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ℓ-Adic properties of the partition function ☆,☆☆

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Celebrating the life of A.O.L. Atkin

Abstract

Ramanujan's famous partition congruences modulo powers of 5, 7, and 11 imply that certain sequences of partition generating functions tend ℓ-adically to 0. Although these congruences have inspired research in many directions, little is known about the ℓ-adic behavior of these sequences for primes $\ell \geqslant 13$. Using the classical theory of "modular forms mod p", as developed by Serre in the 1970s, we show that these sequences are governed by "fractal" behavior. Modulo any power of a prime $\ell \geqslant 5$, these sequences of generating functions ℓ-adically converge to linear combinations of at most $\lfloor\frac{\ell-1}{12}\rfloor - \lfloor\frac{\ell^2-1}{24\ell}\rfloor$ many special q-series. For $\ell \in \{5, 7, 11\}$ we have $\lfloor\frac{\ell-1}{12}\rfloor - \lfloor\frac{\ell^2-1}{24\ell}\rfloor = 0$, thereby giving a conceptual explanation of Ramanujan's congruences. We use the general result to reveal the theory of "multiplicative partition congruences" that Atkin anticipated in the 1960s. His results and observations are examples of systematic infinite families of congruences which exist for all powers of primes $13 \leqslant \ell \leqslant 31$ since $\lfloor\frac{\ell-1}{12}\rfloor - \lfloor\frac{\ell^2-1}{24\ell}\rfloor = 1$. Answering questions of Mazur, in Appendix A we give a new general theorem which fits these results within the framework of *overconvergent* half-integral weight p-adic modular forms. This result, which is based on recent works by N. Ramsey, is due to Frank Calegari.

☆ With Appendix A by Nick Ramsey.
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1. Introduction and statement of results

A *partition* of a positive integer n is any nonincreasing sequence of positive integers which sum to n. The partition function $p(n)$, which counts the number of partitions of n, defines a provocative sequence of integers:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, \ldots,$$
$$\ldots, p(100) = 190569292, \ldots, p(1000) = 24061467864032622473692149727991, \ldots.$$

The study of $p(n)$ has played a central role in number theory. Indeed, Hardy and Ramanujan invented the "circle method" in analytic number theory in their work on $p(n)$ asymptotics. In terms of congruences, $p(n)$ has served as a testing ground for fundamental constructions in the theory of modular forms. Indeed, some of the deepest results on partition congruences have been obtained by making use of modular equations, Hecke operators, Shimura's correspondence, and the Deligne–Serre theory of ℓ-adic Galois representations.

Here we revisit the theory of Ramanujan's celebrated congruences, which assert that

$$p\big(5^m n + \delta_5(m)\big) \equiv 0 \pmod{5^m},$$
$$p\big(7^m n + \delta_7(m)\big) \equiv 0 \pmod{7^{\lfloor m/2 \rfloor + 1}},$$
$$p\big(11^m n + \delta_{11}(m)\big) \equiv 0 \pmod{11^m},$$

where $0 < \delta_\ell(m) < \ell^m$ satisfies the congruence $24\delta_\ell(m) \equiv 1 \pmod{\ell^m}$. To prove these congruences, Atkin, Ramanujan, and Watson [6,30,31,37] made use of special modular equations to produce ℓ-adic expansions of the generating functions

$$P_\ell(b; z) := \sum_{n=0}^{\infty} p\left(\frac{\ell^b n + 1}{24}\right) q^{\frac{n}{24}} \tag{1.1}$$

(note that $q := e^{2\pi i z}$ throughout, $p(0) = 1$, and $p(\alpha) = 0$ if $\alpha < 0$ or $\alpha \notin \mathbb{Z}$).

Little is known about the ℓ-adic properties of the $P_\ell(b; z)$, as $b \to +\infty$, for primes $\ell \geqslant 13$. We address this topic, and we show, despite the absence of modular equations, that these functions are nicely constrained ℓ-adically. They are "self-similar", with resolution that improves as one "zooms in" appropriately. Throughout, if $\ell \geqslant 5$ is prime and $m \geqslant 1$, then we let

$$b_\ell(m) := 2\left(\left\lfloor \frac{\ell - 1}{12} \right\rfloor + 2\right) m - 3. \tag{1.2}$$

To illustrate the general theorem (see Theorem 1.2), we first highlight the phenomenon for powers of the primes $5 \leqslant \ell \leqslant 31$.

Theorem 1.1. *Suppose that $5 \leqslant \ell \leqslant 31$ is prime, and that $m \geqslant 1$. If $b_1 \equiv b_2 \pmod 2$ are integers for which $b_2 > b_1 \geqslant b_\ell(m)$, then there is an integer $A_\ell(b_1, b_2, m)$ such that for every nonnegative integer n we have*

$$p\left(\frac{\ell^{b_2} n + 1}{24}\right) \equiv A_\ell(b_1, b_2, m) \cdot p\left(\frac{\ell^{b_1} n + 1}{24}\right) \pmod{\ell^m}.$$

If $\ell \in \{5, 7, 11\}$, then $A_\ell(b_1, b_2, m) = 0$.

Remark. Boylan and Webb [12] have recently lengthened the range on b in Theorem 1.1. Their work and numerics suggest that one can generically take $b_\ell(m) := 2m - 1$.

Example. Here we illustrate Theorem 1.1 with $\ell = 13$. For $m = 1$, Theorem 1.1 applies for every pair of positive integers $b_1 < b_2$ with the same parity. We let $b_1 := 1$ and $b_2 := 3$. It turns out that $A_{13}(1, 3, 1) = 6$, and so we have that

$$p\left(13^3 n + 1007\right) \equiv 6p(13n + 6) \pmod{13}.$$

By direct calculation, we find that

$$\begin{aligned} 6\sum_{n=0}^{\infty} p(13n+6)q^n &= 66 + 2940q + 50094q^2 + 534804q^3 + 4291320q^4 + 28183230q^5 + \cdots \\ &\equiv 1 + 2q + 5q^2 + 10q^3 + 7q^4 + 10q^5 + \cdots \pmod{13}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} p\left(13^3 n + 1007\right)q^n \\ &\quad = 31724668493728872881006491578226 \\ &\qquad + 50991675504304667711936377645090414961625834061517111251390q + \cdots \\ &\quad \equiv 1 + 2q + 5q^2 + 10q^3 + 7q^4 + 10q^5 + \cdots \pmod{13}. \end{aligned}$$

We zoom in and consider $m = 2$. It turns out that $b_1 := 2$ and $b_2 := 4$ satisfy the conclusion of Theorem 1.1 with $A_{13}(2, 4, 2) = 45$, which in turn implies that

$$p\left(13^4 n + 27371\right) \equiv 45p\left(13^2 n + 162\right) \pmod{13^2}.$$

For $n = 0$, 1, and 2, we find that the smaller partition numbers give

$$\begin{aligned} 45p\left(13^2 \cdot 0 + 162\right) &= 5846125708665 \equiv 99 \pmod{13^2}, \\ 45p\left(13^2 \cdot 1 + 162\right) &= 3546056488619997675 \equiv 89 \pmod{13^2}, \\ 45p\left(13^2 \cdot 2 + 162\right) &= 103507426465844579776215 \equiv 20 \pmod{13^2}. \end{aligned}$$

Although the other partition numbers are way too large to give here, we find

$$p(13^4 \cdot 0 + 27371)$$
$$= 1058165383617801391727085615958122102244407520\ldots \equiv 99 \pmod{13^2},$$
$$p(13^4 \cdot 1 + 27371)$$
$$= 7470616794323243218669697100895332076191362120\ldots \equiv 89 \pmod{13^2},$$
$$p(13^4 \cdot 2 + 27371)$$
$$= 1117777554561273885139609631281557058593813910\ldots \equiv 20 \pmod{13^2}.$$

Theorem 1.1 follows from our general theorem. To make this precise, we recall Dedekind's eta-function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{3k^2+k}{2} + \frac{1}{24}}. \tag{1.3}$$

If $\ell \geqslant 5$ is prime and $m \geqslant 1$, then we let $k_\ell(m) := \ell^{m-1}(\ell - 1)$. In Section 3.3 we consider the action of a special alternating sequence of operators applied to $S_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$, the space of weight $k_\ell(m)$ cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients. We define $\Omega_\ell(m)$ to be the $\mathbb{Z}/\ell^m\mathbb{Z}$-module of the reductions modulo ℓ^m of those forms which arise as images after applying at least the first $b_\ell(m)$ operators. We bound the dimension of $\Omega_\ell(m)$ independently of m, and we relate the partition generating functions to the forms in this space.

Theorem 1.2. *If $\ell \geqslant 5$ is prime and $m \geqslant 1$, then $\Omega_\ell(m)$ is a $\mathbb{Z}/\ell^m\mathbb{Z}$-module with rank $\leqslant \lfloor \frac{\ell-1}{12} \rfloor$. Moreover, if $b \geqslant b_\ell(m)$, then we have that*

$$P_\ell(b; z) \equiv \begin{cases} \frac{F_\ell(b;z)}{\eta(z)} \pmod{\ell^m} & \text{if } b \text{ is even}, \\ \frac{F_\ell(b;z)}{\eta(\ell z)} \pmod{\ell^m} & \text{if } b \text{ is odd}, \end{cases}$$

where $F_\ell(b; z) \in \Omega_\ell(m)$.

Five remarks. (1) As the proof will show, each form $F_\ell(b; z) \in \Omega_\ell(m)$ is congruent modulo ℓ to a cusp form in $S_{\ell-1} \cap \mathbb{Z}[\![q]\!]$. Since these spaces are trivial for $\ell \in \{5, 7, 11\}$, Theorem 1.2 for these ℓ follows immediately from the Ramanujan congruences. Conversely, if $\ell \in \{5, 7, 11\}$ and $m \geqslant 1$, then the proof of Theorem 1.2 will show that for $b \geqslant b_\ell(m)$ that

$$p(\ell^b n + \delta_\ell(b)) \equiv 0 \pmod{\ell^m}.$$

We do not see how to obtain the full strength of the Ramanujan congruences using only the ideas in the proof of Theorem 1.2. The extra information provided by the modular equations employed by Atkin, Ramanujan, and Watson seem to be necessary for this task.

(2) As mentioned earlier, Boylan and Webb [12] have improved the bound for $b_\ell(m)$.

(3) The partition numbers are the coefficients of the generating functions in Theorem 1.2. With the exception of $p(0) = 1$, every value of $p(n)$ appears in at least one of these generating functions. Indeed, if n is positive, then the partition number $p(n)$ occurs (at least with $b = 1$

and $m = 1$) for every prime $\ell \geqslant 5$ which divides $24n - 1$. Obviously, there are such ℓ for every positive n.

(4) Theorem 1.2 shows that the partition numbers are self-similar ℓ-adically with resolutions that improve as one zooms in properly using the stochastic process which defines the $P_\ell(b; z)$. Indeed, the $P_\ell(b; z) \pmod{\ell^m}$, for $b \geqslant b_\ell(m)$, form periodic orbits. This is fractal-type behavior where a simple iteration/induction surprisingly possesses self-similar structure with increasing resolution. Using this metaphor, Theorem 1.2 bounds the corresponding "Hausdorff dimensions", and these dimensions only depend on ℓ. These dimensions are dimensions of $\mathbb{Z}_\ell$ modules. For $\ell \in \{5, 7, 11\}$, the dimension is 0, a fact that is beautifully illustrated by Ramanujan's congruences, and for $13 \leqslant \ell \leqslant 23$, the dimension is 1. Theorem 1.1 summarizes these observations for $5 \leqslant \ell \leqslant 23$ and the proof will show how to include the primes $\ell = 29$ and 31.

(5) In October 2010 Mazur [25] asked the third author questions about the modules $\Omega_\ell(m)$ (see Appendix A). Calegari has answered some of these questions by fitting Theorem 1.2 into the theory of *overconvergent* half-integral weight p-adic modular forms as developed in the recent works of Ramsey. Appendix A by Ramsey includes a detailed discussion of this result.

Theorem 1.2 is inspired by the famous work of Atkin and O'Brien [7,8,10] from the 1960s. Their papers suggested the existence of a richer theory of partition congruences than was known at the time. Although Ramanujan's congruences had already been the subject of many works (for example, see [4,6,7,10,22,27,26,37,38] to name a few),[1] mathematicians had little luck in finding any further partition congruences. Then Atkin and O'Brien [7,10] surprisingly produced congruences modulo the primes $13 \leqslant \ell \leqslant 31$. For example, Atkin proved that

$$p(1977147619n + 815655) \equiv 0 \pmod{19}.$$

In the late 1990s, the third author revisited their work using ℓ-adic Galois representations and the theory of half-integral weight modular forms [28], and he proved that there are such congruences modulo every prime $\ell \geqslant 5$. Ahlgren and the third author [1,3] later extended this to include all moduli coprime to 6. Other recent works by the third author and Lovejoy, Garvan, Weaver,[2] and Yang [16,23,38,39] give more results along these lines, further removing much of the mystery behind the wild congruences of Atkin and O'Brien.

Despite this new knowledge, one important enigmatic problem about $p(n)$ in Atkin's program on "congruence Hecke operators" has remained open. In [7] he writes:

> "*The theory of Hecke operators for modular forms of negative dimension [i.e. positive weight] shows that under suitable conditions their Fourier coefficients possess multiplicative properties… I have overwhelming numerical evidence, and some theoretical support, for the view that a similar theory exists for forms of positive dimension [i.e.negative weight] and functions…; the multiplicative properties being now congruential and not identical.*"

Remark. Guerzhoy [19,20] has confirmed this speculation for level 1 modular functions using the theory of integer weight p-adic modular forms as developed by Hida, and refined by Wan.

[1] Ramanujan's congruences have continued to inspire research. Indeed, the subject of ranks and cranks represents a different thread in number theory which has grown out of the problem of trying to better understand partition congruences (for example, see [5,11,13–15,17,21,24] to name a few).

[2] Weaver found many new congruences using ideas similar to those in [7].

For negative half-integral weights, Atkin offered $p(n)$ as evidence of this theory. Contrary to conventional thinking, he suspected that the $P_\ell(b; 24z) \pmod{\ell^m}$, where the $b, m \to +\infty$, converge to Hecke eigenforms for $\ell = 13$ and 17. Since the $P_\ell(b; 24z)$, as $m \to +\infty$, lie in spaces whose dimensions grow exponentially in m, Atkin believed in the existence of a theory of "congruence Hecke operators", one which depends on ℓ but is independent of m.

To be precise, Atkin considered the weight $-\frac{1}{2}$ Hecke operator with Nebentypus $\chi_{12}(\bullet) = (\frac{12}{\bullet})$. Recall that if λ is an integer and c is prime, then the Hecke operator $T(c^2)$ on the space of forms of weight $\lambda + \frac{1}{2}$ with Nebentypus χ is defined by

$$\left(\sum_n a(n)q^n\right) \Big| T\left(c^2\right) := \sum_n \left(a\left(c^2 n\right) + c^{\lambda-1}\left(\frac{(-1)^\lambda n}{c}\right)\chi(c)a(n) + c^{2\lambda-1}\chi\left(c^2\right)a\left(n/c^2\right)\right)q^n, \tag{1.4}$$

where $a(n/c^2) = 0$ if $c^2 \nmid n$. Atkin and O'Brien found instances in which these series, as b varies, behave like Hecke eigenforms modulo increasing powers of 13 and 17. For 13 (see [10, Theorem 5]) they prove this observation modulo 13 and 13^2, and for 17 Atkin claims (see [7, §6.3]) to have a proof modulo 17, 17^2, and 17^3.

Here we confirm Atkin's speculation for the primes $\ell \leqslant 31$ by decorating Theorem 1.2 with the theory of Hecke operators.

Theorem 1.3. *If $5 \leqslant \ell \leqslant 31$ and $m \geqslant 1$, then for $b \geqslant b_\ell(m)$ we have that $P_\ell(b; 24z) \pmod{\ell^m}$ is an eigenform of all of the weight $k_\ell(m) - \frac{1}{2}$ Hecke operators on $\Gamma_0(576)$.*

As an immediate corollary, we have the following congruences for $p(n)$.

Corollary 1.4. *Suppose that $5 \leqslant \ell \leqslant 31$ and that $m \geqslant 1$. If $b \geqslant b_\ell(m)$, then for every prime $c \geqslant 5$ there is an integer $\lambda_\ell(m, c)$ such that for all n coprime to c we have*

$$p\left(\frac{\ell^b n c^3 + 1}{24}\right) \equiv \lambda_\ell(m, c) p\left(\frac{\ell^b n c + 1}{24}\right) \pmod{\ell^m}.$$

Remark. Atkin [7] found such congruences modulo 13^2, 17^3, 19^2, 23^6, 29, and 31.

To obtain the results in this paper, we begin in Section 2 by defining a sequence of distinguished modular functions. By construction, these functions contain the $P_\ell(b; z)$ as canonical factors. In Section 3 we briefly recall some important facts from the theory of modular forms modulo ℓ as developed by Serre, and we study a special sequence of operators to define a special space $\Omega_\ell(m)$. In Section 4 we relate the $P_\ell(b; z)$ to forms in $\Omega_\ell(m)$, which allows us to prove Theorem 1.2. In Section 5 we prove Theorem 1.3 using recent work of Ahlgren and Boylan [2], Garvan [16] and Yang [39], and in Section 6 we conclude with a discussion of some examples.

2. Partition generating functions

For every prime $\ell \geqslant 5$ we define a sequence of q-series that naturally contain the generating functions $P_\ell(b; z)$ as factors. Throughout, suppose that $\ell \geqslant 5$ is prime, and let

$$\Phi_\ell(z) := \frac{\eta(\ell^2 z)}{\eta(z)}. \tag{2.1}$$

We recall Atkin's $U(\ell)$-operator

$$\left(\sum a(n)q^n\right) \Big| U(\ell) := \sum a(n\ell)q^n, \tag{2.2}$$

and we define $D(\ell)$ by

$$f(z) \mid D(\ell) := \big(\Phi_\ell(z) \cdot f(z)\big) \,\big|\, U(\ell). \tag{2.3}$$

This paper depends on a special sequence of modular functions. We begin by letting

$$L_\ell(0; z) := 1. \tag{2.4}$$

If $b \geqslant 1$, we then let

$$L_\ell(b; z) := \begin{cases} L_\ell(b-1; z) \mid U(\ell) & \text{if } b \text{ is even}, \\ L_\ell(b-1; z) \mid D(\ell) & \text{if } b \text{ is odd}. \end{cases} \tag{2.5}$$

We have the following elementary lemma.

Lemma 2.1. *If b is a nonnegative integer, then*

$$L_\ell(b; z) = \begin{cases} \eta(z) \cdot P_\ell(b; z) & \text{if } b \text{ is even}, \\ \eta(\ell z) \cdot P_\ell(b; z) & \text{if } b \text{ is odd}. \end{cases}$$

Remark. Sequences like $\{L_\ell(b; z)\}$ have played a central role in the papers [6,30,31,37].

Proof of Lemma 2.1. If $F(q)$ and $G(q)$ are formal power series with integer exponents, then

$$\big(F(q^\ell) \cdot G(q)\big) \,\big|\, U(\ell) = F(q) \cdot \big(G(q) \,\big|\, U(\ell)\big).$$

The lemma now follows iteratively by combining the definition of $\Phi_\ell(z)$ with the fact that

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \qquad \square$$

As usual, let $M_k(\Gamma_0(N))$ denote the space of weight k holomorphic modular forms on $\Gamma_0(N)$. We let $M_k^!(\Gamma_0(N))$ denote the space of weight k *weakly holomorphic modular forms* on $\Gamma_0(N)$, those forms whose poles (if any) are supported at the cusps of $\Gamma_0(N)$. When $N = 1$ we use the notation M_k and $M_k^!$.

We have the following crucial lemma about the q-series $L_\ell(b; z)$.

Lemma 2.2. *If b is a nonnegative integer, then $L_\ell(b;z)$ is in $M_0^!(\Gamma_0(\ell)) \cap \mathbb{Z}[\![q]\!]$. In particular, if $b \geqslant 1$, then $L_\ell(b;z)$ vanishes at $i\infty$, and its only pole is at the cusp at* 0*.*

Proof. Using standard facts about Dedekind's eta-function (for example, see [29, Theorems 1.64 and 1.65]), it follows that $\Phi_\ell(z) \in M_0^!(\Gamma_0(\ell^2))$. It is well known that the $U(\ell)$-operator satisfies

$$U(\ell): M_0^!\bigl(\Gamma_0(\ell)\bigr) \to M_0^!\bigl(\Gamma_0(\ell)\bigr).$$

Moreover, Atkin and Lehner (see [9, Lemma 7]) prove that

$$U(\ell): M_0^!\bigl(\Gamma_0\bigl(\ell^2\bigr)\bigr) \to M_0^!\bigl(\Gamma_0(\ell)\bigr).$$

The lemma now follows from the recursive definition of the $L_\ell(b;z)$ and the observation that

$$\Phi_\ell(z) = q^{\frac{\ell^2-1}{24}} + \cdots.$$

In other words, $\Phi_\ell(z)$ vanishes at $i\infty$. □

3. The space $\boldsymbol{\Omega_\ell(m)}$

We shall apply the theory of modular forms mod ℓ to define and study a distinguished space of modular forms modulo ℓ^m, a space we denote by $\Omega_\ell(m)$. It will turn out that $\Omega_\ell(m)$ contains large ranges of the $L_\ell(b;z)$ (mod ℓ^m).

3.1. Modular forms modulo ℓ

We begin by recalling and deriving several important facts about level 1 integer weight modular forms modulo ℓ. We begin with the following well known fact about modular form congruences (for example, see [35, §1.1] or [29, p. 6]).

Lemma 3.1. *(See [35, p. 198].) Suppose that $f_1 \in M_{k_1} \cap \mathbb{Z}[\![q]\!]$ and $f_2 \in M_{k_2} \cap \mathbb{Z}[\![q]\!]$. If $\ell \geqslant 5$ is prime, $f_1 \not\equiv 0 \pmod{\ell}$, and $f_1 \equiv f_2 \pmod{\ell^m}$, then $k_1 \equiv k_2 \pmod{\ell^{m-1}(\ell-1)}$.*

Suppose that $\ell \geqslant 5$ is prime. If f is a modular form with integer coefficients, then define $\omega_\ell(f)$, the *filtration* of f modulo ℓ, by

$$\omega_\ell(f) := \inf_{k \geqslant 0}\bigl\{k:\ f \equiv g \pmod{\ell} \text{ for some } g \in M_k \cap \mathbb{Z}[\![q]\!]\bigr\}. \tag{3.1}$$

The following lemma shall play a central role in the proof of Theorem 1.2.

Lemma 3.2. *If $\ell \geqslant 5$ is prime and $f \in M_k \cap \mathbb{Z}[\![q]\!]$, then the following are true:*

(1) *We have that*

$$\omega_\ell\bigl(f \bigm| U(\ell)\bigr) \leqslant \ell + \frac{(\omega_\ell(f)-1)}{\ell}.$$

(2) *If* $\omega_\ell(f) = \ell - 1$, *then* $\omega_\ell(f \mid U(\ell)) = \ell - 1$.
(3) *If* $\omega_\ell(f) = d(\ell - 1)$, *where* $d \geqslant 2$, *then* $\omega_\ell(f \mid U(\ell)) \leqslant (d-1)(\ell - 1)$.
(4) *If* $\omega_\ell(f) = \ell - 1$ *and* $\Delta(z) := \eta(z)^{24} \in S_{12}$, *then* $\omega_\ell((\Delta(z)^{\frac{\ell^2-1}{24}} \cdot f) \mid U(\ell)) \in \{0, \ell - 1\}$.

Remark. Lemma 3.2(2) says that $U(\ell)$ is a bijection on weight $\ell - 1$ modular forms modulo ℓ.

Proof of Lemma 3.2. Claims (1) and (2) constitute Lemme 2 on p. 213 of [36]. Claim (3) follows from Théorème 6(i) on p. 212 of [36].

To prove (4), we note that for $\ell = 5$, the hypothesis is vacuously true. Indeed, the space S_4 is empty, so f must be congruent modulo 5 to an Eisenstein series of weight 4. But all Eisenstein series of weight 4 with integer coefficients have filtration zero by the Clausen–von Staudt congruences. To prove (4) for $\ell \geqslant 7$, we note that Lemma 3.2(1) implies that

$$\omega_\ell\big(\big(\Delta^{\frac{\ell^2-1}{24}} \cdot f\big) \,\big|\, U(\ell)\big) \leqslant \frac{3\ell}{2} + 1 < 2(\ell - 1).$$

By Lemma 3.1, the filtration must be a multiple of $\ell - 1$, and so we obtain (4). □

3.2. Some consequences

Here we apply the facts from the previous subsection to study the filtrations of special sequences of modular forms. To this end, suppose that $\ell \geqslant 5$ is prime, and suppose that $f(0; z) \in M_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$. In analogy with (2.5), if $b \geqslant 1$, then define $f(b; z)$ by

$$f(b; z) := \begin{cases} f(b-1; z) \mid T(\ell) & \text{if } b \text{ is even}, \\ (\Phi_\ell(z) \cdot f(b-1; z)) \mid T(\ell) & \text{if } b \text{ is odd}. \end{cases} \tag{3.2}$$

Here $T(\ell)$ is the usual ℓth Hecke operator of weight $k_\ell(m)$.

Remark. We also note that since $\Delta(z) := \eta(z)^{24}$, we have the simple congruence

$$\Phi_\ell(z) \equiv \Delta(z)^{\frac{\ell^2-1}{24}} \pmod{\ell}.$$

We obtain the following proposition concerning the filtrations of these forms in the special case where each $f(b; z) \pmod{\ell}$ is the reduction modulo ℓ of a form in $M_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$.

Proposition 3.3. *Assume the notation above, and suppose that each* $f(b; z) \pmod{\ell}$ *is the reduction modulo* ℓ *of a form in* $M_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$. *If* $b \geqslant b_\ell(m)/m$, *then* $\omega_\ell(f(b; z)) \leqslant \ell - 1$.

Proof. On modular forms modulo ℓ, we have that $T(\ell) = U(\ell)$, and so we may use the results of the last section. To ease notation, we let $\omega_b := \omega_\ell(f(b; z))$.

We begin by showing that for integers $0 \leqslant s \leqslant m - 1$, we have the inequality

$$\omega_s \leqslant \big(\ell^{m-1-s} + 2\big)(\ell - 1). \tag{3.3}$$

The inequality trivially holds for $s = 0$ since $f(0; z) \in M_{k_\ell(m)}$. Now suppose that (3.3) is true for an integer $0 \leqslant s < m - 1$. Then we apply Lemma 3.2(1) to obtain

$$\omega_{s+1} \leqslant \ell + \frac{\omega_s + \frac{\ell^2-1}{2} - 1}{\ell} \leqslant \left(\ell^{m-1-(s+1)} + 2\right)(\ell - 1) + \left(\frac{7-\ell}{2\ell}\right)(\ell - 1),$$

where we have taken the right-hand side to be the maximum with respect to both operators $U(\ell)$ and $D(\ell)$. Lemma 3.1 tells us that the filtration must be an integer multiple of $\ell - 1$, and so the inequality (3.3) immediately follows since $(7 - \ell)/2\ell < 1$ for $\ell \geqslant 5$.

Applying inequality (3.3) for $s = m - 1$, we find that $\omega_{m-1} \leqslant 3(\ell - 1)$. If m is even, then

$$\omega_m \leqslant \ell + \frac{3(\ell - 1) - 1}{\ell} = (\ell - 1) + \frac{4}{\ell}(\ell - 1),$$

which gives the result for $\ell \geqslant 5$. On the other hand, if m is odd, then

$$\omega_m \leqslant \ell + \frac{3(\ell - 1) + \frac{\ell^2-1}{2} - 1}{\ell} = (\ell - 1) + \left(\frac{\ell + 9}{2\ell}\right)(\ell - 1),$$

which gives the result for $\ell \geqslant 11$. For $\ell = 5$, we have $\omega_m \leqslant 2(\ell - 1)$ and, analogous to the proof of Lemma 3.2(4), this implies that $\omega_m = 0$. For $\ell = 7$, we have $\omega_m \leqslant 2(\ell - 1)$, and by Lemma 3.2(3), it follows that $\omega_{m+1} \leqslant \ell - 1$.

The argument thus far establishes the truth of the proposition for $b = b_\ell(m)/m$, so by Lemma 3.2(2) and (4), the proposition holds for all $b \geqslant b_\ell(m)/m$. □

3.3. A special sequence of operators and $\Omega_\ell(m)$

We consider the alternating sequence of operators

$$X := \{D(\ell), U(\ell), D(\ell), U(\ell), D(\ell), U(\ell), \ldots\}.$$

For a cusp form $G(z)$, to ease notation, we let $G_\ell(0; z) := G(z)$, and for $b \geqslant 1$ we then let

$$G_\ell(b; z) := \begin{cases} G_\ell(b-1; z) \mid U(\ell) & \text{if } b \text{ is even,} \\ G_\ell(b-1; z) \mid D(\ell) & \text{if } b \text{ is odd.} \end{cases} \tag{3.4}$$

We say that a cusp form $G(z) \in S_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$ is *good for* (ℓ, m) if for each $b \geqslant b_\ell(m)$ we have that $G_\ell(b; z)$ is the reduction modulo ℓ^m of a cusp form in $S_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$. It will turn out that each $L_\ell(b; z)$, for $b \geqslant b_\ell(m)$, is the reduction modulo ℓ^m of a cusp form in $S_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$. This fact then guarantees that Proposition 3.3 can be applied to prove Theorem 4.3, our main technical result about the forms $L_\ell(b; z)$.

Remark. We stress again that $U(\ell) \equiv T(\ell) \pmod{\ell^m}$ on these spaces.

We define the space $\Omega_\ell(m)$ to be the $\mathbb{Z}/\ell^m\mathbb{Z}$-module generated by the set

$$\left\{G_\ell(b; z) \left(\text{mod } \ell^m\right)\text{: where } b \geqslant b_\ell(m) \text{ and } G(z) \text{ is good for } (\ell, m)\right\}. \tag{3.5}$$

Theorem 3.4. *If $\ell \geqslant 5$ and $m \geqslant 1$, then $\Omega_\ell(m)$ is a $\mathbb{Z}/\ell^m\mathbb{Z}$-module with rank $\leqslant \lfloor \frac{\ell-1}{12} \rfloor$.*

Proof. Let $G(z)$ be an element of $\Omega_\ell(m)$. The space $S_{\ell-1} \cap \mathbb{Z}[\![q]\!]$ is well known to be a $\mathbb{Z}$-module of rank $d_\ell := \lfloor \frac{\ell-1}{12} \rfloor$. By Proposition 3.3, it follows that the reduction of the forms in $\Omega_\ell(m)$ modulo ℓ gives a subspace of $S_{\ell-1} \cap \mathbb{Z}[\![q]\!]$ (mod ℓ), which is a $\mathbb{Z}/\ell\mathbb{Z}$-module with rank $\leqslant d_\ell$.

Let $B_\ell(1; z), \ldots, B_\ell(d_\ell(1); z)$, where $d_\ell(1) \leqslant d_\ell$, be elements of $\Omega_\ell(m)$ which form a basis of these reductions modulo ℓ. Therefore, $G(z)$ (mod ℓ) is in the $\mathbb{Z}/\ell\mathbb{Z}$-span of these forms, and so it follows that

$$G_1(z) := G(z) - \sum_{i=1}^{d_\ell(1)} \alpha_1(i) B_\ell(i; z) \equiv 0 \pmod{\ell}$$

for certain $\alpha_1(i) \in \mathbb{Z}/\ell\mathbb{Z}$.

Now we consider $\frac{1}{\ell} \cdot G_1(z)$ (mod ℓ). If this function is not in the $\mathbb{Z}/\ell\mathbb{Z}$-span of the

$$B_\ell(1; z), \ldots, B_\ell\big(d_\ell(1); z\big),$$

then there are forms $B_\ell(d_\ell(1) + 1; z), \ldots, B_\ell(d_\ell(2); z) \in S_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$ for which

$$\big\{B_\ell(1; z), \ldots, B_\ell\big(d_\ell(2); z\big)\big\}$$

is linearly independent over $\mathbb{Z}/\ell^m\mathbb{Z}$, and where

$$B_\ell\big(d_\ell(1) + 1; z\big) \equiv \cdots \equiv B_\ell\big(d_\ell(2); z\big) \equiv 0 \pmod{\ell}.$$

These forms all lie in $\Omega_\ell(m)$, and, of course, we have that $d_\ell(2) \leqslant d_\ell$. We then have that

$$G(z) \equiv \sum_{i=1}^{d_\ell(1)} \big(\alpha_1(i) + \ell\alpha_2(i)\big) B_\ell(i; z) + \sum_{i=d_\ell(1)+1}^{d_\ell(2)} \alpha_2(i) B_\ell(i; z) \pmod{\ell^2},$$

where the $\alpha_j(i)$ are in $\mathbb{Z}/\ell\mathbb{Z}$. The proof follows now in an obvious way. □

4. The forms $L_\ell(b; z)$ and $\Omega_\ell(m)$

Here we apply the previous results to prove Theorem 1.2. First we require some preliminaries involving the ℓ-adic properties of the functions $L_\ell(b; z)$.

4.1. Basic ℓ-adic properties of the $L_\ell(b; z)$

To prove Theorem 1.2, we shall relate the $L_\ell(b; z)$ to forms in $\Omega_\ell(m)$. We begin with some preliminary facts about these level ℓ modular forms.

Lemma 4.1. *If $\ell \geqslant 5$ is prime and $A_\ell(z) := \eta(z)^\ell / \eta(\ell z)$, then for every $m \geqslant 1$ we have that $A_\ell(z)^{2\ell^{m-1}} \in M_{k_\ell(m)}(\Gamma_0(\ell))$, and satisfies the congruence*

$$A_\ell(z)^{2\ell^{m-1}} \equiv 1 \pmod{\ell^m}.$$

Proof. Using facts about Dedekind's eta-function (for example, see [29, Theorems 1.64 and 1.65]), it follows that $A_\ell(z)$ is a weight $(\ell-1)/2$ holomorphic modular form on $\Gamma_0(\ell)$ with Nebentypus $(\frac{\bullet}{\ell})$. To complete the proof, notice that the claimed congruence follows easily from

$$\frac{(1-X)^\ell}{(1-X^\ell)} \equiv 1 \pmod{\ell},$$

and the fact that

$$\bigl(1+\ell\cdot\Psi(q)\bigr)^{\ell^{m-1}} \equiv 1 \pmod{\ell^m},$$

where $\Psi(q)=\sum_{n=1}^{\infty} a(n)q^n$ is a power series with integer coefficients. □

For a fixed $m \geqslant 1$, we define weight $k_\ell(m)$ auxiliary forms $\mathcal{L}_\ell(b;z)$ and $\widehat{\mathcal{L}}_\ell(b;z)$ for $b \geqslant 0$ by

$$\mathcal{L}_\ell(b;z) := L_\ell(b;z)\cdot A_\ell(z)^{2\ell^{m-1}}, \tag{4.1}$$

and

$$\widehat{\mathcal{L}}_\ell(b;z) := \mathcal{L}_\ell(b;z) \mid U(\ell) + \ell^{\frac{k_\ell(m)}{2}-1}\cdot \mathcal{L}_\ell(b;z)\,|_{k_\ell(m)}\, W_\ell. \tag{4.2}$$

Here we used the usual weight k slash operator, which is defined for $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ by

$$(f|_k\gamma)(z) := (\det\gamma)^{\frac{k}{2}}(cz+d)^{-k} f(\gamma z),$$

with the matrix $W_\ell := \left(\begin{smallmatrix} 0 & -1\\ \ell & 0\end{smallmatrix}\right)$.

To use (4.2), we must control $\ell^{\frac{k_\ell(m)}{2}-1}\cdot \mathcal{L}_\ell(b;z)\,|_{k_\ell(m)}\, W_\ell \pmod{\ell^m}$. The next proposition, whose proof mirrors a calculation of Ahlgren and Boylan (see [2, §5]), suffices for our work. In what follows, we define $B_\ell(b,m)$ by

$$B_\ell(b,m) := \begin{cases} 2(\ell^m-m)/3-2/3 & \text{if } b \text{ is even},\\ 2(\ell^m-m)/3-1 & \text{if } b \text{ is odd}.\end{cases}$$

Proposition 4.2. *If $b \leqslant B_\ell(b,m)$ is a nonnegative integer, then we have that*

$$\ell^{\frac{k_\ell(m)}{2}-1}\mathcal{L}_\ell(b;z)\,|_{k_\ell(m)}\, W_\ell \equiv 0 \pmod{\ell^m}.$$

Proof. It is well known that

$$\eta(-1/z)=\sqrt{z/i}\cdot\eta(z) \quad\text{and}\quad \eta(z+1)=\alpha\cdot\eta(z), \tag{4.3}$$

where α is a 24th root of unity. Using the definition of W_ℓ and applying (4.3), we find

$$A_\ell(z)^{2\ell^{m-1}}\,|_{k_\ell(m)}\,W_\ell = \ell^{\frac{(1-\ell)\ell^{m-1}}{2}} z^{(1-\ell)\ell^{m-1}} \left(\frac{\eta^\ell(\frac{-1}{\ell z})}{\eta(\frac{-1}{z})}\right)^{2\ell^{m-1}}$$
$$= (-i)^{\ell^m-\ell^{m-1}} \ell^{\frac{\ell^m+\ell^{m-1}}{2}} \left(\frac{\eta^\ell(\ell z)}{\eta(z)}\right)^{2\ell^{m-1}}.$$

Let $\zeta_\nu := \exp(2\pi i/\nu)$. In view of this identity, by Lemma 4.1 and (4.1), it suffices to show that

$$\ell^{\ell^m-1} L_\ell(b;z)|_0 W_\ell \equiv 0 \pmod{\ell^m} \tag{4.4}$$

in an appropriate power series ring. The case when $b=0$ is trivial, so we first consider the case when $b=1$. We have that

$$\ell^{\ell^m-1} L_\ell(1;z)|_0 W_\ell = \left(\ell^{\ell^m-1}\left(\frac{\eta(\ell^2 z)}{\eta(z)}\right) \,\Big|\, U(\ell)\right)\Big|_0 W_\ell$$
$$= \left(\ell^{\ell^m-2}\sum_{j=0}^{\ell-1} \frac{\eta(\ell^2 z)}{\eta(z)}\Big|_0 \begin{pmatrix}1 & j\\ 0 & \ell\end{pmatrix}\right)\Big|_0 \begin{pmatrix}0 & -1\\ \ell & 0\end{pmatrix}$$
$$= \ell^{\ell^m-2}\sum_{j=0}^{\ell-1} \frac{\eta(\frac{-1}{z}+\ell j)}{\eta(\frac{-1}{\ell^2 z}+\frac{j}{\ell})} = \ell^{\ell^m-2}\sqrt{z/i}\cdot\eta(z)\sum_{j=0}^{\ell-1}\frac{\alpha^{\ell j}}{\eta(\frac{\ell j z-1}{\ell^2 z})}, \tag{4.5}$$

where we have used (4.3), and that

$$f \mid U(\ell) = \ell^{\frac{k}{2}-1}\sum_{j=0}^{\ell-1} f\big|_k \begin{pmatrix}1 & j\\ 0 & \ell\end{pmatrix}. \tag{4.6}$$

Let $1 \leqslant j \leqslant \ell-1$. Because $\gcd(\ell, j) = 1$, there exist integers d_j and b_j such that $b_j j + d_j \ell = -1$. Using this fact we can write

$$\begin{pmatrix}\ell j & -1\\ \ell^2 & 0\end{pmatrix} = \begin{pmatrix} j & d_j\\ \ell & -b_j\end{pmatrix}\begin{pmatrix}\ell & b_j\\ 0 & \ell\end{pmatrix},$$

where the first matrix on the right-hand side is in $\mathrm{SL}_2(\mathbb{Z})$. Thus we deduce that

$$\eta\left(\frac{\ell j z-1}{\ell^2 z}\right) = \epsilon_{j,\ell}(\ell z)^{\frac12}\eta\left(z+\frac{b_j}{\ell}\right)$$

where $\epsilon_{j,\ell}$ is a 24th root of unity, and we have used the general modular transformation law for $\eta(z)$. Thus, we re-write (4.5) as

$$\ell^{\ell^m-5/2}\cdot\sqrt{-i}\cdot\eta(z)\sum_{j=1}^{\ell-1}\frac{\alpha^{\ell j}}{\epsilon_{j,\ell}\cdot\eta(z+\frac{b_j}{\ell})} + \ell^{\ell^m-3}\frac{\eta(z)}{\eta(\ell^2 z)}, \tag{4.7}$$

where again we have used (4.3) to deduce the contribution arising from $j=0$. One then has that the term corresponding to j in (4.7), where $1 \leqslant j \leqslant \ell - 1$, is

$$\ell^{\ell^m-5/2} \cdot \sqrt{-i} \cdot \zeta_{24\ell}^{-b_j} \cdot \frac{\alpha^{\ell j}}{\epsilon_{j,\ell}} \prod_{n\geqslant 1} \frac{(1-q^n)}{(1-\zeta_{24\ell}^{24b_j n} q^n)},$$

and the term corresponding to $j=0$ in (4.7) is

$$\ell^{\ell^m-3} q^{\frac{1-\ell^2}{24}} \prod_{n\geqslant 1} \frac{(1-q^n)}{(1-q^{\ell^2 n})}.$$

These observations combined with the fact that $\ell^m - 5/2 > \ell^m - 3 \geqslant m$ for $\ell \geqslant 5$ and $m \geqslant 1$ indicate vanishing $(\mathrm{mod}\ \ell^m)$ in $q^{\frac{1-\ell^2}{24}} \cdot (\mathbb{Z}[\zeta_{24\ell}])[\![q]\!]$, proving (4.4) when $b=1$.

Although the details are messier, the case for $1 < b < B_\ell(b,m)$ follows similarly. Indeed, for even b, we see that $L(b;z)$ is defined by a nested sequence involving $b/2$ instances of the function $\Phi(z)$, with that same number of applications of $U(\ell^2)$ as follows:

$$L(b;z) = \underbrace{(((\ldots(}_{b-3} \Phi\big|_0 U(\ell^2)) \cdot \Phi)\big|_0 U(\ell^2)) \cdot \Phi)\big|_0 U(\ell^2)) \cdot \Phi \ldots)\big|_0 U(\ell^2) \quad \text{(for } b \text{ even)}. \tag{4.8}$$

Just like $U(\ell)$ introduced an ℓ^{-1} in (4.5) when employing (4.6), each iteration of $U(\ell^2)$ introduces a factor ℓ^{-2}. Moreover, one can show (as with the case $b=1$) that of any additional negative powers of ℓ that occur after multiplying (4.8) by ℓ^{ℓ^m-1} and applying $|_0 W_\ell$, the smallest will be produced for $j=0$ in (4.6) simultaneously for each of the $b/2$ $U(\ell^2)$ operators that appear, each decreasing the power of ℓ by 1. This implies for even b, the smallest negative power of ℓ appearing is $(b/2)\cdot(-2) + (b/2)\cdot(-1) = -3b/2$. For odd b, beginning similarly as in (4.8), we find a total of $(b-1)/2$ instances of the $U(\ell^2)$ operator, and one instance of the $U(\ell)$ operator, yielding a maximal negative power of $-3b/2 - 1/2$. Thus, for even b we require $\ell^m - 1 - 3b/2 \geqslant m$, and for odd b we require $\ell^m - 3/2 - 3b/2 \geqslant m$, which are equivalent to the hypotheses given in the statement of Proposition 4.2. □

4.2. A key theorem and the proof of Theorem 1.2

Using the results in the previous subsection, we can now prove the following crucial fact about the forms $L_\ell(b;z)$.

Theorem 4.3. *If $\ell \geqslant 5$ is prime, $m \geqslant 1$, and $b \geqslant b_\ell(m)$, then $L_\ell(b;z)$ is in $\Omega_\ell(m)$.*

Proof. By Lemma 2.2, we have that $L_\ell(b;z)$ is in $M_0^!(\Gamma_0(\ell)) \cap \mathbb{Z}[\![q]\!]$. Lemma 4.1 gives us the congruence $\mathcal{L}_\ell(b;z) \equiv L_\ell(b;z) \pmod{\ell^m}$. It is clear that $\mathcal{L}_\ell(b;z)$ is a weight $k_\ell(m)$ level ℓ weakly holomorphic modular form. Lemma 7 of [9] asserts that

$$\big(U(\ell) + \ell^{\frac{k_\ell(m)}{2}-1} W_\ell\big) : M^!_{k_\ell(m)}\big(\Gamma_0(\ell)\big) \to M^!_{k_\ell(m)}.$$

From Proposition 4.2, for $b_\ell(m) \leqslant b \leqslant B_\ell(b,m)$, we find that $\widehat{\mathcal{L}}_\ell(b;z) \equiv \mathcal{L}_\ell(b;z) \mid U(\ell)$ (mod ℓ^m), and an inspection of the q-series shows that $\widehat{\mathcal{L}}_\ell(b;z)$ must be congruent modulo ℓ^m to a level 1 cusp form since $L_\ell(b;z)$ vanishes at $i\infty$ by Lemma 2.2.

For $b_\ell(m) \leqslant b \leqslant B_\ell(b,m)$, we employ Lemma 3.2(2), which shows that $U(\ell)$ defines a bijection on level 1 weight $\ell-1$ modular forms modulo ℓ. We also use Proposition 3.3 which determines when the filtration of such forms is $\leqslant \ell-1$. By applying this proposition iteratively modulo increasing powers of ℓ (as in the proof of Theorem 3.4), up to ℓ^m, one obtains the claimed conclusion for these b.

Since $U(\ell) = T(\ell)$ on $\Omega_\ell(m)$, to complete the proof it suffices to prove the claim for odd $b > B_\ell(b,m)$. We make use of the fact that $D(\ell)$ defines a $\mathbb{Z}/\ell^m\mathbb{Z}$-linear map to $\Omega_\ell(m)$ on the submodule of $\Omega_\ell(m)$ generated by the set of functions $L_\ell(b;z)$, where $b_\ell(m) \leqslant b \leqslant B_\ell(b,m)$ is even. If there is an even $b_\ell(m) < b' \leqslant B_\ell(b,m)$ for which $L_\ell(b';z)$ is a $\mathbb{Z}/\ell^m\mathbb{Z}$ linear combination of the previous functions in this set, then the conclusion for b', and hence all even $b \geqslant b'$, follows from the iterative definition of these functions. We have $\lfloor (B_\ell(b,m) - b_\ell(m) + 1)/2 \rfloor$ many functions in this set, and the maximum number of elements in $\Omega_\ell(m)$ one can order before there is such a b' is $\leqslant m\lfloor \frac{\ell-1}{12} \rfloor$ since the dimension of $\Omega_\ell(m)$ is $\leqslant \lfloor \frac{\ell-1}{12} \rfloor$. A short calculation using the definition of $b_\ell(m)$ and $B_\ell(b,m)$ reveals that $\lfloor (B_\ell(b,m) - b_\ell(m) + 1)/2 \rfloor > m\lfloor \frac{\ell-1}{12} \rfloor$ without exception. Therefore, each $L_\ell(b;z)$ is in $\Omega_\ell(m)$. □

Proof of Theorem 1.2. The result follows immediately from Theorems 4.3, 3.4 and Lemma 2.1. □

Proof of Theorem 1.1. The theorem follows trivially from the Ramanujan congruences when $\ell \in \{5, 7, 11\}$. More generally, we consider the two subspaces, $\Omega_\ell^{\text{odd}}(m)$ and $\Omega_\ell^{\text{even}}(m)$, of $\Omega_\ell(m)$ generated by $L_\ell(b;z)$ for odd b and even b, respectively. We observe that applying $D(\ell)$ to a form gives q-expansions satisfying

$$F \mid D(\ell) = \sum_{n > \frac{\ell^2-1}{24\ell}} a(n)q^n.$$

Combining this observation with Theorem 1.2 and the fact that the full space $\Omega_\ell(m)$ is generated by alternately applying $D(\ell)$ and $U(\ell)$, we have that the ranks of $\Omega_\ell^{\text{odd}}(m)$ and $\Omega_\ell^{\text{even}}(m)$ are $\leqslant \lfloor \frac{\ell-1}{12} \rfloor - \lfloor \frac{\ell^2-1}{24\ell} \rfloor$. If $13 \leqslant \ell \leqslant 31$, then direction calculation, when $m = 1$, shows that each of these subspaces has dimension 1. The theorem now follows immediately from Lemma 2.1. □

5. The proof of Theorem 1.3 and Corollary 1.4

To prove results on partition congruences, we made use of the ℓ-adic properties of the sequence of special operators. To prove Theorem 1.3 we combine these ideas with the works of Ahlgren and Boylan, Garvan, and Yang related to Ramanujan-type congruences.

5.1. Theorems of Ahlgren and Boylan, and Garvan and Yang

Here we combine the work of Ahlgren and Boylan, with works by Garvan and Yang to show that our partition generating functions $P_\ell(b;24z)$ are congruent modulo ℓ^m to half-integral weight modular forms in a certain Hecke invariant subspace.

Define the even integer $k_\ell^+(m)$ by

$$k_\ell^+(m) := \begin{cases} \ell^{m-1}(\ell-1) - 12 & \text{if } m \text{ is even,} \\ \frac{(\ell^{m-1}+1)(\ell-1)}{2} - 12(\lfloor \ell/24 \rfloor + 1) & \text{if } m \text{ is odd.} \end{cases}$$

Also define $r_\ell(m)$ by

$$r_\ell(m) := \frac{24\delta_\ell(m) - 1}{\ell^m}.$$

We recall a result of Ahlgren and Boylan [2, Theorem 3] which gives congruence relations between our partition generating functions $P_\ell(b; 24z)$ and a product of a power of the η-function and a level one modular form.

Theorem 5.1 *(Ahlgren and Boylan). For a prime $\ell \geqslant 5$ and an integer $m \geqslant 1$, there exists a modular form $F(k_\ell^+(m); z) \in M_{k_\ell^+(m)} \cap \mathbb{Z}[\![q]\!]$ such that*

$$P_\ell(m; 24z) = \sum_{n=0}^{\infty} p\bigl(\ell^m n + \delta_\ell(m)\bigr) q^{24n + r_\ell(m)} \equiv \eta(24z)^{r_\ell(m)} F\bigl(k_\ell^+(m); 24z\bigr) \pmod{\ell^m}.$$

We also state a result of Garvan [16, Proposition 3.1] which has recently been extended by Yang [39, Theorem 2]. The theorem asserts that the modular forms appearing in the work of Ahlgren and Boylan actually live in a very nice Hecke invariant subspace.

Theorem 5.2 *(Garvan and Yang). Let $0 < r < 24$ be an odd integer and s be a nonnegative even integer. Then*

$$\mathcal{S}_{r,s} := \bigl\{\eta(24z)^r f(24z) \colon f(z) \in M_s\bigr\}$$

is a Hecke invariant subspace of $S_{s+r/2}(\Gamma_0(576), \chi_{12})$ (i.e. for all primes $c \neq 2, 3$ and $F \in \mathcal{S}_{r,s}$, we have $F|_{r+s/2} T(c^2) \in \mathcal{S}_{r,s}$).

5.2. Proof of Theorem 1.3

Combining Lemma 2.1 with Theorem 4.3, arguing in a manner similar to the proof of Theorem 1.2 and using Theorem 5.1, we find that if $\ell \geqslant 5$ is prime and $b_\ell(m) \leqslant b$, then the modular form

$$H_\ell(b; z) := \begin{cases} \eta(z) \cdot \eta(z)^{r_\ell(b)} F(k_\ell^+(b); z) & \text{if } b \text{ is even,} \\ \eta(\ell z) \cdot \eta(z)^{r_\ell(b)} F(k_\ell^+(b); z) & \text{if } b \text{ is odd,} \end{cases}$$

is congruent modulo ℓ^m to a cusp form in $S_{k_\ell(m)} \cap \mathbb{Z}[\![q]\!]$. In particular, we have that

$$H_\ell(b; z) \equiv L_\ell(b; z) \equiv \begin{cases} \eta(z) \cdot P_\ell(b; z) \pmod{\ell^m} & \text{if } b \text{ is even,} \\ \eta(\ell z) \cdot P_\ell(b; z) \pmod{\ell^m} & \text{if } b \text{ is odd.} \end{cases}$$

If $c \nmid 6$ is prime, then we consider $P_\ell(b; z) \mid T(c^2)$. As shown in the proof of Theorem 1.1, we may consider the "even/odd parity" subspaces of $\Omega_\ell(m)$, both of which were determined to have dimension 1 for $13 \leqslant \ell \leqslant 31$. Therefore it suffices to show that

$$K_\ell(b, c; z) := \begin{cases} \eta(z) \cdot (P_\ell(b; z) \mid T(c^2)) & \text{if } b \text{ is even,} \\ \eta(\ell z) \cdot (P_\ell(b; z) \mid T(c^2)) & \text{if } b \text{ is odd} \end{cases}$$

is in $\Omega_\ell(m)$. Since $0 < r_\ell(b) < 24$ is odd, Theorem 5.2 applies, and one obtains the desired conclusion by arguing with the explicit formulas again involving Dedekind's eta-function exactly as in the proof of Theorem 4.3.

Proof of Corollary 1.4. After letting $n \to nc$ in (1.4), the conclusion follows from Theorem 1.3 because $(\frac{nc}{c}) = 0$. □

6. Examples

Here we give examples of Theorem 1.2 for the prime $\ell = 13$. We have that

$$\Phi_{13}(z) := \eta(169z)/\eta(z) = q^7 + q^8 + 2q^9 + \cdots.$$

For $m = 1$, we have that $k_{13}(1) = 12$, and so if $b \geqslant 1$, then by Theorem 4.3 we have that $L_{13}(b; z)$ is congruent modulo 13 to a weight 12 cusp form of level 1, which of course must be a multiple of Ramanujan's $\Delta(z) = \eta(z)^{24}$. The first few terms of $L_{13}(1; z)$ are

$$\begin{aligned} L_{13}(1; z) = \Phi_{13}(z) \mid U(13) &= 11q + 490q^2 + 8349q^3 + 89134q^4 + 715220q^5 + \cdots \\ &\equiv 11q + 9q^2 + 3q^3 + 6q^4 + 12q^5 + 6q^6 + \cdots \pmod{13}. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} 11\Delta(z) &= 11q - 264q^2 + 2772q^3 - 16192q^4 + \cdots \\ &\equiv 11q + 9q^2 + 3q^3 + 6q^4 + 12q^5 + 6q^6 + \cdots \pmod{13}. \end{aligned}$$

Therefore we have that $L_{13}(1; z) \equiv 11\Delta(z) \pmod{13}$, which, by Lemma 2.1, implies that

$$P_{13}(1; z) \equiv 11 \cdot \eta(z)^{11} \pmod{13}.$$

More generally (for example, see [28, §4]), for every nonnegative integer k we have that

$$\begin{aligned} P_{13}(2k + 1; z) &\equiv 11 \cdot 6^k \eta(z)^{11} \pmod{13}, \\ P_{13}(2k + 2; z) &\equiv 10 \cdot 6^k \eta(z)^{23} \pmod{13}. \end{aligned}$$

These congruences illustrate Theorem 1.1 for $\ell^m = 13$.

For $m = 2$, we have, for $b \geqslant 4$, that $L_{13}(b; z)$ is congruent modulo 169 to a form in $S_{156} \cap \mathbb{Z}[\![q]\!]$. If $E_4(z)$ is the usual weight 4 Eisenstein series, then one directly computes and finds that

$$\begin{aligned} L_{13}(2; z) &\equiv 36q + 150q^2 + 154q^3 + 100q^4 + 122q^5 + 22q^6 + 26q^7 + 60q^8 + \cdots \pmod{169} \\ &\equiv 36\Delta E_4^{36} + 89\Delta^2 E_4^{33} + 94\Delta^3 E_4^{30} + 16\Delta^4 E_4^{27} + 36\Delta^5 E_4^{24} + 102\Delta^6 E_4^{21} \\ &\quad + 3\Delta^7 E_4^{18} + 80\Delta^8 E_4^{15} + 166\Delta^9 E_4^{12} + 115\Delta^{10} E_4^9 + 3\Delta^{11} E_4^6 \\ &\quad + 145\Delta^{12} E_4^3 + 88\Delta^{13} \pmod{169}. \end{aligned}$$

Using Lemma 2.1, we find that

$$\begin{aligned} P_{13}(2; z) \equiv \frac{1}{\eta(z)} \cdot \big(&36\Delta E_4^{36} + 89\Delta^2 E_4^{33} + 94\Delta^3 E_4^{30} + 16\Delta^4 E_4^{27} + 36\Delta^5 E_4^{24} \\ &+ 102\Delta^6 E_4^{21} + 3\Delta^7 E_4^{18} + 80\Delta^8 E_4^{15} + 166\Delta^9 E_4^{12} + 115\Delta^{10} E_4^9 + 3\Delta^{11} E_4^6 \\ &+ 145\Delta^{12} E_4^3 + 88\Delta^{13}\big) \quad \big(\mathrm{mod}\ 13^2\big). \end{aligned}$$

Theorem 1.3 is illustrated by the fact that this is a Hecke eigenform modulo 169.

Concerning Theorem 1.1, we have that

$$\begin{aligned} P_{13}(2; 24z) &= 129913904637q^{23} + 78801255302666615q^{47} + \cdots \\ &\equiv 36q^{23} + 17q^{47} + 38q^{71} + 155q^{95} + \cdots \quad \big(\mathrm{mod}\ 13^2\big), \end{aligned}$$

while we have that

$$154P_{13}(4; 24z) \equiv 36q^{23} + 17q^{47} + 38q^{71} + 155q^{95} + \cdots \quad \big(\mathrm{mod}\ 13^2\big).$$

Indeed, it turns out that $P_{13}(4; 24z) \equiv 45P_{13}(2; 24z) \pmod{13^2}$, which in turn implies that

$$p\big(13^4 n + 27371\big) \equiv 45p\big(13^2 n + 162\big) \quad \big(\mathrm{mod}\ 13^2\big).$$

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Appendix A

by Nick Ramsey[3]

Theorem 1.2 in the main text asserts that a certain space of "modular forms modulo ℓ^m" is bounded in rank by a constant independent of m. These forms are obtained by repeated application of two operators on spaces of integral weight modular forms of level one and increasing

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(and increasingly congruent) weights. One of these operators is Atkin's $U(\ell)$, while the other is a "twisted" version of $U(\ell)$ obtained by pre-composing by multiplication by a modular function of weight zero. This is the operator $D(\ell)$.

Ultimately, one is interested in the analog of this sort of result in the setting of half-integral weight modular forms. Indeed, the "twisting" referred to in the previous paragraph is exactly set up to translate a very natural Hecke action in half-integral weight to the integral weight setting under an identification of the form $F \mapsto F/\eta$ between half-integral and integral weight forms. This allowed the authors to employ Serre's theory of integral weight modular forms modulo ℓ.

It is natural to ask if one can work more directly in the half-integral weight setting to arrive at these results. Indeed, we can, and in doing so we provide a very natural answer to the following questions of Mazur [25].

Question 1 *(Mazur).* Do the spaces $\Omega_\ell(m)$ "compile well" to produce a clean free $\mathbb{Z}_\ell$ module? Do the Hecke operators work well on these spaces?

Congruences similar to those discussed in the main text were observed by Atkin in the coefficients of the modular j-invariant. Atkin speculated that such congruences were part of a coherent interplay between Hecke operators and congruences (see his quote in the main text, as well as his quote preceding Theorem 1 of [20]). Since that time, a systematic theory of ℓ-adic (generally referred to as p-adic in the literature) modular forms of integral weight has emerged through the work of Serre, Hida, Katz, Coleman, and Mazur, among others. This theory provides a natural framework for studying this interplay between Hecke operators and congruences. Indeed, using this machinery, Atkin's speculation has been proven in many cases (see [19,20]).

In light of the success of these methods in the integral weight setting, it is natural to frame the half-integral weight question in the context of a theory of overconvergent ℓ-adic modular forms of half-integral weight (see [18] for background on overconvergent p-adic modular forms). Such a theory has been developed by the author in [32,33]. The most relevant features of this work will be summarized in the next subsection. With it, we will answer the questions of Mazur, and we shall prove the following generalization of Theorem 1.2.

Theorem A.1. *Fix a prime ℓ, an odd integer k, a positive integer N, and a Dirichlet character χ modulo $4N$. There exists a module $\Omega \subseteq \mathbb{Z}_\ell[\chi][\![q]\!]$ of finite rank over $\mathbb{Z}_\ell[\chi]$ that is preserved by all half-integral weight Hecke operators and has the following property. Let*

$$F = \sum a(n)q^n$$

be a classical modular form that is holomorphic away from the cusps, and suppose that L is a number field whose ring of integers contains the coefficients of F and the values of χ. Choose an embedding of L into $\overline{\mathbb{Q}}_\ell$. Then for all integers m and s sufficiently large, there is a congruence

$$\sum a\left(\ell^{2s}n\right)q^n \equiv \omega_s \pmod{\ell^m}$$

for some

$$\omega_s \in \Omega \otimes_{\mathbb{Z}_\ell[\chi]} \mathcal{O}_{L,\mathfrak{p}},$$

where $\mathcal{O}_{L,\mathfrak{p}}$ is the completion of the ring of integers of L at the prime determined by the embedding $L \hookrightarrow \overline{\mathbb{Q}}_\ell$ above.

Here, $\mathbb{Z}_\ell[\chi]$ denotes the ring of integers in the finite extension of $\mathbb{Q}_\ell$ generated by the values of χ (thought of as $\overline{\mathbb{Q}}_\ell$-valued).

Three remarks. (1) We note that, given ℓ, k, N, and χ, the module Ω can be explicitly computed.

(2) If this module vanishes, then this result implies that the coefficients $a(\ell^{2s}n)$ tend to zero as $s \to \infty$ *uniformly in n*. This is exactly the situation of Watson's and Atkin's generalization of Ramanujan's congruences.

(3) If the rank of Ω is 1, then since Ω is Hecke-stable, it is spanned by a single nonzero Hecke eigenform. This theorem then implies that the images of F under sufficiently many iterates of $U(\ell^2)$ are congruent modulo ℓ^m to multiples of this single form. This is the phenomenon encapsulated by Theorem 1.1 of the main text.

A.1. ℓ-Adic modular forms of half-integral weight

Let K denote a complete and discretely-valued subfield of $\mathbb{C}_\ell$ such as, for example, a finite extension $K/\mathbb{Q}_\ell$. Purely for the sake of exposition we will assume that $\ell \neq 2$ in what follows. The results of the primary reference [33] hold for all ℓ, but the weight and level book-keeping are slightly different for $\ell = 2$. Fix integers k and N with k odd and N positive, and assume that $\ell \nmid 4N$. One should think of $4N$ as the ℓ-adic "tame level". In [32], the author defined Banach spaces $M_{k/2}(4N, K, r)$ of r-overconvergent modular forms of weight $k/2$ and level $4N$ over K, for any $r \in \mathbb{Q} \cap [0, 1]$. For $r < r'$ there is a natural inclusion

$$M_{k/2}(4N, K, r) \hookrightarrow M_{k/2}\big(4N, K, r'\big).$$

The forms in spaces with $r < 1$ are called *overconvergent*, and the entire space of overconvergent forms of this weight and level is the direct limit

$$M^{\dagger}_{k/2}(4N, K) = \lim_{r \to 1^-} M_{k/2}(4N, K, r).$$

Strictly speaking, there were further restrictions on ℓ, k, and N in [32]. However, the subsequent paper [33] defined these spaces in the current generality, and even greater generality with respect to the weight. Indeed, for any "p-adic weight" κ, a Banach space $M_\kappa(4N, K, r)$ is defined for sufficiently large $r < 1$, and thus an overconvergent space $M^{\dagger}_\kappa(4N, K)$ is defined. The nature of these more general weights is detailed in Section 2.4 of [33]. The classically-minded reader should think of these weights roughly as including all $k/2$ with k odd (positive or negative) and information about the ℓ-part of Nebentypus character "at ℓ" (though they are in fact much more general). This will be explicated in slightly more detail below when we discuss how the classical forms sit among the ℓ-adic forms. Despite these restrictions, it is worth mentioning that the arguments in [32] go through for general odd k (if not for the more general κ of [33]), as that paper is rather less technical than [33] and may provide a more accessible reference for readers unfamiliar with rigid-analytic spaces.

These spaces of ℓ-adic modular forms contain the classical modular forms of half-integral weight in the following sense. Let F be a classical holomorphic modular form of weight $k/2$ and level $4N\ell^s$ (for any integer $s \geqslant 0$) with coefficients in a finite extension $L/\mathbb{Q}$. Suppose for the moment that F is fixed by the diamond operators at ℓ (see [34, Section 8.2]), so F is actually a modular form for the group $\Gamma_1(4N) \cap \Gamma_0(\ell^s)$. Given an embedding $i : L \hookrightarrow \overline{Q}_\ell$, the classical form F gives rise to an ℓ-adic form in $i(f) \in M_{k/2}(4N, i(L), r)$ for all $r < 1$ sufficiently close to 1. For a general classical form on $\Gamma_1(4N\ell^s)$ that is an eigenform for the diamond action at ℓ (but perhaps with nontrivial character), F also gives rise to an ℓ-adic form, but in one of the spaces $M_\kappa(4N, i(L), r)$ of more general weight κ discussed in [33]. One can think of this form as lying in a space of ℓ-adic forms "with Nebentypus character at ℓ", but for technical reasons it is better to package that character as *part of the ℓ-adic weight κ*. The reader can refer to Section 6 of [33] for the details. The upshot is that the space of overconvergent ℓ-adic forms of tame level $4N$ "sees" the classical forms with arbitrarily high powers of ℓ in the level.

To any cusp on $X_1(4N\ell)$ that lies over the cusp ∞ on $X_0(\ell)$, and any ℓ-adic modular form $F \in M_{k/2}(4N, K, r)$, one may associate a q-expansion

$$\sum a(n)q^n$$

whose coefficients lie in a finite extension of K (see [32, Definition 4.1] and [33, Section 5.3]). These q-expansions recover the classical ones if the form f is classical in the above sense.

These spaces are equipped with a collection of commuting continuous Hecke operators, including an operator $U(\ell^2)$ having the effect

$$\sum a(n)q^n \mapsto \sum a(\ell^2 n)q^n \tag{A.1}$$

on q-expansions at the cusps mentioned above. More generally, by the results of Section 8 of [32] and Section 5 of [33], all of these operators have the same effect on q-expansions as their classical counterparts, and therefore recover the classical operators on classical forms. By Theorem 8.2 of [32] and the remarks following Proposition 5.1 of [33], the operator $U(\ell^2)$ is *compact* on the Banach spaces of overconvergent forms. This operator also has the effect of *increasing* the degree of overconvergence on such forms. It follows that, for any $\sigma \in \mathbb{R}$, the slope σ subspace of $M^\dagger_{k/2}(4N, K)$ lies in $M_{k/2}(4N, K, r)$ for some $r < 1$, and hence that this subspace is finite-dimensional by the general theory of compact operators on p-adic Banach spaces (see [35]). Taking $\sigma = 0$, we see that the space of ordinary overconvergent forms is finite-dimensional.

With an eye toward the applications in the main text, the forms in Theorem A.1 are weakly holomorphic (i.e. they are allowed to have poles at the cusps). The construction of the operator $U(\ell^2)$ given in [32] and [33] applies directly to such forms, and this operator has the effect (A.1) on the q-expansions of such forms at cusps lying over the cusp ∞ on $X_0(\ell)$. Indeed, the calculations of Theorem 8.2 of [32] and Proposition 5.5 of [33] carry over easily to this situation.

Examining the effect (A.1) on q-expansions, we see that $U(\ell^2)$ strictly reduces the order of any poles of a form at these cusps. In particular, if F is an ℓ-adic modular form that is regular except perhaps with poles at these cusps, then $U(\ell^2)^t F$ is a genuine ℓ-adic modular form in the above sense for sufficiently large integers t. Moreover, if F is a classical modular form of half-integral weight with poles at *any* cusps, then we arrive at this same conclusion, since the associated ℓ-adic form only sees the cusps lying above ∞ on $X_0(\ell)$ in the first place.

For a Dirichlet character χ_{tm} modulo $4N$, the above spaces contain a Hecke-stable subspace of forms with tame Nebentypus χ_{tm} (denoted by appending χ_{tm} to the list of arguments of the space). Let F be a classical modular form of weight $k/2$, level $4N\ell^s$, and Nebentypus χ. The character χ factors as $\chi = \chi_\ell \chi_{\mathrm{tm}}$ for a character χ_ℓ modulo ℓ^s and a character χ_{tm} modulo $4N$, and the ℓ-adic form that F gives rise to has tame Nebentypus χ_{tm}.

A.2. Proof of Theorem A.1

First note that the "N" below is not the N in the statement of Theorem A.1, but rather its prime-to-ℓ part, so that we may apply to the framework of the previous subsection. Accordingly, χ_{tm} below is the tame part of the χ in the statement (while χ_ℓ is packaged as part of the weight κ).

Let K denote the completion of the number field L in the statement at the prime above ℓ determined by the chosen inclusion $L \hookrightarrow \overline{\mathbb{Q}}_\ell$. Then F gives rise to an element of the Banach space $M_\kappa(4N, K, \chi_{\mathrm{tm}}, r)$ for some $r < 1$. General facts about the spectral theory of compact operators imply that the map

$$e : F \mapsto \lim_{s \to \infty} U\left(\ell^2\right)^{s!} F$$

is defined on $M_\kappa(4N, K, \chi_{\mathrm{tm}}, r)$ (in that the limit exists) and is the projector onto the finite-dimensional ordinary subspace of this Banach space. Since $U(\ell^2)$ increases overconvergence, these projectors form a compatible family of projections onto the *same* finite-dimensional space of ordinary forms for varying r, and hence provide a projector

$$e : M_\kappa^\dagger(4N, K, \chi_{\mathrm{tm}}) \to M_\kappa^\dagger(4N, K, \chi_{\mathrm{tm}})^{\mathrm{ord}}$$

from the full space of overconvergent forms of this weight and level onto this space of ordinary forms.

We can extend the ordinary projector e to overconvergent weakly holomorphic modular forms as follows. As observed above, for any such form F, the form $U(\ell^2)^t F$ (which is of the same weight, tame level and Nebentypus) is a genuine modular form for sufficiently large t. We set

$$e(F) = U\left(\ell^2\right)^{-t} e U\left(\ell^2\right)^t (F).$$

First note that this makes sense as $U(\ell^2)$ is invertible on the space of ordinary forms. It is also independent of t (sufficiently large) as is easily checked. Note that, while this projector is a natural thing to consider, all we really need in the present discussion is that $eU(\ell^2)^t(F)$ lies in the finite-dimensional space of ordinary forms.

The upshot of all this is that, for this fixed weight κ, tame level $4N$, and tame Nebentypus χ_{tm} there is a finite-dimensional vector space of modular forms, namely $M_\kappa^\dagger(4N, K, \chi_{\mathrm{tm}})^{\mathrm{ord}}$, with the following property: For any modular form F of this weight, level, and tame Nebentypus with coefficients in L, perhaps with poles at the cusps, there exists a positive an integer t such that

$$\lim_{k \to \infty} U^{k!+t}\left(\ell^2\right)(F) \in M_\kappa^\dagger(4N, K, \chi_{\mathrm{tm}})^{\mathrm{ord}}.$$

Let $\mathbb{Q}_\ell(\chi_{\mathrm{tm}}, \kappa)$ be the extension of $\mathbb{Q}_\ell$ generated by the values of χ_{tm} and κ, which is finite in the situation of Theorem A.1. Replacing K by its Galois closure over $\mathbb{Q}_\ell(\chi_{\mathrm{tm}}, \kappa)$ if necessary, the finite-dimensional space $M_\kappa^\dagger(4N, K, \chi_{\mathrm{tm}})^{\mathrm{ord}}$ carries an action of $\mathrm{Gal}(K/\mathbb{Q}_\ell(\chi_{\mathrm{tm}}, \kappa))$. By Galois descent, this space has a basis of Galois invariants, which is to say that it is generated over K by

$$M_\kappa^\dagger\left(4N, \mathbb{Q}(\chi_{\mathrm{tm}}, \kappa), \chi_{\mathrm{tm}}\right)^{\mathrm{ord}}.$$

Passing to q-expansions, we arrive at a finite-dimensional subspace of $\mathbb{Q}_\ell(\chi_{\mathrm{tm}}, \kappa)[\![q]\!]$, and the submodule of this space consisting of series with integral coefficients is the Ω that we seek.

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