PERIODIC PARTIAL THETA FUNCTIONS AND q-HYPERGEOMETRIC KNOT MULTISUMS AS QUANTUM JACOBI FORMS

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Abstract. We prove that general two-variable partial theta functions with periodic coefficients are quantum Jacobi forms, and establish their explicit transformation and analytic properties. As applications, we also prove that seven infinite families of q-hypergeometric multisums and related partial theta functions of interest arising from certain knot colored Jones polynomials, Kashaev invariants for torus knots and Virasoro characters, and "strange" identities, appearing in (separate) works of Bijaoui et al., Hikami, Hikami-Kirillov, Lovejoy, and Zagier are quantum Jacobi forms.

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Part I. Introduction, results summary, and preliminaries (§1–2)

1. Introduction and results

Our results in this paper are partially rooted in applications of q-series to topology with connections to modularity. To explain this by way of example, let $T_{(2,3)}$ denote the right-handed trefoil knot, as seen in Figure 1. The N-colored Jones polynomial $J_N(K,q)$ for a knot

Figure 1. Right-handed trefoil knot $T_{(2,3)}$

K is a well-studied knot invariant, and it is known (see e.g., [20, 25, 27]) that this invariant for the aforementioned torus knot may be explicitly given in terms of a (terminating) q-hypergeometric series as follows:

$$J_N(T_{(2,3)};q) = q^{1-N}\sum_{n=0}^{\infty} q^{-nN}(q^{1-N};q)_n. \tag{1.1}$$

It turns out that this topological q-series also possesses modular properties in the following sense. A little over a decade ago, Zagier defined the notion of a quantum modular form (of weight k), a complex function $f:\mathbb{Q}\to\mathbb{C}$ defined on the rationals (as opposed to the upper-half complex plane as in the case of modular forms) which exhibits modular-like transformation properties there, up to the addition of smooth error functions in $\mathbb{R}$. That is, one requires the error to modularity functions

$$h_\gamma(x) := f(x) - \varepsilon_1^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

to satisfy a suitable property in $\mathbb{R}$ such as analyticity or continuity, for all $\gamma := \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ or suitable subgroup. (See [7, 32] and the remainder of this section for more details.) A fundamental example of such a form given by Zagier is obtained using Kontsevich's function

$$F(q) := \sum_{n=0}^{\infty}(q;q)_n, \tag{1.2}$$

so-called "strange" in part due to the fact that it converges nowhere in $\mathbb{C}$ except at roots of unity $q=\zeta_k^h$. Here and throughout we use the notation $\zeta_N := e^{2\pi i/N}$ for roots of unity, and recall that the q-Pochhammer symbol is defined for $n\in\mathbb{N}_0\cup\{\infty\}$ by $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ (taking the empty product to be 1 as usual). Comparing the Kontsevich-Zagier quantum modular form (1.2) to the N-colored Jones polynomial for $T_{(2,3)}$, one finds (and we credit Hikami-Lovejoy [24] for this observation) that they agree (up to a constant multiple) at N-th roots of unity ζ_N^m

$$J_N(T_{(2,3)};\zeta_N^m) = \zeta_N^m F(\zeta_N^m), \tag{1.3}$$

thereby connecting these knot invariants, q-hypergeometric series, and quantum modular forms.

A key ingredient in Zagier's proof of the quantum modularity of[1] $F(q)$ is the following "strange identity":

$$q^{\frac{1}{24}}\sum_{n=0}^{\infty}(q;q)_n \text{ ``}=\text{''} -\frac{1}{2}\sum_{n=0}^{\infty} n\left(\tfrac{12}{n}\right)q^{\frac{n^2}{24}}, \tag{1.4}$$

where the right-hand side features the holomorphic Eichler integral of the modular Dedekind η-function $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, with $q = e^{2\pi i\tau}, \tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ the usual modular variable, and $\left(\frac{12}{\cdot}\right)$ the Kronecker symbol. Zagier's (1.4) is referred to as a (strange) "identity" ("=") versus an identity (=) due to the fact that it holds only asymptotically as q tends towards roots of unity radially from within the complex unit disc. In the limit, we find the left-hand side of (1.4) gives (up to multiplication by $q^{1/24}$) the value of $F(q)$, while the Eichler integral, a kind of partial theta function, governs the behavior on the right-hand side. While it is not modular on $\mathbb{H}$, the right-hand side Eichler integral is shown to possess quantum modular properties in $\mathbb{R}$, which are passed to $F(q)$ via the "identity" (1.4).

Zagier's (1.4) showcases examples of the types of functions which we further study in more generality in this paper in the context of quantum modular (Jacobi) forms: q-hypergeometric series and partial theta functions, and (via (1.3)) associated knot invariants. On their own, q-hypergeometric series like Heine's [18]

$${}_2\phi_1\left(a,b,c;q,x\right) := \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}x^n$$

have made many appearances and led to numerous (other) applications in mathematics. Citing Andrews who quotes Sawyer [1] on their predecessors, hypergeometric functions like ${}_2F_1(a,b,c;x) = \lim_{q\to 1^-} {}_2\phi_1\left(q^a,q^b,q^c;q,x\right)$,

> *"...there are many functions used by engineers or physicists – the Legendre polynomials and the Bessel functions, for example – which are particular cases of the hypergeometric function. In fact, there must be many universities today where 95 per cent, if not 100 per cent, of the functions studied by physicists, engineering and even mathematics students are covered by this single symbol* ${}_2F_1(a,b,c;x)$*."*

Similarly, q-hypergeometric series have seen additional prominent applications to areas including the theory of partitions at the intersection of combinatorics and number theory; modular forms, mock modular forms and harmonic Maass forms; q-difference equations, congruences, and summation formulae in number theory; counting vector spaces and differential operators; understanding torsion in the Bloch group of $\overline{\mathbb{Q}}$; and much much more (see e.g. [1, 7, 9]) – not to mention topology which we are also in part motivated by in this paper.

Holomorphic Eichler integrals such as on the right-hand side of (1.4), and their relatives partial and false theta functions, have also played significant roles in combinatorics, q-series, modular, mock modular and quantum modular forms, and with applications to other areas including quantum topology (see [14] and references therein for more). In these directions, we define and study the following very general class of *periodic partial Jacobi theta functions*,

[1]As is standard in this subject, for simplicity we may slightly abuse terminology in this paper and refer to a function as a modular form or other modular object when in reality it must first be multiplied by a suitable power of q to transform in the right way.

named for their periodic coefficients and their sum over a partial (half) lattice – and also Theorem 1 in Section 3, which reveals (quantum) Jacobi properties.[2]

Definition 1. *Let $r \in \mathbb{N}$, and let $\epsilon_j \in \mathbb{C}$ be a fixed constant for each $1 \leq j \leq r$. Further, let $\beta, \alpha_j \in \mathbb{N}$ with $0 < \alpha_j < \beta$ for each $1 \leq j \leq r$, and $\alpha_j \neq \alpha_k$ for $1 \leq j \neq k \leq r$. Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a periodic function defined explicitly by $\chi(n) = \epsilon_j$ if $n \equiv \alpha_j \pmod{\beta}$ for each $1 \leq j \leq r$ (and $\chi(n) = 0$ otherwise). We define the* **periodic partial Jacobi theta function** *(with respect to χ) by*

$$\Theta_\chi(x;q) := \sum_{n=0}^{\infty} \chi(n) q^{\frac{n^2}{2\beta^2}} x^{\frac{n}{2\beta}}.$$

Explicit examples of such theta functions and their defining characters χ originally appearing in work of others are studied in Sections 5.1–5.7. For example, from [21, 26] we have the (one variable) periodic partial theta function

$$-\frac{1}{2} \sum_{n \geq 0} \chi_{8k+4}^{(a)}(n) q^{\frac{n^2-(2k-2a-1)^2}{8(2k+1)}},$$

where $\chi_{8k+4}^{(a)}(n)$ is the periodic function modulo $8k+4$ given by

$$\chi_{8k+4}^{(a)}(n) = \begin{cases} 1, & \text{if } n \equiv 2k-2a-1 \text{ or } 6k+2a+5 \pmod{8k+4}, \\ -1, & \text{if } n \equiv 2k+2a+3 \text{ or } 6k-2a+1 \pmod{8k+4}, \\ 0, & \text{otherwise.} \end{cases}$$

This (and similar explicit functions and their two variable analogues) is further studied in Section 5, and expressed in the notation of Definition 1 in the proof of Theorem 2 there. The residue class $\alpha_j \equiv 0 \pmod{\beta}$ is not encompassed by 1 nor all proofs and motivating applications offered in Section 5. The reader will find additional general families of one-variable partial theta functions with periodic coefficients in the interesting recent related work [19] which we also discuss below.

Before explaining our main results, we recall that Bringmann and the author defined a formal notion of a quantum Jacobi form and offered the first example (arising from combinatorics) in [6]. In words, quantum Jacobi forms take values in $\mathbb{C}$, are defined in $\mathbb{Q} \times \mathbb{Q}$ (as opposed to $\mathbb{C} \times \mathbb{H}$ in the case of Jacobi forms as developed by Eichler and Zagier [11]), and exhibit Jacobi transformation properties there, up to the addition of smooth error functions in $\mathbb{R} \times \mathbb{R}$. Precisely, we have the following definition.

Definition 2 (Bringmann-Folsom [6])**.** *A weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \frac{1}{2}\mathbb{Z}$* **quantum Jacobi form** *is a complex-valued function ϕ on $\mathbb{Q} \times \mathbb{Q}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$, the functions $h_\gamma : \mathbb{Q} \times (\mathbb{Q} \setminus \gamma^{-1}(i\infty)) \to \mathbb{C}$ and $g_{(\lambda,\mu)} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ defined by*

$$h_\gamma(z;\tau) := \phi(z;\tau) - \varepsilon_1^{-1}(\gamma)(c\tau+d)^{-k} e^{\frac{-2\pi i m c z^2}{c\tau+d}} \phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right),$$

[2] Two notes on notation. While the periodic χ are dependent on and defined by the additional parameters $\{\alpha_j\}, \{\epsilon_j\}$, r and β, we suppress them in the naming of χ for ease of notation. We also choose to define $\Theta_\chi(x;q)$ with the shorter name *periodic partial Jacobi theta functions* instead of the more immediately clear yet longer name *partial Jacobi theta functions with periodic coefficients* again for ease and convenience.

$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \varepsilon_2^{-1}((\lambda,\mu))e^{2\pi im(\lambda^2\tau+2\lambda z)}\phi(z+\lambda\tau+\mu;\tau),$$

satisfy a "suitable" property of continuity or analyticity in a subset of $\mathbb{R}\times\mathbb{R}$.

Remarks.

(1) The complex numbers $\varepsilon_1(\gamma)$ and $\varepsilon_2((\lambda,\mu))$ satisfy $|\varepsilon_1(\gamma)| = |\varepsilon_2((\lambda,\mu))| = 1$; in particular, the $\varepsilon_1(\gamma)$ are such as those appearing in the theory of half-integral weight modular forms.
(2) We may modify the definition to allow modular transformations on appropriate subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domain to be a suitable subset of $\mathbb{Q}\times\mathbb{Q}$.
(3) The "suitable" property of continuity or analyticity required is intentionally left somewhat vague in order to mimic Zagier's definition of a quantum modular form [32].

Quantum modular forms have been well-studied since the time of their definition roughly 10 years ago; they have been shown to be related to the diverse areas of harmonic Maass forms, partial theta functions, colored Jones polynomials, meromorphic Jacobi forms, and vertex algebras, among other things (see, e.g., [7] and references therein). We also now know that the notion of a quantum modular form is related to Ramanujan's original notion of a mock theta function (see, e.g., [7, 8, 16]). The subject of quantum Jacobi forms also continues to develop; the known examples of quantum Jacobi forms to date have been established in [3, 4, 6, 10, 13, 17], and like quantum modular forms, quantum Jacobi forms have been shown to emerge in the diverse areas of number theory, combinatorics, topology, and mathematical physics. In both cases, a comprehensive theory is still lacking, and obtaining explicit and natural sources of quantum forms is a problem of interest.

1.1. **Results summary and outline of paper.** The aforementioned subjects have seen a great deal of research activity: papers such as [8, 13, 15, 19, 24] extend the example in the previous section into broader frameworks and families of examples, establishing quantum modular properties of Eichler integrals, partial theta functions, q-series, and colored Jones polynomials. Papers such as [2, 17, 29] examine similar problems from the perspective of q-hypergeometric series and sums-of-tails identities (noting that such an identity leads to (1.4)). Our main results in this paper extend these research directions as follows in items (1) - (4):

(1) In Theorem 1 in Section 3, we show that the general family of periodic partial theta functions in two variables (see Definition 1)

$$\Theta_\chi(x;q) = \sum_{n=0}^{\infty}\chi(n)q^{\frac{n^2}{2\beta^2}}x^{\frac{n}{2\beta}}$$

are quantum Jacobi forms, and establish their explicit transformation and analytic properties. A key component to our proof of this result involves establishing two-variable mock Jacobi forms in the (Jacobi) domain $\mathbb{C}\times\mathbb{H}$ which are "dual" to partial Jacobi theta functions, and subsequently establishing their suitable and explicit analytic properties in the "boundary" domain $\mathbb{R}\times\mathbb{R}$.

We also draw the reader's attention to interesting related work of Goswami-Osburn in [19] referenced above, in which they establish one-variable quantum modular properties of partial theta functions with even or odd periodic coefficients using different methods.

(2) As applications to Theorem 1, in a series of five theorems (Theorems 2–6), we prove that five rather general infinite families of q-hypergeometric multisums and related partial theta functions of interest are quantum Jacobi forms. One such family we study is this one

$$\sum_{n_1,\dots,n_k\geq 0}(xq;q)_{n_k}q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}}x^{2n_1+\cdots+2n_{k-1}+n_k}\prod_{j=1}^{k-1}\begin{bmatrix} n_{j+1}+\delta_{a,j} \\ n_j \end{bmatrix}_q, \tag{1.5}$$

which is used by Hikami [21] and Lovejoy [26] to establish Hikami's [21] elegant generalization of Zagier's (1.4) given by

(1.6)

$$\sum_{n_1,\dots,n_k\geq 0}(q;q)_{n_k}q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}}\prod_{j=1}^{k-1}\begin{bmatrix} n_{j+1}+\delta_{j,a} \\ n_j \end{bmatrix}_q \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 0}n\chi_{8k+4}^{(a)}(n)q^{\frac{n^2-(2k-2a-1)^2}{8(2k+1)}}.$$

Here and throughout, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, & 0\leq k\leq n, \\ 0, & \text{else,} \end{cases}$$

and $\chi_{8k+4}^{(a)}$ is a certain periodic function (see Section 5.1). The four additional q-hypergeometric multisums including (1.5) which we study in Sections 5.1–5.5 appear in recent related work of Lovejoy [26] on Bailey pairs and "strange" identities, motivated by Hikami's generalization of Zagier's (1.4) in [21], and the important Andrews-Gordon identities, which have implications in number theory, combinatorics, algebra, and physics.

(3) As another application to Theorem 1, we establish in Theorem 7 in Section 5.6 that a certain infinite family of q-hypergeometric multisums appearing in work of Bijaoui et al. [5] in their study of Kontsevich-Zagier series for torus knots $T(3,2^t)$ $(t\geq 2)$, along with a related family of partial Jacobi theta functions, are quantum Jacobi forms.

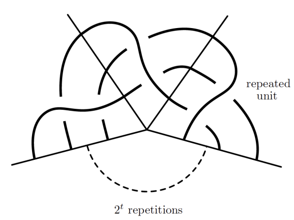

FIGURE 2. Torus knots $T(3,2^t)$ (image credited to [5])

(4) As a final application, in Theorem 8 in Section 5.7, we establish that a certain doubly infinite family of functions appearing in [22] in their study of Kashaev invariants for torus knots $T(s,t)$ and Virasoro characters

$$\mathrm{ch}_{n,m}^{s,t}(\tau)=\eta^{-1}(\tau)\sum_{k=0}^{\infty}\chi_{2st}^{(n,m)}(k)q^{\frac{k^2}{4st}}$$

of the minimal models $\mathcal{M}(s,t)$ $(s,t\in\mathbb{N},\gcd(s,t)=1)$, are quantum Jacobi forms. That is, work of Hikami and Kirillov explains that Kashaev invariants for the torus knots $T(s,t)$ coincide with Eichler integrals of Virasoro characters for the minimal model $\mathcal{M}(s,t)$, leading to new q-identities. E.g., it is well known that the Virasoro character for $\mathcal{M}(2,2m+1)$ is related to the aforementioned Andrews-Gordon identity generalizing the famous Rogers–Ramanujan identities. Particular attention is paid to the case of $s=3$ in [22], where it is shown that $(s,t)=(3,4)$ gives rise to famous identities of Slater [31]. Our work here establishes the quantum Jacobi properties of two variable extensions of the general characters for the minimal models $\mathcal{M}(s,t)$.

The remainder of the paper is structured as follows. Section 2 on modular preliminaries completes Part I of the paper. Sections 3–4 constitute the paper's Part II: Quantum periodic partial Jacobi theta functions, in which we state and prove Theorem 1 by way of intermediate results including Proposition 2, which extends earlier work in [13] and requires analysis in the complex Jacobi domain $\mathbb{C}\times\mathbb{H}$ and in $\mathbb{R}\times\mathbb{R}$ in order to (explicitly) establish quantum Jacobi properties in $\mathbb{Q}\times\mathbb{Q}$. Finally, in Theorems 2 - 8 in Part III: Applications to q-hypergeometric multisum knot families (Section 5) we apply Theorem 1 in order to establish the quantum Jacobi properties of seven infinite families of interest with associated q-hypergeometric knot sums and minimal model characters appearing in [5, 21, 22, 23, 26], and as described in our results summaries **(2)**–**(4)** above.

2. Modular preliminaries

2.1. **Modular and mock modular (Jacobi) forms.** In this section we define some modular-type functions used in the remainder of the paper. We let $q=e(\tau)$ and $w=e(z)$, where $e(\alpha):=e^{2\pi i\alpha}$, and begin with the Jacobi theta function of weight $1/2$, defined for $z\in\mathbb{C},\tau\in\mathbb{H}$ by

$$\vartheta(z;\tau):=\sum_{n\in\mathbb{Z}+\frac{1}{2}}e^{\pi i n^2\tau+2\pi i n(z+\frac{1}{2})}.$$

Lemma 1 (see [28])**.** *For $\lambda,\mu\in\mathbb{Z},\gamma=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, and $(z,\tau)\in\mathbb{C}\times\mathbb{H}$,*

(i) $\vartheta\left(z+\lambda\tau+\mu;\tau\right)=(-1)^{\lambda+\mu}q^{-\frac{\lambda^2}{2}}e^{-2\pi i\lambda z}\vartheta(z;\tau)$,
(ii) $\vartheta\left(\dfrac{z}{C\tau+D};\gamma\tau\right)=\varepsilon^3(\gamma)(C\tau+D)^{\frac{1}{2}}e^{\frac{\pi iCz^2}{C\tau+D}}\vartheta(z;\tau)$,
(iii) $\vartheta(z;\tau)=-iq^{\frac{1}{8}}w^{-\frac{1}{2}}\displaystyle\prod_{n=1}^{\infty}(1-q^n)(1-wq^{n-1})(1-w^{-1}q^n)$,

where for $C>0$ we have that

$$\varepsilon(\gamma)=\begin{cases}\frac{1}{\sqrt{i}}\left(\frac{D}{C}\right)i^{(1-C)/2}e^{\pi i(BD(1-C^2)+C(A+D))/12}, & \text{if } C \text{ is odd,}\\ \frac{1}{\sqrt{i}}\left(\frac{C}{D}\right)e^{\pi iD/4}e^{\pi i(AC(1-D^2)+D(B-C))/12}, & \text{if } D \text{ is odd.}\end{cases}\tag{2.1}$$

We also use the weight-3/2 theta functions defined for $\tau\in\mathbb{H}$ and $A,B\in\mathbb{R}$ by

$$g_{A,B}(\tau):=\sum_{\nu\in A+\mathbb{Z}}\nu e^{\pi i\nu^2\tau+2\pi i\nu B}.\tag{2.2}$$

Lemma 2 (see [30, 33])**.** *With hypotheses as above, we have:*
(i) $g_{A+1,B}(\tau)=g_{A,B}(\tau)$,
(ii) $g_{A,B+1}(\tau)=e^{2\pi iA}g_{A,B}(\tau)$,
(iii) $g_{A,B}(\tau+1)=e^{-\pi iA(A+1)}g_{A,A+B+\frac{1}{2}}(\tau)$,

(iv) $g_{A,B}\left(-\frac{1}{\tau}\right) = ie^{2\pi iAB}(-i\tau)^{\frac{3}{2}}g_{B,-A}(\tau)$,
(v) $g_{-A,-B}(\tau) = -g_{A,B}(\tau)$.

Next we define the level 2 Appell function for $z_1, z_2 \in \mathbb{C}$, $\tau \in \mathbb{H}$ (after Zwegers [7]) by

$$A_2(z_1, z_2; \tau) := \xi_1 \sum_{n\in\mathbb{Z}} \frac{\xi_2^n q^{n(n+1)}}{1-\xi_1 q^n}, \tag{2.3}$$

where $\xi_j = e(z_j), j \in \{1, 2\}$. While A_2 is not in general modular, it can be completed to a nonholomorphic Jacobi form $\widehat{A}_2$ defined by

$$\widehat{A}_2(z_1, z_2; \tau) := \tag{2.4}$$
$$A_2(z_1, z_2; \tau) + \frac{i}{2}\sum_{j=0}^{1} e^{2\pi ijz_1}\vartheta\left(z_2 + j\tau + \frac{1}{2}; 2\tau\right) R\left(2z_1 - z_2 - j\tau - \frac{1}{2}; 2\tau\right),$$

where R is defined by

$$R(z;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left\{\operatorname{sgn}(\nu) - E\left((\nu+\lambda)\sqrt{2y}\right)\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i\nu^2\tau-2\pi i\nu z}, \tag{2.5}$$

with $y := \operatorname{Im}(\tau)$, $\lambda := \frac{\operatorname{Im}(z)}{\operatorname{Im}(\tau)}$ and

$$E(z) := 2\int_0^z e^{-\pi u^2} du.$$

Lemma 3 (see [7]). *With hypotheses as above, for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the functions $\widehat{A}_2$ satisfy:*

(i) $\widehat{A}_2(-z_1, -z_2; \tau) = -\widehat{A}_2(z_1, z_2; \tau)$,

(ii) $\widehat{A}_2(z_1 + n_1\tau + m_1, z_2 + n_2\tau + m_2; \tau) = \xi_1^{2n_1-n_2}\xi_2^{-n_1}q^{n_1^2-n_1n_2}\widehat{A}_2(z_1, z_2; \tau)$,

(iii) $\widehat{A}_2\left(\frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}; \gamma\tau\right) = (c\tau + d)e^{\frac{\pi ic}{c\tau+d}(-2z_1^2+2z_1z_2)}\widehat{A}_2(z_1, z_2; \tau)$.

Lastly, we define the Mordell integral h, defined for $z \in \mathbb{C}, \tau \in \mathbb{H}$, by

$$h(z;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau u^2-2\pi zu}}{\cosh(\pi u)} du. \tag{2.6}$$

The following result relates the functions h and $g_{A,B}$ [33].

Lemma 4. *For $A, B \in (-\frac{1}{2}, \frac{1}{2})$,*

$$\int_0^{i\infty} \frac{g_{A+\frac{1}{2},B+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}} dz = -e^{-\pi iA^2\tau+2\pi iA\left(B+\frac{1}{2}\right)}h(A\tau - B;\tau).$$

2.2. **Groups and sets.** Here we define a number of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and study their Jacobi action on various subsets of $\mathbb{Q}\times\mathbb{Q}$ in Lemmas 5 and 6 below. We use the notation $\langle S\rangle$ to denote the group generated by the set S. We define the groups

$$G_{B,u} := \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 2Bu & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2B^2/u \\ 0 & 1\end{smallmatrix}\right)\right\rangle, \qquad G'_{B,u} := \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 2Bu & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 8B^2/u \\ 0 & 1\end{smallmatrix}\right)\right\rangle,$$

and for $\beta\in\mathbb{N}$

$$\begin{aligned}
\mathcal{G}(\beta,f_\beta) &:= \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 2f_\beta & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2\beta^2 \\ 0 & 1\end{smallmatrix}\right)\right\rangle,\\
H_{\ell_{2,\beta}} &:= \left\{\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})\colon A,D\equiv 1 \pmod{2\ell_{2,\beta}}, B,C\equiv 0 \pmod{\ell_{2,\beta}}\right\},\\
H'_{\ell_{2,\beta}} &:= \left\{\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in H_{\ell_{2,\beta}}\colon B\equiv 0 \pmod{2\beta^2}\right\},
\end{aligned}$$

where

$$f_\beta := \begin{cases} \ell_{2,\beta}, & 4\mid\beta,\\ 2\ell_{2,\beta}, & 4\nmid\beta,\end{cases} \qquad \text{and} \qquad \ell_{2,B} := \mathrm{lcm}(2,B) \quad (B\in\mathbb{N}).$$

We will later specify pairs $(B,u)\in\mathbb{Q}^2$ which yield subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Observe that $\mathcal{G}(\beta,f_\beta)$ is a subgroup of $G_{\beta,1}$, $G'_{B,u}$ is a subgroup of $G_{B,u}$, and $\mathcal{G}(\beta,f_\beta)$ is a subgroup of $H_{\ell_{2,\beta}}$.

We point out that these groups generalize certain groups appearing in [13] as follows. When $4\mid\beta$ as is required in [13], we have that $\ell_{2,\beta}=f_\beta=\beta$, and $\mathcal{G}(\beta,f_\beta)=\mathcal{G}(\beta,\beta)=G_{\beta,1}$. In this case, this group also equals the group G_β defined in [13]. Further, when $4\mid\beta$ the group $H_{\ell_{2,\beta}}=H_\beta$ above agrees with the group H_β of the same name in [13].

Next we define the several subsets of $\mathbb{Q}\times\mathbb{Q}$. Here and throughout, unless otherwise indicated, we assume all fractions c/d in $\mathbb{Q}$ are *reduced*, meaning that $c\in\mathbb{Z}$ and $d\in\mathbb{N}$, with $\gcd(c,d)=1$. We define

$$\mathcal{Q}^e_{B,u} := \left\{ \left(\tfrac{a}{b},\tfrac{h}{\kappa}\right)\in\mathbb{Q}\times\mathbb{Q}\colon \begin{array}{l} b\mid\kappa,\ \kappa \text{ is even, and if } \kappa\equiv 0 \pmod{2Bu},\\ \text{then } h\not\equiv\pm1 \pmod{4B^3/u}\end{array}\right\},$$

$$\mathcal{Q}^o_{B,u} := \left\{ \left(\tfrac{a}{b},\tfrac{h}{\kappa}\right)\in\mathbb{Q}\times\mathbb{Q}\colon \begin{array}{l} b\mid\kappa,\ \kappa \text{ is odd, and if } \kappa\equiv 0 \pmod{2Bu},\\ \text{then } h\not\equiv\pm1 \pmod{4B^3/u}\end{array}\right\},$$

$$\mathcal{Q}_{B,u} := \mathcal{Q}^e_{B,u}\cup\mathcal{Q}^o_{B,u},$$

$$\mathcal{Q}'_{B,u} := \left\{ \left(\tfrac{a}{b},\tfrac{h}{\kappa}\right)\in\mathbb{Q}\times\mathbb{Q}\colon \begin{array}{l} b\mid\kappa, \text{ and either } \kappa \text{ is odd, or } 4\mid\kappa \text{ and } \kappa/b \text{ is even},\\ \text{and if } \kappa\equiv 0 \pmod{2Bu}, \text{ then } h\not\equiv\pm1 \pmod{4B^3/u}\end{array}\right\},$$

and

$$Q_{\alpha,\beta,u} := \left\{ \left(\tfrac{a}{b},\tfrac{h}{\kappa}\right)\in\mathbb{Q}\times\mathbb{Q}\colon \begin{array}{l} \kappa \text{ is even, } \exists\, m\in\mathbb{Z} \text{ s.t. } \frac{a}{2b}+\frac{h}{\kappa}\left(\frac{\alpha}{\beta}+\frac12+2m\right)\in\mathbb{Z},\\ \text{and if } \kappa\equiv 0 \pmod{2\beta u}, \text{ then } h\not\equiv\pm1 \pmod{4\beta^3/u}\end{array}\right\}.$$

Finally, dependent on periodic χ as in Definition 1, and using groups and sets above, we further define

$$G_\chi := \bigcap_{j=1}^{r}\mathcal{G}(\beta'_j,f_{\beta'_j}), \qquad H'_\chi := \bigcap_{j=1}^{r} H'_{\ell_{2,\beta'_j}},$$

and

$$Q_\chi := \bigcap_{j=1}^{r} Q_{\alpha_j', \beta_j', 1},$$

where

$$\alpha_j' := \alpha_j/\gcd(\alpha_j, \beta), \quad \text{and} \quad \beta_j' := \beta/\gcd(\alpha_j, \beta). \tag{2.7}$$

We now establish the closure of the sets above under the Jacobi action of some of the specific groups above, as needed later to establish quantum Jacobi properties of the many families being studied in this paper.

Lemma 5. *Let $\beta, u \in \mathbb{N}$ with $\beta \equiv 0 \pmod{u}$. Then $\mathcal{Q}_{\alpha,\beta,u}$ is closed under the Jacobi action of $G_{\beta,u} \ltimes (4\mathbb{Z} \times 2\mathbb{Z})$.*

Proof. We begin by considering the modular Jacobi action

$$\left(\begin{smallmatrix} 1 & 0 \\ 2\beta u & 1 \end{smallmatrix}\right) \cdot (z, \tau) := \left(\frac{z}{2\beta u\tau + 1}, \frac{\tau}{2\beta u \tau + 1} \right)$$

under the first generator, where we take $(z, \tau) = (\frac{a}{b}, \frac{h}{\kappa}) \in \mathcal{Q}_{\alpha,\beta,u}$. Then we have that

$$\frac{\tau}{2\beta u\tau + 1} = \frac{h}{2\beta u h + \kappa} =: \frac{h'}{\kappa'},$$

where $h' := \operatorname{sgn}(2\beta u h + \kappa) h$ and $\kappa' := |2\beta u h + \kappa|$. Note that h'/κ' is reduced because h/κ is. In particular, $\kappa' \neq 0$, for otherwise $\kappa = -2\beta u h$, and since $\gcd(h, \kappa) = 1$ we have $h = \pm 1$, contradicting the third condition defining $\mathcal{Q}_{\alpha,\beta,u}$. Moreover, we have that

$$\frac{z}{2\beta u \tau + 1} = \frac{a\kappa}{b(2\beta u h + \kappa)} =: \frac{a'}{b'},$$

where the integers a' and b' are defined to be such that their fraction a'/b' is reduced and equal to $\frac{a\kappa}{b(2\beta u h + \kappa)}$.

We now verify that $(a'/b', h'/\kappa')$ is in $\mathcal{Q}_{\alpha,\beta,u}$. We have that κ' is even because κ is. Next, since $(a/b, h/\kappa) \in \mathcal{Q}_{\alpha,\beta,u}$, there exist integers m and x such that $a/(2b) + (h/\kappa)(\alpha/\beta + 1/2 + 2m) = x$. We let $\ell := m + \beta u x$. Then

$$\begin{aligned}
\frac{a'}{2b'} + \frac{h'}{\kappa'}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2\ell\right) &= \pm\frac{\kappa}{\kappa'}\left(\frac{a}{2b} + \frac{h}{\kappa}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m + 2\beta u x\right)\right) \\
&= \pm\frac{\kappa x}{\kappa'} \pm \frac{2\beta u x h}{\kappa'} \\
&= \pm x
\end{aligned}$$

which is an integer, so we have verified the second condition defining $\mathcal{Q}_{\alpha,\beta,u}$ for $(a'/b', h'/\kappa')$. Finally, Suppose $\kappa' \equiv 0 \pmod{2\beta u}$. Then $\kappa \equiv 0 \pmod{2\beta u}$ so that $h \not\equiv \pm 1 \pmod{4\beta^3/u}$. Since $h' = h$, we find that the third condition defining $\mathcal{Q}_{\alpha,\beta,u}$ is satisfied for $(a'/b', h'/\kappa')$.

As for the modular Jacobi action of the second generator on $(z, \tau) = (\frac{a}{b}, \frac{h}{\kappa}) \in \mathcal{Q}_{\alpha,\beta,u}$, we find that

$$\left(\begin{smallmatrix} 1 & 2\beta^2/u \\ 0 & 1 \end{smallmatrix}\right) \cdot (z, \tau) := (z, \tau + 2\beta^2/u) = \left(\frac{a}{b}, \frac{h + 2\beta^2\kappa/u}{\kappa}\right) =: \left(\frac{a'}{b'}, \frac{h'}{\kappa'}\right),$$

where $a' := a, b' := b, h' := h + 2\beta^2\kappa/u$, and $\kappa' := \kappa$. The fractions a'/b' and h'/κ' are reduced because a/b and h/κ are, and $\kappa' = \kappa$ is even. We take m and x as in the argument above, and find that

$$\begin{aligned}\frac{a'}{2b'} + \frac{h'}{\kappa'}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right) &= x + 2\beta^2\kappa/u\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right)\\ &= x + \tfrac{\beta\kappa}{u}\left(2\alpha + \beta + 4\beta m\right)\end{aligned}$$

which is an integer, verifying the second condition defining $\mathcal{Q}_{\alpha,\beta,u}$ for $(a'/b', h'/\kappa')$. To verify the third condition, if $\kappa' \equiv 0 \pmod{2\beta u}$ then $h \not\equiv \pm 1 \pmod{4\beta^3/u}$ since $\kappa' = \kappa$. For the sake of contradiction, suppose $h' \equiv \pm 1 \pmod{4\beta^3/u}$. Then $h + 2\beta^2\kappa/u \equiv \pm 1 \pmod{4\beta^3/u}$. But $2\beta^2\kappa/u \equiv 0 \pmod{4\beta^3/u}$ because, again, $\kappa = \kappa'$ and $\kappa' \equiv 0 \pmod{2\beta u}$. This implies that $h \equiv \pm 1 \pmod{4\beta^3/u}$ which is a contradiction.

That the set is closed under the Jacobi elliptic action, e.g. that $(z + \lambda\tau + \mu, \tau)$ is in $\mathcal{Q}_{\alpha,\beta,u}$ for $(\lambda, \mu) \in (4\mathbb{Z} \times 2\mathbb{Z})$, where $(z, \tau) = (a/b, h/\kappa) \in \mathcal{Q}_{\alpha,\beta,u}$, is similarly checked explicitly, and omitted here for brevity's sake. (For a similar proof, see part 2 of the proof of [13, Lemma 11].) □

Lemma 6. *Let $B, u \in \mathbb{N}$ with $2B^2 \equiv 0 \pmod{u}$. Then $\mathcal{Q}^e_{B,u}, \mathcal{Q}^o_{B,u}, \mathcal{Q}_{B,u}, \mathcal{Q}'_{B,u}$ are closed under the Jacobi action of $G_{B,u} \ltimes (2\mathbb{Z} \times \mathbb{Z})$.*

Proof. Similar to the proof of Lemma 5 and first considering the Jacobi modular action, under the first generator, we find that $(a/b, h/\kappa) \in \mathbb{Q} \times \mathbb{Q}$ maps to

$$\left(\frac{a\kappa/b}{\kappa'}, \frac{h}{2Buh + \kappa}\right) =: \left(\frac{a'}{b'}, \frac{h'}{\kappa'}\right),$$

where $h' := \pm h, k' := \pm(2Buh + \kappa)$, and a' and b' are integers such that $a'/b' = a(\kappa/b)/\kappa'$, and a'/b' is reduced. We have that h'/k' is also reduced, and in particular, that $\kappa' \neq 0$ as in the proof of Lemma 5. For each of the groups $\mathcal{Q}^e_{B,u}, \mathcal{Q}^o_{B,u}, \mathcal{Q}_{B,u}, \mathcal{Q}'_{B,u}$, because $b \mid \kappa$, we specifically have that $\pm b'g = \kappa'$ and $\pm a'g = a(\kappa/b)$, where $g := \gcd(a(\kappa/b), \kappa')$. Thus, $b' \mid \kappa'$ as wanted. Moreover, the parity of κ' is dictated by the parity of κ. For the last condition defining the sets of pairs of rationals in question, if we suppose for contradiction's sake that $\kappa' \equiv 0 \pmod{2Bu}$ and $h' \equiv \pm 1 \pmod{4B^3/u}$, then $k \equiv 0 \pmod{2Bu}$ and $h \equiv \pm 1 \pmod{4B^3/u}$, a contradiction. This establishes closure of $\mathcal{Q}^e_{B,u}, \mathcal{Q}^o_{B,u}$, and $\mathcal{Q}'_{B,u}$ under the first generator. To finish closure of $\mathcal{Q}'_{B,u}$ under this generator, there are two scenarios to check: if κ is odd, then as just mentioned κ' is odd; if $4 \mid \kappa$ then κ/b is even, and if it is additionally the case that $4 \mid \kappa'$ (a situation that only occurs when $2 \mid Bu$) then $2 \mid \pm g = \kappa'/b'$ as wanted.

Under the second generator, $(a/b, h/\kappa) \in \mathbb{Q} \times \mathbb{Q}$ maps to

$$\left(\frac{a}{b}, \frac{h + 2(B^2/u)\kappa}{\kappa}\right) =: \left(\frac{a'}{b'}, \frac{h'}{\kappa'}\right),$$

where $h' := h + 2(B^2/u)\kappa, \kappa' := \kappa, a' := a, b' := b$. The conditions prescribed to b and κ in the definitions of the sets in question are thus obviously preserved by b' and κ' (noting that h'/κ' and a'/b' as just defined are reduced). If it is the case that $\kappa' \equiv 0 \pmod{2Bu}$, then $\kappa \equiv 0 \pmod{2Bu}$ so that $h \not\equiv \pm 1 \pmod{4B^3/u}$. Then $h' = h + 2(B^2/u)\kappa \equiv h \pmod{4B^3/u}$ (since $\kappa \equiv 0 \pmod{2Bu}$). Thus, $h' \not\equiv \pm 1 \pmod{4B^3/u}$ (because $h \not\equiv \pm 1 \pmod{4B^3/u}$) as wanted.

To check closure under the prescribed Jacobi elliptic action, we seek to show that for pairs $(a/b, h/\kappa)$ of rationals in the sets in question that

$$\left(\frac{a}{b}+\lambda\frac{h}{k}+\mu, \frac{h}{k}\right) =: \left(\frac{a'}{b'}, \frac{h'}{\kappa'}\right)$$

is also in the appropriate set of pairs of rationals for any $(\lambda, \mu) \in 2\mathbb{Z}\times\mathbb{Z}$. Here, we have that $h' := h, \kappa' := \kappa$, and the integers a' and b' are defined such that a'/b' is reduced and equal to $\frac{a}{b}+\lambda\frac{h}{k}+\mu$. Since $b \mid k$ we may rewrite a'/b' as the ratio of integers

$$\frac{a(\kappa/b)+\lambda h+\mu\kappa}{\kappa},$$

implying that b' (in reduced form) satisfies $b' \mid \kappa = \kappa'$ as wanted. The parity of κ' is obviously given by the parity of κ. In the case of $\mathcal{Q}'_{B,u}$ in which $4 \mid \kappa$, then $4 \mid \kappa'$ and κ/b is even. Since λ is also even, we have that $2 \mid \gcd(a(\kappa/b)+\lambda h+\mu\kappa, \kappa)$ and hence κ'/b' is even as wanted. Finally, we suppose that $\kappa' \equiv 0 \pmod{2Bu}$. Then $\kappa \equiv 0 \pmod{2Bu}$ and hence $h \not\equiv \pm 1 \pmod{4B^3/u}$. Because $h = h'$, we see that the last condition defining the sets of pairs of rationals in question for $(a'/b', h'/\kappa')$ is also satisfied. $\square$

Part II. Quantum periodic partial Jacobi theta functions (§3–4)

3. Periodic partial Jacobi theta functions and Theorem 1

Recall from Definition 1 the periodic partial Jacobi theta functions

$$\Theta_\chi(x;q) := \sum_{n=0}^{\infty}\chi(n)q^{\frac{n^2}{2\beta^2}}x^{\frac{n}{2\beta}}.$$

We let $(z,\tau)\in\mathbb{C}\times\mathbb{H}$, and define Θ_χ on this domain by

$$\widetilde{\Theta}_\chi(z;\tau) := \Theta_\chi(e(z);e(\tau)).$$

We recall G_χ and Q_χ from Section 2.2, and define the character

$$\chi_{C,D} := \zeta_8^{1-D}\left(\frac{C}{D}\right). \tag{3.1}$$

Throughout the paper, we write the (Nebentypus) characters appearing in modular-type transformation results as dependent on the parameters A, B, C and/or D from a matrix $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ in the relevant modular group. e.g. in (3.1) and Theorem 1, the character $\chi_{C,D}$ is defined using matrices $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in G_\chi$.

Our first main theorem is as follows.

Theorem 1. *With notation and hypotheses as above, the periodic partial Jacobi theta functions $\widetilde{\Theta}_\chi(z;\tau)$ are quantum Jacobi forms of weight $1/2$, index $-1/8$, on Q_χ, with Jacobi group $G_\chi \ltimes (4\mathbb{Z}\times 2\mathbb{Z})$, and character $\chi_{C,D}$. Moreover, under the same assumptions, $\widetilde{\Theta}_\chi(z;-\tau)$ is a mock Jacobi form of weight $1/2$ and index $-1/8$, with Jacobi group $H'_\chi \ltimes (4\mathbb{Z}\times 2\mathbb{Z})$, and character $\chi_{C,D}$.*

Remarks. (1) Further explicit transformation and analytic properties of these functions may be deduced from Proposition 2 and the proof of this result.

(2) Quantum modular properties of these and related functions when viewed as one-variable

functions of τ for fixed $z = a/b$ are studied in [8, 16]. We also refer the reader to interesting related work of Goswami-Osburn in [19] (also noted in Section 1.1), in which they establish one-variable quantum modular properties of partial theta functions with even or odd periodic coefficients using different methods.

4. Proof of Theorem 1

To prove Theorem 1 we extend some results and methods from our prior work [13]; these are stated as Propositions 1 - 3 below. Analogous results and definitions in [13] require $4 \mid \beta$ and $\gcd(\alpha, \beta) = 1$, while here this is not necessarily the case, and a number of nontrivial technical modifications are required to prove our results here. (E.g. it's not in general true that $4 \mid \beta$ in Theorem 6 nor that $\gcd(\alpha, \beta) = 1$ in Theorem 5, etc. and hence we have established and proved Theorem 1 and related results accordingly.) For consistency and convenience, we will often reuse notation from [13] in this section, and point out that the functions of the same name there agree with the ones here when $4 \mid \beta$, but not necessarily in general.

4.1. **A nonholomorphic Jacobi family.** To prove Theorem 1, we will use, among other things, the level 2 Appell function A_2 defined in (2.3) as well as its completed version $\widehat{A}_2$ from (2.4), and define for integers α and β satisfying $0 < \alpha < \beta$

$$B_{\alpha,\beta}(z;\tau) := e\left(\frac{\alpha z}{2\beta}\right) q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}} A_2\left(\frac{-z}{2} + \frac{\alpha}{\beta}\tau - \frac{\tau}{2}, -\tau; 2\tau\right),$$

and

$$\widehat{B}_{\alpha,\beta}(z;\tau) := e\left(\frac{\alpha z}{2\beta}\right) q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}} \widehat{A}_2\left(\frac{-z}{2} + \frac{\alpha}{\beta}\tau - \frac{\tau}{2}, -\tau; 2\tau\right).$$

In Proposition 1 below, we establish the Jacobi transformation properties of $\widehat{B}_{\alpha,\beta}$, which are ultimately deduced from those of the completed Appell function $\widehat{A}_2$. The proof of this proposition is similar to the proof of [13, Prop. 1] which it extends, so we refer the reader there for details. In particular, we point out that n_1, m_1, n_2, m_2 defined after [13, (4.1)] are still integers given the new conditions on A, B, C, D and α, β here.

Proposition 1. *The function $\widehat{B}_{\alpha,\beta}(z;\tau)$ is a nonholomorphic Jacobi form of weight 1, index $-1/8$, group $H_{2\ell_\beta}$, and character $\zeta_8^{AB}\zeta_{2\beta^2}^{-AB\alpha^2}$.*

4.2. **A quantum and mock Jacobi family.** We next establish quantum and mock Jacobi properties of the functions

$$C_{\alpha,\beta}(z;\tau) := q^{\frac{\alpha^2}{2\beta^2}} w^{\frac{\alpha}{2\beta}} \sum_{n=0}^{\infty} q^{\frac{n^2}{2}} \left(w^{\frac{1}{2}} q^{\frac{\alpha}{\beta}}\right)^n,$$

where α, β are integers satisfying $0 < \alpha < \beta$, extending our work in [13]. To state (part (2) of) the result, we define the additional character

$$\psi_{B,C,D}(\alpha,\beta) := \zeta_8^{1-D}\zeta_{2\beta}^{-\alpha^2 B/\beta}\left(\frac{C}{D}\right).$$

Proposition 2. *The following are true.*

(1) *The function $C_{\alpha,\beta}(z;\tau)$ is a quantum Jacobi form on $Q_{\alpha,\beta,1}$ of weight $1/2$, index $-1/8$, group $\mathcal{G}(\beta, f_\beta)$, and character $\chi_{C,D}$.*

In particular, for any $\epsilon_{\alpha,\beta} > 0$ satisfying $\frac{\beta-\alpha}{\beta^2} < \epsilon_{\alpha,\beta} < \frac{1}{\beta}$, if $z \in (-\frac{\alpha}{\beta f_\beta}, \frac{1}{f_\beta} - \frac{\alpha}{\beta f_\beta} - \frac{\beta}{f_\beta}\epsilon_{\alpha,\beta})$, we have that

$$
\begin{aligned}
(4.1)\quad & C_{\alpha,\beta}(z;\tau) - (-2f_\beta\tau+1)^{-\frac{1}{2}}\chi_{2f_\beta,1}^{-1} e\left(\frac{2f_\beta z^2}{8(-2f_\beta\tau+1)}\right) C_{\alpha,\beta}\left(\frac{z}{-2f_\beta\tau+1}; \frac{\tau}{-2f_\beta\tau+1}\right) \\
&= \frac{-1}{2}\int_0^\infty \frac{\sum_\pm g_{-\frac{\alpha}{2\beta}+\frac{3\mp1}{4},-z}\left(\frac{2}{f_\beta}+it\right)}{\sqrt{-i(\frac{2}{f_\beta}+it-4\tau)}}dt,
\end{aligned}
$$

and the difference in (4.1) *extends to a C^∞ function on*

$$\left(\mathbb{R} \setminus \left(\tfrac{2}{\beta}\mathbb{Z} - \tfrac{\alpha}{\beta^2} + \{0, \tfrac{1}{\beta}, \tfrac{\alpha}{\beta^2}, \tfrac{1}{\beta} \pm \epsilon_{\alpha,\beta}\}\right)\right) \times \left(\mathbb{R} \setminus \{\tfrac{1}{2f_\beta}\}\right).$$

(2) *The function $C_{\alpha,\beta}(z;-\tau)$ is a mock Jacobi form of weight $1/2$, index $-1/8$, group $H_{\ell_{2,\beta}}$, and character $\psi_{B,C,D}(\alpha,\beta)$.*

4.3. **Proof of Proposition 2: transformation properties.** Towards the proof of Proposition 2, we establish various technical lemmas and key transformation properties of $C_{\alpha,\beta}$ in this section. We note that part (1) of the proposition will be proved assuming part (2) (see the proof of Proposition 3 below and the end of this section (Sec. 4.3)). To this end, we define

$$r_\pm(z;\tau) = r_{\pm,\alpha,\beta}(z;\tau) := R\left(-z + 2\tau\frac{\alpha}{\beta} - (1\mp1)\tau - \frac{1}{2}; 4\tau\right),$$

and

$$
\begin{aligned}
z_1^\pm = z_1^\pm(\alpha,\beta,z,\tau) &:= -\frac{z}{2f_\beta\tau+1} + \frac{2\tau\alpha}{\beta(2f_\beta\tau+1)} - (1\mp1)\frac{\tau}{2f_\beta\tau+1} - \frac{1}{2}, \\
z_2^\pm = z_2^\pm(\alpha,\beta,z,\tau) &:= \frac{1}{2} - \frac{2\alpha\tau}{\beta} + (1\mp1)\tau + z, \\
\tau_1 = \tau_1(\beta,\tau) &:= \frac{-1}{4\tau} - \frac{f_\beta}{2}.
\end{aligned}
$$

(As noted above, here and throughout this section we reuse notation from [13] for consistency and convenience; functions of the same name there agree with the ones here when $4 \mid \beta$, but not necessarily in general.)

In Lemma 7 we establish Jacobi-type transformation properties of $r_\pm(z;\tau)$. Its proof is similar to the proof of [13, Lemma 13] which it generalizes, so we omit its proof for brevity's sake.

Lemma 7. *We have that*

$$r_\pm\left(\frac{z}{2f_\beta\tau+1}; \frac{\tau}{2f_\beta\tau+1}\right) = a_\pm(z;\tau)h(z_1^\pm\tau_1;\tau_1) - b_\pm(z;\tau)h(z_2^\pm;4\tau) + b_\pm(z;\tau)r_\pm(z;\tau),$$

where

$$a_\pm(z;\tau) := \sqrt{-i\tau_1}e\left(\frac{-(z_1^\pm)^2\tau_1}{2}\right),$$
$$b_\pm(z;\tau) := a_\pm(z;\tau)(-1)^{f_\beta/4}\zeta_8^{f_\beta/2}\sqrt{-4i\tau}e\left(\frac{-(z_2^\pm)^2}{8\tau}\right).$$

We now determine the Jacobi transformation properties of the function $C_{\alpha,\beta}(z;\tau)$ under the group $\mathcal{G}(\beta,f_\beta)$. A direct calculation reveals invariance under the generator $\left(\begin{smallmatrix}1 & 2\beta^2\\ 0 & 1\end{smallmatrix}\right)$ using its definition. For transformation properties under the other generator $\left(\begin{smallmatrix}1 & 0\\ 2f_\beta & 1\end{smallmatrix}\right)$, we define

$$f_\pm(z;\tau) = f_{\pm,\alpha,\beta}(z;\tau) := \frac{i}{2}e\left(\left(\frac{1\mp 1}{2}\right)\left(-\frac{z}{2}+\frac{\alpha\tau}{\beta}-\frac{\tau}{2}\right)\right),$$

and establish the following key proposition, used to establish Proposition 2 part (1).

Proposition 3. *We have that*

$$\begin{aligned}
&C_{\alpha,\beta}(z;-\tau) - (2f_\beta\tau+1)^{-\frac{1}{2}}\chi_{2f_\beta,1}^{-1}e\left(\frac{2f_\beta z^2}{8(2f_\beta\tau+1)}\right)C_{\alpha,\beta}\left(\frac{z}{2f_\beta\tau+1};\frac{-\tau}{2f_\beta\tau+1}\right)\\
&\quad= q^{-\frac{1}{8}-\frac{f_\beta^2}{8}+\frac{f_\beta}{4}}(2f_\beta\tau+1)^{-1/2}\varepsilon^3\left(\left(\begin{smallmatrix}1 & 0\\ f_\beta/2 & 1\end{smallmatrix}\right)\right)e\left(\frac{f_\beta z^2}{4(2f_\beta\tau+1)}+\frac{\alpha z}{2\beta(2f_\beta\tau+1)}\right)\\
&\quad\times e\left(\frac{\tau(-4\alpha^2+\beta^2)}{8\beta^2(2f_\beta\tau+1)}+\frac{f_\beta(-\tau+\frac{1}{2}(2f_\beta\tau+1))^2}{4(2f_\beta\tau+1)}\right)\\
&\quad\times\sum_\pm f_\pm\left(\frac{z}{2f_\beta\tau+1};\frac{\tau}{2f_\beta\tau+1}\right)\left(a_\pm(z;\tau)h(z_1^\pm\tau_1;\tau_1)-b_\pm(z;\tau)h(z_2^\pm;4\tau)\right),
\end{aligned}\tag{4.2}$$

and for $z \in (-\frac{\alpha}{\beta f_\beta}, -\frac{\alpha}{\beta f_\beta}+\frac{1}{f_\beta}-\frac{\epsilon\beta}{f_\beta})$, the right hand side of equation (4.2) equals

$$\frac{-1}{2}\int_0^\infty \frac{\sum_\pm g_{-\frac{\alpha}{2\beta}+\frac{3\mp1}{4},-z}\left(\frac{2}{f_\beta}+it\right)}{\sqrt{-i(\frac{2}{f_\beta}+it+4\tau)}}dt.\tag{4.3}$$

Proof of Proposition 3. We modify the proof of [13, Proposition 3], and point out that it is not a simple change of variable in β throughout – some dependence on β in [13] arises from the functions $C_{\alpha,\beta}$ and remains intact here, while some dependence on β arises from the groups G_β and H_β there, which are generalized here to $\mathcal{G}(\beta,f_\beta)$ and $H_{\ell_{2,\beta}}$. We divide the proof into two parts.

Part 1. From Proposition 2 (2), using [13, Proposition 1, Proposition 2] for suitable $\gamma = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$, we have that

$$\begin{aligned}
&C_{\alpha,\beta}(z;-\tau) - (C\tau+D)^{-\frac{1}{2}}\psi_{B,C,D}^{-1}(\alpha,\beta)e\left(\frac{Cz^2}{8(C\tau+D)}\right)C_{\alpha,\beta}\left(\frac{z}{C\tau+D};-\gamma\tau\right)\\
&= -\widetilde{C}_{\alpha,\beta}^-(z;\tau) + (C\tau+D)^{-\frac{1}{2}}\psi_{B,C,D}^{-1}(\alpha,\beta)e\left(\frac{Cz^2}{8(C\tau+D)}\right)\widetilde{C}_{\alpha,\beta}^-\left(\frac{z}{C\tau+D};\gamma\tau\right),
\end{aligned}\tag{4.4}$$

where $\widetilde{C}^-_{\alpha,\beta}(z;\tau) := q^{-\frac{1}{8}}T^{-1}(\tau)B^-_{\alpha,\beta}(z;\tau)$, with

$$B^-_{\alpha,\beta}(z;\tau) = \frac{i}{2}e\left(\frac{\alpha z}{2\beta}\right)q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}}T(\tau)\sum_{k=0}^{1}e\left(k\left(-\frac{z}{2}+\frac{\alpha\tau}{\beta}-\frac{\tau}{2}\right)\right)R\left(-z+2\tau\left(\frac{\alpha}{\beta}-k\right)-\frac{1}{2};4\tau\right)$$

and

$$T(\tau) := \vartheta(-\tau+\tfrac{1}{2};4\tau). \tag{4.5}$$

We have used that $\vartheta(z+1;\tau)=-\vartheta(z;\tau)$, and that $\vartheta(z;\tau)$ is an odd function in z.

With respect to $\gamma = \left(\begin{smallmatrix}1 & 0\\ 2f_\beta & 1\end{smallmatrix}\right)$, we find that

$$\begin{aligned}
&B^-_{\alpha,\beta}(z;\tau) - (2f_\beta\tau+1)^{-1}e\left(\frac{2f_\beta z^2}{8(2f_\beta\tau+1)}\right)B^-_{\alpha,\beta}\left(\frac{z}{2f_\beta\tau+1};\gamma\tau\right) \\
&= T(\tau)\Bigg[e\left(\frac{\alpha z}{2\beta}\right)q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}}\sigma(z;\tau) - (2f_\beta\tau+1)^{-1}e\left(\frac{f_\beta z^2}{4(2f_\beta\tau+1)}\right)e\left(\frac{\alpha z}{2\beta(2f_\beta\tau+1)}\right) \\
&\qquad\times e\left(\frac{\tau}{2f_\beta\tau+1}\cdot\frac{-4\alpha^2+\beta^2}{8\beta^2}\right)\varepsilon^3\left(\left(\begin{smallmatrix}1 & 0\\ \frac{f_\beta}{2} & 1\end{smallmatrix}\right)\right) \\
&\qquad\times (2f_\beta\tau+1)^{\frac{1}{2}}e^{\frac{\pi i f_\beta\left(-\tau+\frac{1}{2}(2f_\beta\tau+1)\right)^2}{2(2f_\beta\tau+1)}}q^{-\frac{f_\beta^2}{8}+\frac{f_\beta}{4}}\sigma\left(\frac{z}{2f_\beta\tau+1};\gamma\tau\right)\Bigg],
\end{aligned}\tag{4.6}$$

where

$$\sigma(z;\tau) := \sum_{\pm} f_\pm(z;\tau)r_\pm(z;\tau).$$

We rewrite the term in brackets $[\,\cdot\,]$ in (4.6) in terms of $f_\pm$ and $r_\pm$, and apply Lemma 7, to obtain

$$\begin{aligned}
&e\left(\frac{\alpha z}{2\beta}\right)q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}}\sum_{\pm} f_\pm(z;\tau)r_\pm(z;\tau) \\
&- (2f_\beta\tau+1)^{-1/2}\varepsilon^3\left(\left(\begin{smallmatrix}1 & 0\\ \frac{f_\beta}{2} & 1\end{smallmatrix}\right)\right)e\left(\frac{f_\beta z^2}{4(2f_\beta\tau+1)}\right)e\left(\frac{\alpha z}{2\beta(2f_\beta\tau+1)}\right)e\left(\frac{\tau}{2f_\beta\tau+1}\cdot\frac{-4\alpha^2+\beta^2}{8\beta^2}\right)
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
&\times e\left(\frac{\beta(-\tau+\frac{1}{2}(2f_\beta\tau+1))^2}{4(2f_\beta\tau+1)}\right)q^{-\frac{f_\beta^2}{8}+\frac{f_\beta}{4}}\sum_{\pm} f_\pm\left(\frac{z}{2f_\beta\tau+1};\frac{\tau}{2f_\beta\tau+1}\right)b_\pm(z;\tau)\,r_\pm(z;\tau) \\
&- (2f_\beta\tau+1)^{-1/2}\varepsilon^3\left(\left(\begin{smallmatrix}1 & 0\\ \frac{f_\beta}{2} & 1\end{smallmatrix}\right)\right)e\left(\frac{f_\beta z^2}{4(2f_\beta\tau+1)}\right)e\left(\frac{\alpha z}{2\beta(2f_\beta\tau+1)}\right)e\left(\frac{\tau}{2f_\beta\tau+1}\cdot\frac{-4\alpha^2+\beta^2}{8\beta^2}\right)
\end{aligned}\tag{4.8}$$

$$\times e\left(\frac{f_\beta(-\tau+\frac{1}{2}(2f_\beta\tau+1))^2}{4(2f_\beta\tau+1)}\right)q^{-\frac{f_\beta^2}{8}+\frac{f_\beta}{4}}\sum_{\pm} f_\pm\left(\frac{z}{2f_\beta\tau+1};\frac{\tau}{2f_\beta\tau+1}\right)G_\pm(z;\tau), \tag{4.9}$$

where

$$G_{\pm}(z;\tau) = G_{\pm,\alpha,\beta}(z;\tau) := a_{\pm}(z;\tau)h(z_1^{\pm}\tau_1;\tau_1) - b_{\pm}(z;\tau)h(z_2^{\pm};4\tau). \tag{4.10}$$

Observe that lines (4.7) and (4.8) both involve the function $r_{\pm}(z;\tau)$. Using that

$$\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0 \\ \frac{f_\beta}{2} & 1\end{smallmatrix}\right)\right) = \zeta_8^{-f_\beta/2},$$

and after a long explicit calculation, we find that these two lines (appearing in the the large expression in (4.7), (4.8), and (4.9)) completely cancel with each other. That is, we have shown that the term in brackets $[\,\cdot\,]$ in (4.6) equals the expression in (4.9). This, along with the fact that $\psi_{B,C,D}(\alpha,\beta) = \chi_{C,D}$ when $B \equiv 0 \pmod{2\beta^2}$, yields (4.2) in Proposition 3.

Part 2. In order to establish (4.3) in Proposition 3, we study $G_{\pm}(z;\tau)$ (see (4.10)) and re-write

$$z_1^{\pm}\tau_1 = a_2\tau_1 - b_1^{\pm}, \quad z_2^{\pm} = a_1^{\pm}4\tau - a_2$$

where $a_2 = a_2(z) := -\frac{1}{2} - z$, $b_1^{\pm} = b_1^{\pm}(\alpha,\beta;z) := \frac{f_\beta z}{2} + \frac{\alpha}{2\beta} - \frac{1}{4}(1 \mp 1)$, and $a_1^{\pm} = a_1^{\pm}(\alpha,\beta) := \frac{-\alpha}{2\beta} + \frac{(1\mp 1)}{4}$.

Lemma 8. *For α, β, and $a_1^{\pm}$ as above, we have*

(i) $a_1^{+} \in (-\frac{1}{2}, 0)$ *and* $a_1^{-} \in (0, \frac{1}{2})$.

Further, let $\epsilon = \epsilon_{\alpha,\beta} > 0$ satisfy

$$\frac{\beta-\alpha}{\beta^2} < \epsilon < \frac{1}{\beta},$$

and suppose

$$z \in \frac{\beta}{f_\beta}\left(-\frac{\alpha}{\beta^2}, \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon\right).$$

Then under these additional hypotheses, we have that

(ii) $b_1^{+} \in (0, \frac{1}{2} - \frac{\epsilon\beta}{2}) \subset (0, \frac{1}{2})$, *and* $b_1^{-} \in (-\frac{1}{2}, -\frac{\epsilon\beta}{2}) \subset (-\frac{1}{2}, 0)$,

(iii) $-a_2 \in \left(\frac{1}{2} - \frac{\alpha}{\beta f_\beta}, \frac{1}{2} + \frac{1}{f_\beta} - \frac{\alpha}{\beta f_\beta} - \frac{\epsilon\beta}{f_\beta}\right) \subset \left(\frac{1}{4}, \frac{1}{2}\right)$.

Proof of Lemma 8. We omit the detailed proof for brevity's sake, as it is similar to the one given in [13], and makes particular use of the new definition above of $b_1^{\pm}$ and the new range for z. $\square$

To continue the proof of (4.3) we have, with ϵ and z as in Lemma 8, using Lemma 4, that

$$h(z_1^{\pm}\tau_1;\tau_1) = h(a_2\tau_1 - b_1^{\pm};\tau_1) = -e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^{\pm} + \tfrac{1}{2})\right)\int_0^{i\infty} \frac{g_{a_2+\frac{1}{2}, b_1^{\pm}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau_1)}}du, \tag{4.11}$$

$$h(z_2^{\pm};4\tau) = h(a_1^{\pm}4\tau - a_2;4\tau) = -e\left(\frac{(a_1^{\pm})^2 4\tau}{2} - a_1^{\pm}(a_2 + \tfrac{1}{2})\right)\int_0^{i\infty} \frac{g_{a_1^{\pm}+\frac{1}{2}, a_2+\frac{1}{2}}(u)}{\sqrt{-i(u+4\tau)}}du. \tag{4.12}$$

We make the change of variable $u = f_\beta/2 - 1/\rho$ in the integral in (4.11); the right hand side of (4.11) is thus equal to

$$-e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^{\pm} + \tfrac{1}{2})\right)\int_{\frac{2}{f_\beta}}^{0} \frac{g_{a_2+\frac{1}{2}, b_1^{\pm}+\frac{1}{2}}\left(\frac{f_\beta}{2} - \frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}}\frac{\sqrt{4\rho\tau}d\rho}{\rho^2}$$

$$(4.13)\quad = -e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^\pm + \tfrac{1}{2})\right) e\left(-\frac{\frac{f_\beta}{2}(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right)$$

$$\times \int_{\frac{2}{f_\beta}}^{0} \frac{g_{a_2+\frac{1}{2}, \frac{f_\beta}{2}(a_2+\frac{1}{2})+b_1^\pm+\frac{1}{2}+\frac{f_\beta}{4}}\left(-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}} \frac{\sqrt{4\rho\tau}d\rho}{\rho^2}.$$

We have also used that

$$g_{A,B}(\tau+n) = e\left(-\frac{nA(A+1)}{2}\right) g_{A,nA+B+\frac{n}{2}}(\tau)$$

for $n \in \mathbb{N}_0$, which we deduce from Lemma 2. We have

$$\frac{f_\beta}{2}\left(a_2+\frac{1}{2}\right) + b_1^\pm + \frac{1}{2} + \frac{f_\beta}{4} = -a_1^\pm + \frac{1}{2} + \frac{f_\beta}{4},$$

and obtain that (4.13) is equal to

$$- e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^\pm + \tfrac{1}{2})\right) e\left(-\frac{\frac{f_\beta}{2}(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right) \int_{\frac{2}{f_\beta}}^{0} \frac{g_{a_2+\frac{1}{2}, -a_1^\pm+\frac{1}{2}+\frac{f_\beta}{4}}\left(-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}} \frac{\sqrt{4\rho\tau}d\rho}{\rho^2}$$

$$= e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^\pm + \tfrac{1}{2})\right) e\left(-\frac{\frac{f_\beta}{2}(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right) \sqrt{\frac{4\tau}{-1}} i(-i)^{\frac{3}{2}}$$

$$\times e\left(\left(a_2+\frac{1}{2}\right)\left(-a_1^\pm+\frac{1}{2}+\frac{f_\beta}{4}\right)\right) \int_{\frac{2}{f_\beta}}^{0} \frac{g_{a_1^\pm+\frac{1}{2}, a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}} d\rho,$$

using further properties from Lemma 2. We further simplify and obtain

$$a_\pm(z;\tau)h(z_1^\pm\tau_1;\tau_1)$$

$$(4.14)\qquad = a_\pm(z;\tau)\sqrt{\frac{4\tau}{-1}} i(-i)^{\frac{3}{2}} \zeta_{4\beta}^{\alpha}\zeta_{16}^{-f_\beta} e^{\pi i\frac{(1\pm 1)}{4}} e\left(-\frac{1}{32\tau} - \frac{z}{8\tau} - \frac{z^2}{8\tau}\right) \int_{\frac{2}{f_\beta}}^{0} \frac{g_{a_1^\pm+\frac{1}{2}, a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}} d\rho.$$

We also have using (4.12) and simplifying that

$$-b_\pm(z;\tau)h(z_2^\pm;4\tau) = a_\pm(z;\tau)(-1)^{f_\beta/4}\zeta_8^{f_\beta/2}\zeta_{4\beta}^{\alpha} e^{-\pi i\frac{1\mp 1}{4}} e\left(-\frac{1}{32\tau} - \frac{z}{8\tau} - \frac{z^2}{8\tau}\right)\sqrt{-4i\tau}$$

$$(4.15)\qquad \times \int_0^{i\infty} \frac{g_{a_1^\pm+\frac{1}{2}, a_2+\frac{1}{2}}(u)}{\sqrt{-i(u+4\tau)}} du.$$

We simplify the constants in (4.14) and (4.15) to obtain

$$G_\pm(z;\tau) = \zeta_4^{-1-\frac{f_\beta}{4}}\zeta_8^{\pm 1} a_\pm(z;\tau)\sqrt{4\tau}\zeta_{4\beta}^{\alpha} e\left(-\frac{1}{32\tau} - \frac{z}{8\tau} - \frac{z^2}{8\tau}\right) \int_{\frac{2}{f_\beta}}^{i\infty} \frac{g_{a_1^\pm+\frac{1}{2}, a_2+\frac{1}{2}}(u)}{\sqrt{-i(u+4\tau)}} du$$

$$= \zeta_4^{-1-\frac{f_\beta}{4}}\zeta_8^{\pm 1}\sqrt{i}\sqrt{1+2f_\beta\tau}\, e\left(\frac{-(z_1^\pm)^2\tau_1}{2}\right)\zeta_{4\beta}^{\alpha} e\left(-\frac{1}{32\tau} - \frac{z}{8\tau} - \frac{z^2}{8\tau}\right) \int_{\frac{2}{f_\beta}}^{i\infty} \frac{g_{-\frac{\alpha}{2\beta}+\frac{3\mp 1}{4}, -z}(u)}{\sqrt{-i(u+4\tau)}} du.$$

Finally, we conclude after some further simplifications that (4.9) equals

$$\frac{-i}{2}q^{\frac{1}{8}}\int_{\frac{2}{f_\beta}}^{i\infty}\frac{\sum_{\pm} g_{-\frac{\alpha}{2\beta}+\frac{3\mp 1}{4},-z}(u)}{\sqrt{-i(u+4\tau)}}du = \frac{1}{2}q^{\frac{1}{8}}\int_0^{\infty}\frac{\sum_{\pm} g_{-\frac{\alpha}{2\beta}+\frac{3\mp 1}{4},-z}\left(\frac{2}{f_\beta}+it\right)}{\sqrt{-i(\frac{2}{f_\beta}+it+4\tau)}}dt.$$

Here, we integrate from $2/f_\beta \to 2/f_\beta + i\infty$ then $2/f_\beta + i\infty \to i\infty$ (the latter vanishes), and then make the change of variable $u = 2/f_\beta + it$ where t runs from $0 \to \infty$. We multiply by $-q^{-\frac{1}{8}}$ to obtain (4.3) in Proposition 3. This completes the proof of Proposition 3. $\square$

Next, to prove Proposition 2 (2) we modify the proof of [13, Theorem 1 (2)] and use [13, Proposition 2], which holds under the relaxed hypotheses given here, along with Proposition 1 below. Another ingredient of the proof of (2) requires for $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in H_{\ell_{2,\beta}}$ that

$$\varepsilon\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\varepsilon^{-2}\left(\begin{smallmatrix} A & 2B \\ C/2 & D\end{smallmatrix}\right)\zeta_8^{AB}\zeta_{2\beta^2}^{-AB\alpha^2} = \psi_{B,C,D}(\alpha,\beta)$$

(where we recall $\varepsilon(\gamma)$ from Lemma 1). We verify this by direct calculation using definitions of characters and congruence conditions on A, B, C, D, and note that the results analogous to [13, (5.6), (5.7)] here are established a bit differently given our hypotheses. We refer the reader to [13] for additional explicit details for brevity's sake.

4.4. **Proof of Proposition 2: quantum properties.** That $C_{\alpha,\beta}$ is defined on $Q_{\alpha,\beta,1}$ follows as explained in [13, (5.5), Proposition 2 and its proof, see also Theorem 2]. That $Q_{\alpha,\beta,1}$ is closed under $\mathcal{G}(\beta, f_\beta) \ltimes (4\mathbb{Z}\times 2\mathbb{Z})$ follows from Lemma 5 and the fact that $\mathcal{G}(\beta, f_\beta)$ is a subgroup of $G_{\beta,1}$. The C^∞ properties and the Jacobi elliptic transformation properties follow as in Section 5.2.2 and 5.2.3 of [13].

This completes the proof of Proposition 2. $\square$

4.5. **Proof of Theorem 1.** Using the results above we are ready to prove Theorem 1. First, we write

(4.16)

$$\Theta_\chi(x;q) := \sum_{n\geq 0}\chi(n)q^{\frac{n^2}{2\beta^2}}x^{\frac{n}{2\beta}} = \sum_{j=1}^{r}\epsilon_j \sum_{\substack{n=0 \\ n\equiv\alpha_j \pmod{\beta}}}^{\infty} q^{\frac{n^2}{2\beta^2}}x^{\frac{n}{2\beta}} = \sum_{j=1}^{r}\epsilon_j\, q^{\frac{\alpha_j^2}{2\beta^2}}x^{\frac{\alpha_j}{2\beta}}\sum_{n=0}^{\infty}q^{\frac{n^2}{2}+n\frac{\alpha_j}{\beta}}x^{\frac{n}{2}}.$$

With $x = e(z)$ and $q = e(\tau)$, $(z,\tau)\in\mathbb{C}\times\mathbb{H}$, we have that (4.16) equals

$$\sum_{j=1}^{r}\epsilon_j C_{\alpha_j',\beta_j'}(z;\tau), \tag{4.17}$$

where we recall α_j' and β_j' from (2.7). Theorem 1 now follows from Proposition 2.

Part III. Applications to q-hypergeometric multisum knot families (§5)

5. Quantum Jacobi q-series and knot families

In this section we use Theorem 1 to establish the quantum Jacobi properties of several q-hypergeometric multisum and partial theta families of interest arising from knot colored Jones polynomials, Kashaev invariants for torus knots and Virasoro characters, and "strange" identities, appearing in work of Bijaoui et al. [5], Hikami [21], Hikami-Kirillov [22, 23], Lovejoy [26], and Zagier [32]. This also adds to related work in [13, 17].

5.1. **On Hikami's generalization of Zagier's "strange" identity and Theorem 2.** We define the functions

$$T_{a,k}^{(1)}(x;q) := \sum_{n=0}^{\infty}(-1)^n x^{(2k+1)n} q^{\binom{n+1}{2}+(a+1)n^2+(k-a-1)(n^2+n)}(1-x^{2(a+1)}q^{(a+1)(2n+1)}),$$

$$H_{a,k}^{(1)}(x;q) := (1-x)\sum_{n_1,\dots,n_k\geq 0}(xq;q)_{n_k} q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}} x^{2n_1+\cdots+2n_{k-1}+n_k} \prod_{j=1}^{k-1}\begin{bmatrix} n_{j+1}+\delta_{a,j} \\ n_j \end{bmatrix}_q$$

appearing in work of Hikami [21] and Lovejoy [26, p1029], defined for $k\in\mathbb{N}$ and $0\leq a\leq k-1$. These functions are used (in [21] and [26]) to establish Hikami's generalization (1.6) of Zagier's "strange identity" (1.4), which is used to establish quantum modularity of $F(q)$ (and related to the colored Jones polynomials $J_N(T_{(2,3)};q)$ for the $T(2,3)$ torus knots via (1.3)). We define modest normalizations of these functions with $(x,q)=(e(z),e(\tau))$ by

$$\widetilde{T}_{a,k}^{(1)}(z;\tau) := q^{\frac{(2k-2a-1)^2}{8(2k+1)}} x^{\frac{2k-2a-1}{2}} T_{a,k}^{(1)}(x;q),$$

$$\widetilde{H}_{a,k}^{(1)}(z;\tau) := q^{\frac{(2k-2a-1)^2}{8(2k+1)}} x^{\frac{2k-2a-1}{2}} H_{a,k}^{(1)}(x;q),$$

and establish their quantum Jacobi properties.

Theorem 2. *The functions $\widetilde{H}_{a,k}^{(1)}(z;\tau)$ and $\widetilde{T}_{a,k}^{(1)}(z;\tau)$ are quantum Jacobi forms of weight of weight $1/2$ and index $-k-\frac{1}{2}$ on $\mathcal{Q}_{8k+4,8k+4}\subset\mathbb{Q}\times\mathbb{Q}$, with Jacobi group $G_{8k+4,8k+4}\ltimes(4\mathbb{Z}\times\mathbb{Z})$ and character $\chi_{C/(8k+4),D}$.*

Proof. We define the periodic function χ as in Definition 1 with $r=4$,

$$\epsilon_j := \begin{cases} 1, & j=1,2,\\ -1, & j=3,4, \end{cases} \qquad \alpha_j := \begin{cases} 2k-2a-1, & j=1,\\ 6k+2a+5, & j=2,\\ 2k+2a+3, & j=3,\\ 6k-2a+1, & j=4, \end{cases}$$

and $\beta := 4(2k+1)$. By way of [26, Proof of (5) p1029], we find that $\widetilde{H}_{a,k}^{(1)}(z;\tau)=\widetilde{T}_{a,k}^{(1)}(z;\tau)=\widetilde{\Theta}_\chi(\beta z;\beta\tau)$, where χ and β are as above. In this case, we have that $4\mid\beta_j'$ and $\alpha_j'\neq\beta_j'$ for each $1\leq j\leq 4$. Thus, we have by Theorem 1 that $\widetilde{\Theta}_\chi(z;\tau)$ transforms with weight $1/2$ and index $-1/8$ under $G_{8k+4,1}\ltimes(4\mathbb{Z}\times 2\mathbb{Z})$, noting that $G_{8k+4,1}$ is a subgroup in the intersection $\cap_{j=1}^4\mathcal{G}(\beta_j',f_{\beta_j'})=\cap_{j=1}^4 G_{\beta_j',1}$. With this, it is not difficult to verify that $\widetilde{\Theta}_\chi(\beta z;\beta\tau)$ transforms appropriately on $G_{8k+4,8k+4}\ltimes(4\mathbb{Z}\times\mathbb{Z})$ with weight $1/2$ and index $-k-1/2$. Finally, a direct calculation reveals that $\widetilde{H}_{a,k}^{(1)}(z;\tau)$ is defined on the subset $\mathcal{Q}_{8k+4,8k+4}\subset\mathbb{Q}\times\mathbb{Q}$, which is closed under the Jacobi action of $G_{8k+4,8k+4}\ltimes(4\mathbb{Z}\times\mathbb{Z})$ by Lemma 6, and can be expressed as an explicit polynomial in roots of unity there. This completes the proof. $\square$

5.2. **On Lovejoy's generalized "strange identities" I and Theorem 3.** Using the machinery of Bailey pairs, Lovejoy establishes several other multisum "strange" identities including Hikami's generalization of Zagier's studied in the previous subsection. In this and the following three subsections, we establish the quantum Jacobi properties of these functions. To begin, we define the functions

$$
\begin{aligned}
&T^{(2)}_{a,k}(x;q) := \sum_{n=0}^{\infty}(-1)^n x^{2kn} q^{2kn^2+(2k-2a-1)n}(1-x^{2a+1}q^{(2a+1)(2n+1)}),\\
&H^{(2)}_{a,k}(x;q)\\
&\quad := (1-x)\sum_{n_1,\dots,n_k\geq 0}\frac{(xq^2;q^2)_{n_k} q^{2n_1^2+\cdots+2n_{k-1}^2+2n_{a+1}+\cdots+2n_{k-1}} x^{2n_1+\cdots+2n_{k-1}+n_k}}{(-xq;q^2)_{n_1+\delta_{a,0}}}\prod_{j=1}^{k-1}\begin{bmatrix} n_{j+1}+\delta_{j,a} \\ n_j\end{bmatrix}_{q^2},
\end{aligned}
$$

as appearing in [26, p1037-1038], defined for $k \in \mathbb{N}$ and $0 \leq a \leq k-1$. These functions are used by Lovejoy to establish further interesting "strange" identities similar to Hikami's generalization (1.6) of Zagier's (1.4). We normalize the functions (again with $(x,q) = (e(z), e(\tau))$) by

$$
\begin{aligned}
\widetilde{T}^{(2)}_{a,k}(z;\tau) &:= q^{\frac{(2k-2a-1)^2}{8k}} x^{\frac{2k-2a-1}{2}} T^{(2)}_{a,k}(x;q),\\
\widetilde{H}^{(2)}_{a,k}(z;\tau) &:= q^{\frac{(2k-2a-1)^2}{8k}} x^{\frac{2k-2a-1}{2}} H^{(2)}_{a,k}(x;q).
\end{aligned}
$$

Theorem 3. *The functions $\widetilde{H}^{(2)}_{a,k}(z;\tau)$ and $\widetilde{T}^{(2)}_{a,k}(z;\tau)$ are quantum Jacobi forms of weight of weight $1/2$ and index $-k/2$ on $\mathcal{Q}'_{8k,16k} \subset \mathbb{Q}\times\mathbb{Q}$, with Jacobi group $G_{8k,16k} \ltimes (8\mathbb{Z}\times\mathbb{Z})$, and character $\chi_{C/16k,D}$.*

Proof. The proof is similar to the proof of Theorem 2 above. We define the periodic function χ as in Definition 1 with $r = 4$,

$$
\epsilon_j := \begin{cases} 1, & j = 1,2,\\ -1, & j = 3,4,\end{cases} \qquad \alpha_j := \begin{cases} 2k-2a-1, & j=1,\\ 6k+2a+1, & j=2,\\ 2k+2a+1, & j=3,\\ 6k-2a-1, & j=4,\end{cases}
$$

and $\beta := 8k$. With this χ and β, from [26, Proof of (11) p1037-1038], we have that $\widetilde{H}^{(2)}_{a,k}(z;\tau) = \widetilde{T}^{(2)}_{a,k}(z;\tau) = \widetilde{\Theta}_\chi(\beta z; 2\beta\tau)$. In this case, we have that $4 \mid \beta'_j$ and $\alpha'_j \neq \beta'_j$ for each $1 \leq j \leq 4$. Thus, we have by Theorem 1 that $\widetilde{\Theta}_\chi(z;\tau)$ transforms with weight $1/2$ and index $-1/8$ under $G_{8k,1} \ltimes (4\mathbb{Z}\times 2\mathbb{Z})$, noting that $G_{8k,1}$ is a subgroup in the intersection $\cap_{j=1}^4 \mathcal{G}(\beta'_j, f_{\beta'_j}) = \cap_{j=1}^4 G_{\beta'_j,1}$. With this, it is not difficult to verify that $\widetilde{\Theta}_\chi(\beta z; 2\beta\tau)$ transforms appropriately on $G_{8k,16k} \ltimes (4\mathbb{Z}\times\mathbb{Z})$ with weight $1/2$ and index $-k/2$. Finally, a direct calculation reveals that $\widetilde{H}^{(2)}_{a,k}(z;\tau)$ is defined on the subset $\mathcal{Q}'_{8k,16k} \subset \mathbb{Q}\times\mathbb{Q}$, which is closed under the Jacobi action of $G_{8k,16k} \ltimes (8\mathbb{Z}\times\mathbb{Z})$ by Lemma 6, and can be expressed as an explicit rational function in roots of unity there. This completes the proof. □

5.3. **On Lovejoy's generalized "strange identities" II and Theorem 4.** We define the functions

$$
T^{(3)}_k(x;q) := \sum_{n=0}^{\infty}(-1)^n x^{2kn} q^{kn^2+(k-1)n}(1-x^2q^{(2n+1)}),
$$

$$H_k^{(3)}(x;q) := (1-x)\sum_{n_1,\dots,n_k\geq 0}\frac{(xq;q)_{n_k}q^{n_1^2+\cdots+n_{k-1}^2+n_1+\cdots+n_{k-1}}x^{2n_1+\cdots+2n_{k-1}+n_k}}{(-xq;q)_{n_1}}\prod_{j=1}^{k-1}\begin{bmatrix} n_{j+1} \\ n_j \end{bmatrix}_q$$

($k\in\mathbb{N}$), another family along with those in the previous two subsections studied by Lovejoy [26, p1032] in his work on Bailey pairs and "strange identities". Define a normalization of these functions (again with $(x,q)=(e(z),e(\tau))$) by

$$\widetilde{T}_k^{(3)}(z;\tau) := q^{\frac{(k-1)^2}{4k}}x^{k-1}T_k^{(3)}(x;q),$$

$$\widetilde{H}_k^{(3)}(z;\tau) := q^{\frac{(k-1)^2}{4k}}x^{k-1}H_k^{(3)}(x;q).$$

Theorem 4. *For integers $k\geq 2$, the functions $\widetilde{H}_k^{(3)}(z;\tau)$ and $\widetilde{T}_k^{(3)}(z;\tau)$ are quantum Jacobi forms of weight of weight $1/2$ and index $-k$ on $\mathcal{Q}^o_{4k,32k}$, with Jacobi group $G'_{4k,32k}\ltimes(4\mathbb{Z}\times\mathbb{Z})$, and character $\chi_{C/8k,D}$.*

Remark. The quantum and mock Jacobi properties of the functions in this theorem in the case $k=1$ are given by [17, Theorem 1.4]. (Note that the functions $H_1(x;q)$ in [17, (1.7)]) and $\theta_1(x;q)$ in [17, (1.4)] are the same as $(1-x)^{-1}H_1^{(3)}(x;q)$ and $T_1^{(3)}(x;q)$ here (respectively).)

Proof. The proof is similar to the prior two proofs. We let $r=4$, and define the periodic χ as in Definition 1 by

$$\epsilon_j := \begin{cases} 1, & j=1,2, \\ -1, & j=3,4, \end{cases} \qquad \alpha_j := \begin{cases} k-1, & j=1, \\ 3k+1, & j=2, \\ k+1, & j=3, \\ 3k-1, & j=4, \end{cases}$$

and $\beta:=4k$. Using [26, Proof of (8) p1032] we find that $\widetilde{H}_k^{(3)}(z;\tau)=\widetilde{T}_k^{(3)}(z;\tau)=\widetilde{\Theta}_\chi(2\beta z;2\beta\tau)$. The result now follows using Theorem 1 as in the proofs of Theorems 2 and 3 above. □

5.4. **On Lovejoy's generalized "strange identities" III and Theorem 5.** To add to the group of functions studied in Sections 5.1–5.3, we define the functions

$$T_k^{(4)}(x;q) := \sum_{n=0}^{\infty}(-1)^n(1-xq^{2n+1})x^{(2k-1)n}q^{(2k-1)n^2+(2k-2)n},$$

$$H_k^{(4)}(x;q)$$
$$:= (1-x)\sum_{n_1,\dots,n_k\geq 0}\frac{(xq^2;q^2)_{n_k}q^{2n_1^2+2n_1+\cdots+2n_{k-1}^2+2n_{k-1}}x^{2n_1+\cdots+2n_{k-1}+n_k}(q;q^2)_{n_1}}{(-xq;q)_{2n_1+1}}\prod_{j=1}^{k-1}\begin{bmatrix} n_{j+1} \\ n_j \end{bmatrix}_{q^2}$$

appearing in [26, p1034], defined for $k\in\mathbb{N}$. Define a normalization of these functions (again with $(x,q)=(e(z),e(\tau))$) by

$$\widetilde{T}_k^{(4)}(z;\tau) := q^{\frac{(k-1)^2}{2k-1}}x^{k-1}T_k^{(4)}(x;q),$$

$$\widetilde{H}_k^{(4)}(z;\tau) := q^{\frac{(k-1)^2}{2k-1}}x^{k-1}H_k^{(4)}(x;q).$$

Theorem 5. *For integers $k \geq 2$, the functions $\widetilde{H}_k^{(4)}(z;\tau)$ and $\widetilde{T}_k^{(4)}(z;\tau)$ are quantum Jacobi forms of weight of weight $1/2$ and index $-k/2+1/4$ on $\mathcal{Q}^o_{8k-4,64k-32}$, with Jacobi group $G'_{8k-4,64k-32} \ltimes (8\mathbb{Z}\times\mathbb{Z})$, and character $\chi_{C/(16k-8),D}$.*

Remark. The quantum and mock Jacobi properties of the functions in this theorem in the case $k=1$ may be deduced from [17, Theorem 1.4]. Note that the function $\theta_1(x^{\frac{1}{2}};q)$ in [17, (1.4)] is the same as $T_1^{(4)}(x;q)$ here, and by virtue of [12, (14.31)] and Lovejoy's identity at the bottom of p. 1034 in [26] for $k=1$, we find that $(1-x^{\frac{1}{2}})H_1(x^{\frac{1}{2}};q)$ in [17, (1.7)] equals $H_1^{(4)}(x;q)$ here.

Proof. The proof is similar to the prior three proofs. We define χ as in Definition 1 with $r=4$,

$$\epsilon_j := \begin{cases} 1, & j=1,2, \\ -1, & j=3,4, \end{cases} \qquad \alpha_j := \begin{cases} 2k-2, & j=1, \\ 6k-2, & j=2, \\ 2k, & j=3, \\ 6k-4, & j=4, \end{cases}$$

and $\beta := 8k-4$. From [26, Proof of (9) p1034] we find that $\widetilde{H}_k^{(4)}(z;\tau) = \widetilde{T}_k^{(4)}(z;\tau) = \widetilde{\Theta}_\chi(\beta z; 2\beta\tau)$. The result now follows using Theorem 1 as in the proofs of Theorems 2–4 above. $\square$

5.5. **On Lovejoy's generalized "strange identities" IV and Theorem 6.** Similar to the families studied in Sections 5.1–5.4, we define

$$T_{a,k}^{(5)}(x;q) := \sum_{n=0}^{\infty} x^{(2k-1)n} q^{\binom{n+1}{2}+an^2+(k-a-1)(n^2+n)}(1+x^{2a}q^{a(2n+1)}),$$

$$H_{a,k}^{(5)}(x;q) := (1-x)\sum_{n_1,\ldots,n_k\geq 0} (xq;q)_{n_k} q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}} x^{2n_1+\cdots+2n_{k-1}+n_k}$$
$$\times \frac{(-1;q)_{n_1+\delta_{a,0}}}{(x^2q;q^2)_{n_1+\delta_{a,0}}} \prod_{j=1}^{k-1} \begin{bmatrix} n_{j+1}+\delta_{j,a} \\ n_j \end{bmatrix}_q$$

appearing in [26, p1041], defined for $k\in\mathbb{N}$, $0\leq a\leq k-1$. Define a normalization of these functions (again with $(x,q)=(e(z),e(\tau))$) by

$$\widetilde{T}_{a,k}^{(5)}(z;\tau) := q^{\frac{(2k-2a-1)^2}{8(2k-1)}} x^{\frac{2k-2a-1}{2}} T_{a,k}^{(5)}(x;q),$$

$$\widetilde{H}_{a,k}^{(5)}(z;\tau) := q^{\frac{(2k-2a-1)^2}{8(2k-1)}} x^{\frac{2k-2a-1}{2}} H_{a,k}^{(5)}(x;q).$$

Theorem 6. *The functions $\widetilde{H}_{a,k}^{(5)}(z;\tau)$ and $\widetilde{T}_{a,k}^{(5)}(z;\tau)$ are quantum Jacobi forms of weight of weight $1/2$ and index $-k+1/2$ on $\mathcal{Q}^e_{4k-2,2k-1}$, with Jacobi group $G_{4k-2,2k-1}\ltimes(2\mathbb{Z}\times\mathbb{Z})$, and character $\chi_{C/(2k-1),D}$.*

Proof. The proof is similar to the prior four proofs. We again define a periodic function χ as in Definition 1 with $r = 2$, $\epsilon_j := 1$ for $1 \leq j \leq 2$,

$$\alpha_j := \begin{cases} 2k - 2a - 1, & j = 1, \\ 2k + 2a - 1, & j = 2, \end{cases}$$

and $\beta := 4k - 2$. In [26, (12)], the parameter given corresponding to α_2 is $-(2k - 2a - 1)$. Above we define instead $\alpha_2 := 2k + 2a - 1$, which is congruent to $-(2k - 2a - 1) \mod 4k - 2$, and satisfies $0 < \alpha_2 < \beta = 4k - 2$. From [26, Proof of (13) p1041], we find with χ and β as just defined that $\widetilde{H}^{(5)}_{a,k}(z;\tau) = \widetilde{T}^{(5)}_{a,k}(z;\tau) = (1 + \delta_{a,0})\widetilde{\Theta}_\chi(\beta z; \frac{\beta}{2}\tau)$. The result now follows using Theorem 1 as in the proofs of Theorems 2–5 above. $\square$

5.6. **On the Bijaoui et al. Kontsevich-Zagier series for torus knots $T(3, 2^t)$ and Theorem 7.** In this section we establish quantum Jacobi properties of q-hypergeometric and partial theta families studied by Bijaoui et al. related to Kontsevich-Zagier series for torus knots $T(3, 2^t)$ ($t \geq 2$) (see also [19]). Specifically, we consider the series from [5, p6] defined for $t \geq 2$:

$$T^{(6)}_t(x;q) := \sum_{n=0}^{\infty} \chi_t(n) q^{\frac{n^2 - (2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{n-(2^{t+1}-3)}{2}},$$

where χ_t is a periodic function defined as in Definition 1 with $r = 4$, $\beta := 3 \cdot 2^{t+1}$, and

$$\epsilon_j := \begin{cases} 1, & j = 1, 2, \\ -1, & j = 3, 4, \end{cases} \qquad \alpha_j := \begin{cases} 2^{t+1} - 3, & j = 1, \\ 3 + 2^{t+2}, & j = 2, \\ 2^{t+1} + 3, & j = 3, \\ 2^{t+2} - 3, & j = 4, \end{cases},$$

as well as the q-hypergeometric series from [5, RHS of (2.9)]

$$\begin{aligned} H^{(6)}_t(x;q) :=& (-1)^{h''(t)} q^{-h'(t)} x^{-h(t)} \sum_{n=0}^{\infty} (x;q)_{n+1} x^{nm(t)} \\ &\times \sum_{3\sum_{\ell=1}^{m(t)-1} j_\ell \ell \equiv 1 \pmod{m(t)}} (-x)^{\sum_{\ell=1}^{m(t)-1} j_\ell} q^{\frac{-a(t) + \sum_{\ell=1}^{m(t)-1} j_\ell \ell}{m(t)} + \sum_{\ell=1}^{m(t)-1} \binom{j_\ell}{2}} \\ &\times \sum_{k=0}^{m(t)-1} x^k \prod_{\ell=1}^{m(t)-1} \begin{bmatrix} n + I(\ell \leq k) \\ j_\ell \end{bmatrix}_q . \end{aligned}$$

Define a normalization of these functions (again with $q = e(\tau), x = e(z)$) by

$$\widetilde{T}^{(6)}_t(z;\tau) := q^{\frac{(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{(2^{t+1}-3)}{2}} T^{(6)}_t(x;q),$$

$$\widetilde{H}^{(6)}_t(z;\tau) := q^{\frac{(2^{t+1}-3)^2}{3\cdot 2^{t+2}}} x^{\frac{(2^{t+1}-3)}{2}} H^{(6)}_t(x;q).$$

Theorem 7. *The functions $\widetilde{H}^{(6)}_t(z;\tau)$ and $\widetilde{T}^{(6)}_t(z;\tau)$ are quantum Jacobi forms of weight of weight $1/2$ and index $-3 \cdot 2^t/4$ on $\mathcal{Q}_{3\cdot 2^{t+1}, 3\cdot 2^{t+1}}$, with Jacobi group $G_{3\cdot 2^{t+1}, 3\cdot 2^{t+1}} \ltimes (4\mathbb{Z} \times \mathbb{Z})$, and character $\chi_{C/(3\cdot 2^{t+1})}$.*

Proof. By [5, Proposition 2.3], we have that $\widetilde{H}_t^{(6)}(z;\tau) = \widetilde{T}_t^{(6)}(z;\tau) = \widetilde{\Theta}_{\chi_t}(\beta z;\beta\tau)$, where $\beta = 3\cdot 2^{t+1}$. The result now follows using Theorem 1 as in the proofs of Theorems 2–6. $\square$

5.7. **On the Hikami-Kirillov Virasoro characters of minimal models $\mathcal{M}(s,t)$ and Kashaev invariants for torus knots $T(s,t)$ and Theorem 8.** Consider the function in the identity in [22, (5.5)], defined for positive s,t satisfying $\gcd(s,t)=1$ and $0<n<s, 0<m<t$. Denote the partial Jacobi theta function appearing there by

$$T^{(7)}_{s,t,n,m}(x;q) := \sum_{k=0}^{\infty} \chi_{2st}^{(n,m)}(k) q^{\frac{k^2-(nt-ms)^2}{4st}} x^{\frac{k-|nt-ms|}{2}},$$

where $\chi_{2st}^{(n,m)}$ is a periodic function defined by

$$\chi_{2st}^{(n,m)}(k) := \begin{cases} 1, & k \equiv \widetilde{\alpha}_1 \text{ or } \widetilde{\alpha}_2 \pmod{2st}, \\ -1, & k \equiv \widetilde{\alpha}_3 \text{ or } \widetilde{\alpha}_4 \pmod{2st}, \end{cases}$$

where

$$\widetilde{\alpha}_j := \begin{cases} nt-ms, & j=1, \\ 2st-(nt-ms), & j=2, \\ nt+ms, & j=3, \\ 2st-(nt+ms), & j=4. \end{cases}$$

Define a normalization of this function (again with $(x,q)=(e(z),e(\tau))$) by

$$\widetilde{T}^{(7)}_{s,t,n,m}(z;\tau) := q^{\frac{(nt-ms)^2}{4st}} x^{\frac{|nt-ms|}{2}} T^{(7)}_{s,t,n,m}(x;q).$$

For specific choices of s,t,n,m, corresponding to Virasoro characters of minimal models $\mathcal{M}(s,t)$, associated q-hypergeometric sums are given in [22]; in particular, Hikami-Kirillov consider the q-series identities associated with (the Eichler integral of) the minimal model $\mathcal{M}(3,t)$ in [22, Section 5], and also establish that the case of $t=4$ for $\mathcal{M}(3,4)$ is associated to Slater's famous identities [31]. For example, from [22, Proposition 8] (which pertains to the case $(s,t)\mapsto(3,2t)$) we have that

$$\sum_{a=0}^{t-1} (-1)^{a-1} q^{\ell_t(a)} x^{\frac{|2t-6a-3|-1}{2}} T^{(7)}_{3,2t,1,2a+1}(x;q) = \chi_{12}(2t+3) \sum_{k=0}^{\infty} (x;q^{\frac{1}{t}})_{k+1} x^k, \tag{5.1}$$

where $\ell_t(a) := (t-3a-1)(t-3a-2)/(6t)$, and $\chi_{12}(\cdot) := (\frac{12}{\cdot})$ is defined by the Kronecker symbol. See also [23]. We remark that the quantum Jacobi properties of the function

$$\sum_{k=0}^{\infty} (x;q)_{k+1} x^k$$

appearing on the right-hand side of (5.1) may be deduced from [13, Theorem 4, $t=1$ case]. We therefore focus here on establishing quantum Jacobi properties for the more general $\widetilde{T}^{(7)}_{s,t,n,m}(z;\tau)$ series as follows.

Theorem 8. *For any positive, relatively prime, integers s and t, and all integers n and m such that $0<n<s$ and $0<m<t$, the functions $\widetilde{T}^{(7)}_{s,t,n,m}(z;\tau)$ are quantum Jacobi forms of weight $1/2$ and index $-st/4$ on $\mathcal{Q}_{\chi_{2st}^{(n,m)}}$, with Jacobi group $G'_{2st,8st} \ltimes (4\mathbb{Z}\times\mathbb{Z})$, and character $\chi_{C/2st,D}$.*

Proof. First we observe that $\widetilde{\alpha}_j \not\equiv 0 \pmod{2st}$ for each j because of the hypotheses on s, t, n and m. Thus, for each $1 \leq j \leq 4$ we may define α_j satisfying $\alpha_j \equiv \widetilde{\alpha}_j \pmod{\beta}$ where $\beta := 2st$, and also $0 < \alpha_j < \beta$. We further define

$$\epsilon_j := \begin{cases} 1, & j = 1, 2, \\ -1, & j = 3, 4, \end{cases}$$

and hence have that $\chi_{2st}^{(n,m)}$ defining $T_{s,t,n,m}^{(7)}$ may be written as in Definition 1 with hypotheses imposed there. With this, we define α_j' and β_j' as usual, and find that $\widetilde{T}_{s,t,n,m}^{(7)}(z;\tau) = \widetilde{\Theta}_{\chi_{2st}^{(n,m)}}(\beta z; \beta\tau)$. The result now follows as in the proofs of Theorems 2–7 above. $\square$

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