$, we must first obtain a new expression for $\widehat{G}(\omega, q)$ to which Proposition 2.2 applies. We do this by showing that the function

$$\widehat{G}(\omega, t; q) = \sum_{n\geq 0} \frac{t^n}{(\omega^{-1}q; q)_n} = \sum_{n=0}^{\infty} g_n t^n$$

satisfies a certain recurrence relation. Iterating the recurrence relation, and letting $t = -\omega^{-1}$, reveals the following.

Proposition 3.1 *The function $\widehat{G}(\omega, q)$ satisfies*

$$\widehat{G}(\omega, q) = \frac{1}{1+\omega^{-1}} \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{-2n} q^{n^2}}{(\omega^{-2}q^2; q^2)_n}.$$

Proof One can see by definition of g_n that $g_{n+1}(1-\omega^{-1}q^{n+1}) = g_n$. If one multiplies this equation by t^{n+1} and sums on n, one finds that

$$\widehat{G}(\omega, t; q) = \frac{-1+\omega^{-1}}{t-1} - \frac{\omega^{-1}}{t-1}\widehat{G}(\omega, tq, q). \tag{3.1}$$

Moreover, the fact that $(1-a)(aq;q)_{n-1} = (a;q)_n$ implies

$$\widehat{G}(\omega, t; q) = 1 + \frac{t}{1-\omega^{-1}q}\widehat{G}\big(\omega q^{-1}, t; q\big). \tag{3.2}$$

Combining (3.1) and (3.2) shows that

$$\widehat{G}(\omega, t; q) = \frac{1}{1-t} + \frac{tq\omega^{-1}}{(1-t)(1-\omega^{-1}q)}\widehat{G}\big(\omega q^{-1}, tq; q\big). \tag{3.3}$$

Iterating (3.3) two more times, we see

$$\widehat{G}(\omega,t;q)=\frac{1}{1-t}\left(1+\frac{tq\omega^{-1}}{(tq;q)_1(\omega^{-1}q;q)_1}+\frac{t^2q^4\omega^{-2}}{(tq;q)_1(\omega^{-1}q)_2}\widehat{G}\big(\omega q^{-2},tq^2,q\big)\right),$$

$$\widehat{G}(\omega,t;q)=\frac{1}{1-t}\left(1+\frac{tq\omega^{-1}}{(tq;q)_1(\omega^{-1}q;q)_1}+\frac{t^2q^4\omega^{-2}}{(tq;q)_2(\omega^{-1}q;q)_2}\right.$$
$$\left.+\frac{t^3q^9\omega^{-3}}{(tq;q)_2(\omega^{-1}q;q)_3}\widehat{G}\big(\omega q^{-3},tq^3,q\big)\right).$$

Continuing in this manner and iterating (3.3) an infinite number of times proves the proposition, after letting $t=-\omega^{-1}$. □

Next, we make use of another expression for basic hypergeometric series of the shape appearing in Proposition 3.1.

Lemma 3.2 *For any parameters a,t, and q such that $F(a,0;t,q)$ is well defined, we have*

$$F(a,0;t,q)=\frac{1}{1-t}\sum_{n\geq 0}\frac{(-at)^n q^{(n^2+n)/2}}{(tq;q)_n}.$$

Proof See [3], Chap. 1, Sects. 4–6. □

By Lemma 3.2 and Proposition 3.1, we see that

$$\widehat{G}(\omega,q)=\big(1-\omega^{-1}\big)F\big(q^{-1},0;\omega^{-2},q^2\big), \tag{3.4}$$

$$\big(-1+\psi\big(-\omega^2,q\big)\big)=-\omega^2 qF\big(\omega^2,0;q,q^2\big). \tag{3.5}$$

Thus,

$$\begin{aligned}\widehat{D}(\omega,q)\big(1-\omega^{-2}\big)^{-1}&=\widehat{G}(\omega,q)\cdot\big(1-\omega^{-1}\big)^{-1}+\big(-1+\psi\big(-\omega^2,q\big)\big)\\&=D_F\big(q^{-1},\omega^{-2},q^2\big).\end{aligned} \tag{3.6}$$

After a short calculation using (2.16), one finds that

$$D_F\big(q^{-1},\omega^2,q^2\big)=D_F\big(q,\omega^2,q^2\big)\cdot\frac{(1-q^{-1})(1-q\omega^2)}{(1-q^{-1}\omega^{-2})}, \tag{3.7}$$

or equivalently (after applying (3.6)) that

$$\begin{aligned}\widehat{D}(\omega,q)\widehat{D}\big(\omega^{-1},q\big)&=D_F\big(q^{-1},\omega^{-2},q^2\big)D_F\big(q,\omega^2,q^2\big)\\&\quad\times\frac{(1-q\omega^2)(1-\omega^{-2})(1-\omega^2)(1-q^{-1})}{(1-q^{-1}\omega^{-2})}.\end{aligned} \tag{3.8}$$

We compute the relevant factor

$$\kappa := \kappa \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \begin{pmatrix} q^{-1}, \omega^{-2} & \\ 1, \omega^{-2} & ; \ q^2 \\ -\frac{1}{2}, 0 & \end{pmatrix} = P\big(q^{-1}, \omega^{-2}\big) \cdot \frac{(c_{-1}^1(\omega^{-2}, -\frac{1}{2}, q^2))^2}{c_{-1}^1(1, -\frac{1}{2}, q^2)}$$
$$= q^{\frac{1}{8}} \frac{(1 - q^{-1}\omega^{-2})}{(1 - q^{-1})(1 - \omega^{-2})(1 - \omega^2)(1 - \omega^2 q)}. \tag{3.9}$$

Next, we apply Proposition 2.2 to the expression in (3.8) and find using (3.9) that

$$\widehat{D}(\omega, q)\widehat{D}\big(\omega^{-1}, q\big) = \kappa \cdot q^{-\frac{1}{6}+2p(-\frac{1}{2},0)} \frac{(1 - \omega^2)(1 - \omega^{-2})(1 - q^{-1})(1 - q\omega^2)}{(1 - q^{-1}\omega^{-2})}$$
$$\times \eta^2\big(q^2\big) \frac{N_2(\omega^{-2}, -\frac{1}{2}, 0; q^2)}{N_1(1, \omega^{-2}, -\frac{1}{2}, 0; q^2)}$$
$$= q^{\frac{1}{12}} \frac{\eta^4(q^2)\eta^2(\omega^2, q)}{\eta^2(q)\eta^3(\omega^2, q^2)}.$$

This proves Theorem 1.1.

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