ACTA ARITHMETICA
159.3 (2013)

Mock modular forms and singular combinatorial series

by

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1. Introduction and statement of results. Let $p(n) := \#\{\text{integer partitions of } n\}$, where a *partition* of $n \in \mathbb{N}$ is defined to be any non-increasing sequence of positive integers that sums to n. For example, the partitions of 4 are: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$, so that $p(4) = 5$. It is well known that the partition generating function satisfies

$$1 + \sum_{n\geq 1} p(n)q^n = q^{1/24}\eta^{-1}(\tau)$$

upon specializing $q = q_\tau := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ the upper complex half-plane, where $\eta(\tau) := q^{1/24}\prod_{n\geq 1}(1-q^n)$ is Dedekind's η-function, a weight 1/2 modular form. More recently, Bringmann and Ono [8] studied the generating function for partition *ranks*, where the rank of a partition, after Dyson, is defined to be the largest part of the partition minus the number of parts. For example, the rank of the partition $2+1+1$ is $2-3=-1$. If $N(m,n) := \#\{\text{partitions of } n \text{ with rank equal to } m\}$, it is well known that the associated two-variable generating function satisfies

$$1 + \sum_{m\in\mathbb{Z}}\sum_{n\geq 1} N(m,n)w^m q^n = 1 + \sum_{n\geq 1}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}, \tag{1.1}$$

where the q-Pochhammer symbol is defined for integers $n \geq 1$ by $(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$, and $(a;q)_0 := 1$. In particular,

$$R(1;q) = 1 + \sum_{n\geq 1} p(n)q^n = q^{1/24}\eta^{-1}(\tau), \tag{1.2}$$

$$R(-1;q) = 1 + \sum_{n\geq 1}\frac{q^{n^2}}{(-q;q)_n^2} = f(q), \tag{1.3}$$

where $f(q)$ is not a modular form, but one of Ramanujan's original third

2010 *Mathematics Subject Classification*: 11F37, 11P82, 05A17.
Key words and phrases: weak Maass forms, mock modular forms, non-holomorphic modular forms, integer partitions, partition ranks, generating functions.

DOI: 10.4064/aa159-3-4 © Instytut Matematyczny PAN, 2013

order mock theta functions. Two major questions in number theory surrounding $f(q)$ and the other Ramanujan mock theta functions, 17 peculiar q-series similar in shape to (1.3), persisted in the decades following Ramanujan's death in 1920:

1. How do Ramanujan's mock theta functions fit into the theory of modular forms?
2. Is there an *exact* formula for the Fourier coefficients of the mock theta function $f(q)$?

It was not until the groundbreaking 2002 thesis of Zwegers [26] that the answer to the first question was finally provided: Ramanujan's mock theta functions exhibit suitable modular transformation properties after they are *completed* by the addition of certain non-holomorphic functions. (See [19, 24, 26] for example, and §3 for more detail.) Unifying (1.2) and (1.3), Bringmann and Ono remarkably showed more generally in [8] that upon specialization of the parameter w to certain complex roots of unity, the rank generating function of (1.1) can be completed by the addition of a suitable non-holomorphic integral to exhibit appropriate modular transformation properties. In particular, they establish the following theorem.

Theorem ([8, Theorem 1.1]). *If* $0 < a < c$, *then*

$$q^{-\ell_c/24} R(\zeta_c^a; q^{\ell_c}) + \frac{i \sin(\pi a/c) \ell_c^{1/2}}{\sqrt{3}} \int_{-\overline{z}}^{i\infty} \frac{\Theta(a/c; \ell_c \tau)}{\sqrt{-i(\tau + z)}}\, d\tau$$

is a weak Maass form of weight $1/2$ *on* Γ_c.

Here, $\zeta_n := e^{2\pi i/n}$ is an nth root of unity, $\Theta(a/c; \ell_c\tau)$ is a certain weight $3/2$ cusp form, $\ell_c := \mathrm{lcm}(2c^2, 24)$, and Γ_c is a particular subgroup of $\mathrm{SL}_2(\mathbb{Z})$. *Weak Maass forms*, originally defined by Bruiner and Funke [10], are (non-holomorphic) generalizations of ordinary modular forms that in addition to satisfying appropriate modular transformations, must be eigenfunctions of a certain weight k Laplacian operator. The theory of weak Maass forms has been substantially developed in the wake of Zwegers's thesis in recent years. (See [19] for a detailed history, and §3 for more detail and explicit defintions.)

Turning to the second question above, Andrews and Dragonette [1, 12] established a detailed asymptotic formula for the Fourier coefficients of the $f(q)$ mock theta function that was in fact conjectured to give an exact formula for the coefficients. Namely, let $\alpha(n)$ denote the nth Fourier coefficient of $f(q)$, that is,

$$f(q) = 1 + \sum_{n \geq 1} \alpha(n) q^n = 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + \cdots .$$

Andrews and Dragonette conjectured for $n \geq 1$ that the coefficients $\alpha(n)$ are equal to

$$(1.4)\qquad \pi(24n-1)^{-1/4}\sum_{\ell\geq 1}\frac{(-1)^{\lfloor(\ell+1)/2\rfloor}A_{2\ell}(n-\ell(1+(-1)^{\ell})/4)}{\ell}\times I_{1/2}\left(\frac{\pi\sqrt{24n-1}}{12\ell}\right),$$

where $I_\alpha(x)$ denotes the usual I-Bessel function, and the Kloosterman sum $A_\ell(n)$ is defined in (6.24) below. A second celebrated result of Bringmann and Ono [7] establishes that indeed, the Andrews–Dragonette conjecture is true: (1.4) gives an exact formula for the Fourier coefficients $\alpha(n)$ of the mock theta function $f(q)$, thus answering the second major question above.

Here, we turn our attention to the problem of understanding the automorphic properties of certain combinatorial q-series arising from k-marked Durfee symbols, as originally defined by Andrews in [2]. To each partition, Andrews associates a *Durfee symbol*. For example, the Durfee symbol

$$\begin{pmatrix} 2 & \\ 2 & 1 \end{pmatrix}_4$$

represents the partition 5+5+4+4+2+1 of 21. Using k copies of the integers, Andrews more generally defines *k-marked Durfee symbols*. Analogous to the rank of a partition, Andrews defines a notion of rank for each of the k copies of the integers used to define the k-marked Durfee symbols. (See §2 for more detailed definitions and descriptions.) If $\mathcal{D}_k(m_1,\ldots,m_k;n)$ denotes the number of k-marked Durfee symbols arising from partitions of n with ith rank equal to m_i, Andrews [2] shows that the $k+1$-variable generating function may be expressed as follows.

THEOREM ([2, Theorem 10]). *For $k\geq 1$,*

$$(1.5)\qquad \sum_{n_1,\ldots,n_k=-\infty}^{\infty}\sum_{n\geq 0}\mathcal{D}_k(n_1,\ldots,n_k;n)x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}q^n = R_k(\boldsymbol{x};q),$$

where

$$(1.6)\qquad R_k(\boldsymbol{x};q) := \sum_{\substack{m_1\geq 0\\ m_2,\ldots,m_k\geq 0}}\frac{q^{(m_1+\cdots+m_k)^2+(m_1+\cdots+m_{k-1})+(m_1+\cdots+m_{k-2})+\cdots+m_1}}{(x_1q;q)_{m_1}(q/x_1;q)_{m_1}(x_2q^{m_1};q)_{m_2+1}(q^{m_1}/x_2;q)_{m_2+1}}$$
$$\times (x_3q^{m_1+m_2};q)_{m_3+1}^{-1}\left(\frac{q^{m_1+m_2}}{x_3};q\right)_{m_3+1}^{-1}\cdots(x_kq^{m_1+\cdots+m_{k-1}};q)_{m_k+1}^{-1}\left(\frac{q^{m_1+\cdots+m_{k-1}}}{x_k};q\right)_{m_k+1}^{-1}$$

and $\boldsymbol{x}:=(x_1,\ldots,x_k)$.

When $k = 1$, one recovers Dyson's rank, that is, $\mathcal{D}_1(n_1; n) = N(n_1, n)$. The modularity of the associated two-variable generating function $R_1(x; q) = R(x; q)$ was studied in [8] as described above. When $k = 2$, the modularity of $R_2(1, 1; q)$ was originally studied by Bringmann in [4], who shows that

$$R_2(1,1;q) := \frac{1}{(q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n-1} q^{3n(n+1)/2}}{(1-q^n)^2},$$

where $(a; q)_\infty := \prod_{j \geq 0}(1 - aq^j)$ and $(q)_\infty := (q; q)_\infty$, is a *quasimock theta function* (see §3). In [5], Bringmann, Garvan, and Mahlburg show more generally that $R_k(1, \ldots, 1; q)$ is a quasimock theta function for $k \geq 2$. (See [4] and [5] for precise details of these statements.)

Here, we establish the automorphic properties of general infinite familes of combinatorial q-series $R_k(x_1, \ldots, x_k; q)$, for more arbitrary parameters $(x_1, \ldots, x_k)$, thereby treating families of k-marked Durfee functions with additional singularities to those of $R_k(1, \ldots, 1; q)$. We point out that the techniques of Andrews [2] and Bringmann [4] are not directly applicable in our setting due to the presence of such additional singularities. We show that these singular combinatorial families are essentially mixed mock and quasimock modular forms, and we provide their explicit non-holomorphic completions. To this end, we define a non-holomorphic completion $\widehat{B}_k(\boldsymbol{\zeta}_k; q)$ of the combinatorial series $R_k(\boldsymbol{\zeta}_k; q)$ by

$$\widehat{B}_k(\boldsymbol{\zeta}_k; q) := q^{-1/24}(B_k^+(\boldsymbol{\zeta}_k; q) + B_k^-(\boldsymbol{\zeta}_k; q)). \tag{1.7}$$

Here, the "holomorphic part" $q^{-1/24}B_k^+(\boldsymbol{\zeta}_k; q)$ of $\widehat{B}_k(\boldsymbol{\zeta}_k; q)$ is defined by

$$B_k^+(\boldsymbol{\zeta}_k; q) := R_k(\boldsymbol{\zeta}_k; q) + b_k(\boldsymbol{\zeta}_k; q), \tag{1.8}$$

where the combinatorial series $R_k(\boldsymbol{\zeta}_k; q)$ is defined in (1.6), and the holomorphic function $b_k(\boldsymbol{\zeta}_k; q)$ is defined in (4.4) below. The "non-holomorphic part" $q^{-1/24}B_k^-(\boldsymbol{\zeta}_k; q)$ of $\widehat{B}_k(\boldsymbol{\zeta}_k; q)$ is defined in (4.3). Analogous to Theorem 1.1 of [8] stated above, we establish the following theorem.

THEOREM 1.1. *If $k \geq 2$ is an integer, then*

$$\widehat{B}_k(\boldsymbol{\zeta}_k; q) = \widehat{\mathcal{H}}(\boldsymbol{\zeta}_k; q) + \widehat{\mathcal{A}}(\boldsymbol{\zeta}_k; q),$$

where $\widehat{\mathcal{H}}(\boldsymbol{\zeta}_k; q)$, defined in (4.25), is a non-holomorphic modular form of weight $3/2$ on $\Gamma_{k,N}$ with character χ_γ^{-1}, and $\widehat{\mathcal{A}}(\boldsymbol{\zeta}_k; q)$, defined in (4.26), is a non-holomorphic modular form of weight $1/2$ on $\Gamma_{k,N}$ with character χ_γ^{-1}.

Here, $\boldsymbol{\zeta}_k = \boldsymbol{\zeta}_{k,N}$ is a length k vector consisting of roots of unity defined in (4.21), the subgroup $\Gamma_{k,N} \subseteq \mathrm{SL}_2(\mathbb{Z})$ under which $\widehat{B}_k(\boldsymbol{\zeta}_k; q)$ transforms is defined in (4.28), and Nebentypus character χ_γ^{-1} is given in Lemma 3.2.

REMARK 1. Loosely speaking, Zagier has recently defined a *mixed mock modular form* [25] to be a finite sum of products of mock modular forms and modular forms. Other recent works in which this notion of a mixed mock modular form appears include [6], [11], [14], and [21], for example. Here, the holomorphic parts B_k^+ of $\widehat{B}_k$, which are defined in (1.8) in terms of R_k, consist of linear combinations of mixed mock modular forms, and also terms consisting of derivatives $\frac{d}{du}\phi(u,\tau)\big|_{u=0}$ of mock Jacobi forms $\phi(u,\tau)$ in the Jacobi u variable evaluated at $u=0$, multiplied by modular forms. See §4 for explicit details, and the modular transformations of the associated forms.

Remark 2. We point out that the dependence of the vector $\boldsymbol{\zeta}_k$ on k is reflected only in the length of the vector, and not (necessarily) in the roots of unity that are its components. In particular, the vector components may be chosen to be nth roots of unity for different values of n (see (4.21), and Examples 1 and 2 below).

The techniques we use to prove Theorem 1.1 may be adapted appropriately to study other singular families. As an example, we consider a second infinite family of $k = 2^r+1$-marked Durfee symbols for integers $r \geq 1$, with $\boldsymbol{x} = \boldsymbol{\zeta}'_{2^r}$, a length 2^r+1 vector of roots of unity defined in (5.1), on a subgroup $\Gamma'_r \subseteq \mathrm{SL}_2(\mathbb{Z})$ defined in (5.14). To this end, we define a non-holomorphic completion $\widehat{C}_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24})$ of the combinatorial series $R_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24})$ by

$$\widehat{C}_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24}) := q^{-1}C^+_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24}) + C^-_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24}). \tag{1.9}$$

Here, the "holomorphic part" $q^{-1}C_k^+(\boldsymbol{\zeta}'_{2^r}; q^{24})$ of $\widehat{C}_k(\boldsymbol{\zeta}'_{2^r}; q^{24})$ is defined by

$$C^+_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24}) := R_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24}) + c_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24}), \tag{1.10}$$

where the combinatorial series $R_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24})$ is defined in (1.6), and the holomorphic function $c_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24})$ is defined in (5.3). The "non-holomorphic part" $C^-_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24})$ of the function $\widehat{C}_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24})$ is defined in (5.4).

Theorem 1.2. *If $r \geq 1$ is an integer, then $q^{-1}\widehat{C}_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q^{24})$ is a non-holomorphic modular form of weight $3/2$ on Γ'_r with character $(\frac{12}{\cdot})$.*

Remark 3. The function $\widehat{C}_{2^r+1}(\boldsymbol{\zeta}'_{2^r}; q)$ is of similar shape to the function $\widehat{B}_k(\boldsymbol{\zeta}_k; q)$ as described in Remark 1, yet also includes the quasimodular form $E_2(\tau)$ (defined in (3.1)). See §4 for explicit details.

We illustrate the diversity of the combinatorial series to which Theorems 1.1 and 1.2 apply in the following examples.

Example 1. We begin with a more colorful example, chosen to contrast with Example 2. We let $k = 7$, and choose $\boldsymbol{\zeta}_7 = (\zeta_{23}^4, \zeta_{23}^4, \zeta_5, \zeta_5, \zeta_{108}^{11}, \zeta_3, i)$. In this case, the combinatorial generating function $R_7(\boldsymbol{\zeta}_k; q)$ is of the form

$$\frac{1}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^{n-1}q^{(3n^2+11n)/2}(1-q^{2n})^2(1-q^n)^2(1-q^{3n})^{-1}(1-q^{4n})^{-1}}{\left(1-2\cos\left(\frac{8\pi}{23}\right)q^n+q^{2n}\right)^2\left(1-2\cos\left(\frac{2\pi}{5}\right)q^n+q^{2n}\right)^2\left(1-2\cos\left(\frac{11\pi}{54}\right)q^n+q^{2n}\right)}.$$

By Theorem 1.1, this function is essentially a sum of mixed mock modular forms. The completed non-holomorphic combinatorial function is given explicitly by (1.7).

Example 2. A perhaps tamer example to be contrasted with Example 1 is the combinatorial generating function

$$R_3(1,1,-1;q) = R_3(q) := \frac{1}{(q)_\infty} \sum_{n\neq 0} \frac{(-1)^{n-1} q^{3n(n+1)/2}}{(1-q^{2n})^2} =: \sum_{n=1}^{\infty} a_3(n) q^n, \tag{1.11}$$

which arises after specializing $r = 1$ in Theorem 1.2. If we let $N_{i,k}(r,t;n)$ denote the number of k-marked Durfee symbols of n with ith rank congruent to r modulo t, then (as explained in [2] and §2) the series $R_3(q)$ is the combinatorial generating function

$$R_3(q) = \sum_{n=1}^{\infty} N_{3,3}(0,2;n) q^n - \sum_{n=1}^{\infty} N_{3,3}(1,2;n) q^n,$$

so that

$$a_3(n) = N_{3,3}(0,2;n) - N_{3,3}(1,2;n). \tag{1.12}$$

We point out the similar combinatorial description of the coefficients $a_3(n)$ described in (1.12) and the coefficients $\alpha(n)$ of Ramanujan's mock theta function $f(q)$ discussed above, which may be expressed as

$$f(q) = 1 + \sum_{n=1}^{\infty} N(0,2;n) q^n - \sum_{n=1}^{\infty} N(1,2;n) q^n,$$

where $N(r,t;n) := N_{1,1}(r,t;n)$. Here, by Theorem 1.2, $R_3(q)$ is essentially a mixed quasimock modular form. The completed non-holomorphic combinatorial function is given by (1.9).

Next we address an open problem of Andrews.

Open Problem (Andrews [2, Problem 11]). *In light of Bringmann's asymptotics for the rank partition functions... and the recent breakthroughs by Bringmann and Ono on the exact formulas for the coefficients in the power series of the mock theta functions... provide similar results for* $\mathcal{D}_k(n)$...

Bringmann has addressed the special case pertaining to $R_2(1,1;q)$ in [4]. Here we address another special case of Andrews's open problem pertaining to $R_3(1,1,-1;q)$ as defined in (1.11), which arises after setting $r = 1$ in our family R_{2^r+1} of Theorem 1.2. We establish the following asymptotic result.

THEOREM 1.3. *Let $\varepsilon > 0$. Then for all $n \geq 1$,*

(1.13)
$$a_3(n) = \sum_{\ell=1}^{\lfloor n^{1/2} \rfloor} A_\ell(n) \left[\frac{\pi(24n-1)^{1/4}}{48\ell} I_{-1/2}\left(\frac{\pi}{6\ell}\sqrt{24n-1}\right) - \frac{3}{8(24n-1)^{1/4}} \times I_{1/2}\left(\frac{\pi}{6\ell}\sqrt{24n-1}\right) - \frac{5\pi}{48\ell(24n-1)^{3/4}} I_{3/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right)\right] - \sum_{\substack{\ell=1 \\ \ell \text{ even}}}^{\lfloor n^{1/2} \rfloor} A_\ell^e(n) \left[\frac{\pi}{8\ell(24n-1)^{1/4}} I_{1/2}\left(\frac{\pi}{6\ell}\sqrt{24n-1}\right)\right] + O(n^{1+\varepsilon}).$$

Here, $I_\alpha(x)$ denotes the usual I-Bessel function of order α, and the Kloosterman sums $A_\ell(n)$ and $A_\ell^e(n)$ are defined in (6.24) and (6.25) respectively. In particular, the first term in (1.13) gives the main term in the asymptotic expansion for $a_3(n)$.

COROLLARY 1.4. *Assuming the notation above, as $n \to \infty$, we have*
$$a_3(n) \sim \frac{\sqrt{3}}{48} e^{\frac{\pi}{6}\sqrt{24n-1}}.$$

The remainder of the paper is structured as follows. In §2, we provide preliminary results and definitions pertaining to k-marked Durfee symbols. In §3 we describe relevant automorphic objects and various associated properties. In §4 we prove Theorem 1.1, and in §5 we prove Theorem 1.2. In §6 we prove Theorem 1.3, and as a corollary (Corollary 6.4) provide modular transformation laws for a family of universal mock theta functions after Gordon-McIntosh [16].

2. k-marked Durfee symbols. Here we recall the definition of k-marked Durfee symbols as defined by Andrews in [2] and their connection to the functions $R_k(\boldsymbol{x}; q)$ as defined in (1.6). Recall that the *Durfee square* (named by Sylvester) is the largest square of nodes in the Ferrers graph of a partition (see [3]). Andrews associates to each partition a *Durfee symbol* where the top row consists of the columns to the right of the Durfee square, the bottom row consists of the rows below the Durfee square, and the subscript denotes the side length of the Durfee square. The number being partitioned is equal to the sum of the rows plus the size of the Durfee square. The example considered in §1,

$$\begin{pmatrix} 2 & \\ 2 & 1 \end{pmatrix}_4, \tag{2.1}$$

represents the partition $5+5+4+4+2+1$ of the number $2+2+1+4^2 = 21$.

To define the *k-marked Durfee symbol*, Andrews uses k copies of the integers ($k \in \mathbb{N}$), denoted by $\{1_1, 2_1, 3_1, \dots\}$, $\{1_2, 2_2, 3_2, \dots\}$, $\dots$, $\{1_k, 2_k, 3_k, \dots\}$ (see [2]). The k-marked Durfee symbols are formed as before, but use the k copies of the integers as the parts in both rows. The number being partitioned is equal to the sum of the rows plus the size of the Durfee square. In addition, the following restrictions are imposed:

(1) the sequence of subscripts in each row is non-increasing;
(2) each of the subscripts $1, \dots, k-1$ must occur at least once in the top row; and
(3) if $M_1, \dots, M_{k-1}$ are the largest parts with their respective subscripts in the top row, then all parts in the bottom row with subscript 1 lie in $[1, M_1]$, with subscript 2 lie in $[M_1, M_2]$, $\dots$, with subscript $k-1$ lie in $[M_{k-2}, M_{k-1}]$, and with subscript k lie in $[M_{k-1}, S]$, where S is the side of the Durfee square.

For example, three of the 133 3-marked Durfee symbols for 7 are:

$$\begin{pmatrix} 2_2 & 1_1 \\ & \end{pmatrix}_2, \qquad \begin{pmatrix} 1_3 & 1_2 & 1_1 \\ 1_3 & 1_3 & 1_1 \end{pmatrix}_1, \qquad \begin{pmatrix} 1_2 & 1_1 \\ 1_1 & \end{pmatrix}_2.$$

For partitions of n formed using one copy of the integers, Dyson's notion of the rank of a partition (i.e. the largest part minus the number of parts of the partition as defined in §1) can be obtained using the Durfee symbol. For example, the rank of the partition $5+5+4+4+2+1$ of 21 is $5-6=-1$. Alternatively, the rank of a partition can be obtained from the Durfee symbol as the number of entries in the top row minus the number of entries in the bottom row. Using the Durfee symbol for $5+5+4+4+2+1$ given in (2.1), we find that the rank is indeed $1-2=-1$.

Andrews associates a similar notion of rank to each of the k copies of the integers used to define the k-marked Durfee symbols. Suppose δ is a k-marked Durfee symbol, and denote by $\tau_i(\delta)$ (respectively $\beta_i(\delta)$) the number of entries in the top (resp. bottom) row of δ with subscript i. Then Andrews defines $\rho_i(\delta)$, the *ith rank* of δ, by

$$\rho_i(\delta) = \begin{cases} \tau_i(\delta) - \beta_i(\delta) - 1 & \text{for } 1 \le i < k, \\ \tau_i(\delta) - \beta_i(\delta) & \text{for } i = k. \end{cases}$$

Andrews notes that for $k=1$, $\rho_1(\delta)$ is Dyson's rank. Defining

$$\mathcal{D}_k(m_1, \dots, m_k; n)$$

to be the number of k-marked Durfee symbols arising from partitions of n with ith rank equal to m_i, Andrews [2] established that the $k+1$-variable generating function is given by (1.5).

3. Automorphic forms. In this section, we define relevant automorphic objects, and give various associated properties.

3.1. Weak Maass forms. Assume that $\kappa \in \frac{1}{2}\mathbb{Z}$, and Γ is a congruence subgroup of either $\mathrm{SL}_2(\mathbb{Z})$ or $\Gamma_0(4)$, depending on whether or not $\kappa \in \mathbb{Z}$. The weight κ *slash operator*, defined for a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and any function $f : \mathbb{H} \to \mathbb{C}$, is given by

$$f\big|_\kappa \gamma(\tau) := j(\gamma,\tau)^{-2\kappa} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

where

$$j(\gamma,\tau) := \begin{cases} \sqrt{c\tau+d} & \text{if } \kappa \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{c\tau+d} & \text{if } \kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Here

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

To define weak Maass forms, we also require the *weight κ hyperbolic Laplace operator* $(\tau = x + iy)$

$$\Delta_\kappa := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\kappa y\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

DEFINITION 3.1. Let $\kappa \in \frac{1}{2}\mathbb{Z}$, N a positive integer, χ a Dirichlet character modulo N, $\lambda \in \mathbb{C}$. A *weak Maass form of weight κ for Γ with Nebentypus character χ and Laplace eigenvalue λ* is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have $f\big|_\kappa \gamma(\tau) = \chi(d)f(\tau)$.
(2) $\Delta_\kappa f = \lambda f$.
(3) The function f has at most linear exponential growth at all cusps.

Harmonic weak Maass forms (originally defined by Bruinier–Funke [10]) are those weak Maass forms with eigenvalue $\lambda = 0$, and have been of particular interest. (In [10], one can find a more precise description of condition (3) in the definition above.) It is known that harmonic weak Maass forms naturally decompose into two parts: a *holomorphic part* and a *non-holomorphic part*. For example, it is known [17, 23] that the generating function for Hurwitz class numbers $H(n)$ of binary quadratic forms of discriminant $-n$ is (essentially) the holomorphic part of the following weak Maass form of weight 3/2 and level 4, the Zagier–Eisenstein series:

$$-\frac{1}{12} + \sum_{\substack{n\geq 1 \\ n\equiv 0,3 \pmod 4}} H(n)q^n + \frac{1+i}{16\pi}\int_{-\overline{\tau}}^{i\infty} \frac{\Theta(z)}{(z+\tau)^{3/2}}\,dz,$$

where $\Theta(\tau) := \sum_n q^{n^2}$. The holomorphic parts of harmonic weak Maass forms are called *mock modular forms* [24].

3.2. Holomorphic and almost holomorphic modular forms. A harmonic weak Maass form with trivial non-holomorphic part is a *weakly holomorphic modular form* (i.e. is holomorphic on $\mathbb{H}$, but may have poles in cusps). If the aforementioned modular forms are also holomorphic in the cusps, they are called *holomorphic modular forms.* A special ordinary modular form we require here is Dedekind's η-function, defined by

$$\eta(\tau) := q^{1/24} \prod_{n\geq 1} (1-q^n).$$

This function is well known to satisfy the following transformation law [20].

LEMMA 3.2. *For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$\eta(\gamma\tau) = \chi_\gamma (c\tau + d)^{1/2} \eta(\tau),$$

where

$$\chi_\gamma := \begin{cases} e^{\pi i b/12} & \text{if } c = 0, d = 1, \\ \sqrt{-i}\, \omega_{d,c}^{-1} e^{\pi i \frac{a+d}{12c}} & \text{if } c > 0, \end{cases}$$

with $\omega_{d,c} := e^{\pi i s(d,c)}$, and the Dedekind sum $s(m,t)$ is given for coprime integers m and t by

$$s(m,t) := \sum_{j \bmod t} \left(\left(\frac{j}{t}\right)\right) \left(\left(\frac{mj}{t}\right)\right),$$

where $((x)) := x - \lfloor x \rfloor - 1/2$ if $x \in \mathbb{R} \setminus \mathbb{Z}$, and $((x)) := 0$ if $x \in \mathbb{Z}$.

We also encounter *almost holomorphic modular forms*, which as originally defined by Kaneko–Zagier [18], transform like usual modular forms, but are polynomials in $1/y$, where $y = \mathrm{Im}(\tau)$, with holomorphic coefficients. Well known examples of almost holomorphic modular forms include derivatives of holomorphic modular forms, as well as the non-holomorphic Eisenstein series $\widehat{E}_2$, defined by $\widehat{E}_2(\tau) := E_2(\tau) - 3/\pi y$, with "holomorphic part" $E_2(\tau)$ given by

$$E_2(\tau) := 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n, \tag{3.1}$$

where $\sigma_1(n)$ is the sum of positive integer divisors of n. In general, the holomorphic part of an almost holomorphic modular form is called a *quasi-modular form.*

3.3. Holomorphic and mock Jacobi forms. In [26], Zwegers studied another type of automorphic object commonly referred to as a *mock Jacobi* *form*. Before describing these forms, we recall the definition of a holomorphic Jacobi form, after Eichler and Zagier [13].

DEFINITION 3.3. A *holomorphic Jacobi form of weight κ and index M* ($\kappa, M \in \mathbb{N}$) on a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ of finite index is a holomorphic function $\varphi(z;\tau) : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ which for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\lambda, \mu \in \mathbb{Z}$ satisfies

(1) $\varphi\left(\frac{z}{c\tau+d}; \gamma\tau\right) = (c\tau+d)^{\kappa} e^{\frac{2\pi i M c z^2}{c\tau+d}} \varphi(z;\tau)$,
(2) $\varphi(z+\lambda\tau+\mu;\tau) = e^{-2\pi i M(\lambda^2\tau+2\lambda z)}\varphi(z;\tau)$,
(3) $\varphi(z;\tau)$ has a Fourier development of the form $\sum_{n,r} c(n,r) q^n e^{2\pi i r z}$ with $c(n,r) = 0$ unless $n \geq r^2/4M$.

Jacobi forms with multipliers and of half-integral weight, meromorphic Jacobi forms, and weak Jacobi forms are defined similarly with suitable modifications made, and have been studied in [13] and [26], for example. A canonical example of a (weight 1/2) Jacobi form is Jacobi's theta function, defined by

$$\vartheta(z;\tau) = \vartheta(z) := \sum_{\nu \in 1/2+\mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu(z+1/2)}; \tag{3.2}$$

here and throughout, we may omit the dependence of various functions on the variable τ when the context is clear. In [27], Zwegers considers a family of "level ℓ" Appell–Lerch functions, $\ell \in \mathbb{N}$, extending work in [26] pertaining to the case $\ell = 1$. For $u, v \in \mathbb{C}$, $\tau \in \mathbb{H}$ these functions are defined by

$$A_\ell(u,v;\tau) := e^{\pi i \ell u} \sum_{n\in\mathbb{Z}} \frac{(-1)^{\ell n} q^{\ell n(n+1)/2} e^{2\pi i n v}}{1 - q^n e^{2\pi i u}}.$$

Alone, the $A_\ell(u,v;\tau)$ do not transform like Jacobi forms as in Definition 3.3. Zwegers completes the series $A_\ell(u,v;\tau)$ to non-holomorphic functions $\widehat{A}_\ell(u,v;\tau)$ defined by

$$\widehat{A}_\ell(u,v;\tau) :=$$
$$A_\ell(u,v;\tau) + \frac{i}{2}\sum_{j=0}^{\ell-1} e^{2\pi i j u}\vartheta\left(v + j\tau + \frac{\ell-1}{2}; \ell\tau\right) R\left(\ell u - v - j\tau - \frac{\ell-1}{2}; \ell\tau\right),$$

where $\vartheta(z;\tau)$ is as in (3.2), and

$$R(u;\tau) := \sum_{\nu\in 1/2+\mathbb{Z}} \left\{ \mathrm{sgn}(\nu) - E\left(\left(\nu + \frac{\mathrm{Im}(u)}{\mathrm{Im}(\tau)}\right)\sqrt{2\,\mathrm{Im}(\tau)}\right)\right\} \times (-1)^{\nu-1/2} q^{-\nu^2/2} e^{-2\pi i \nu u},$$

with $u \in \mathbb{C}, \tau \in \mathbb{H}$, and

$$E(z) := 2\int_0^z e^{-\pi u^2}\, du, \qquad z \in \mathbb{R}.$$

PROPOSITION (Zwegers [27]). *For all* $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ *and* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *the completed level* ℓ *Appell functions* $\widehat{A}_\ell$ *satisfy*

$$\begin{aligned}(3.3)\quad &\widehat{A}_\ell(u + n_1\tau + m_1, v + n_2\tau + m_2; \tau) \\ &\qquad = (-1)^{\ell(n_1+m_1)} e^{2\pi i(u(\ell n_1 - n_2) - v n_1)} q^{\ell n_1^2/2 - n_1 n_2} \widehat{A}_\ell(u, v; \tau),\end{aligned}$$

$$(3.4)\quad \widehat{A}_\ell\left(\frac{u}{c\tau + d}, \frac{v}{c\tau + d}; \gamma\tau\right) = (c\tau + d) e^{\pi i c(-\ell u^2 + 2uv)/(c\tau + d)} \widehat{A}_\ell(u, v; \tau).$$

4. Proof of Theorem 1.1. In §4.1, we define three auxiliary functions $\widehat{F}_{m,s}(z, \tau)$, $\widehat{G}_{m,s}(z, \tau)$ and $\widehat{H}_{m,s}(z; \tau)$, and establish their modular transformation properties. In §4.2, we relate the singular combinatorial generating functions $R_k(\boldsymbol{\zeta}_k; q)$ to the auxiliary functions. In §4.3, we prove Theorem 1.1.

4.1. Auxiliary functions I. Let $\boldsymbol{z} := (z_1, \ldots, z_k) \in \mathbb{R}^k$, for some fixed integer $k \geq 2$, and let N be a fixed integer satisfying $0 \leq N \leq \lfloor k/2 \rfloor$. For such fixed pairs (k, N) we define for each $1 \leq m \leq N$ and $N+1 \leq n \leq k-N$ respectively

$$\begin{aligned}(4.1)\quad \Pi_m(\boldsymbol{z}, w) := {}& (1 - e(-2z_m)) \\ &\times \prod_{\substack{j=1 \\ j \neq m}}^{N} (e(w + z_m) - e(z_j))^2 \left(1 - \frac{1}{e(w + z_m + z_j)}\right)^2 \\ &\times \prod_{\ell=N+1}^{k-N} (e(w + z_m) - e(z_\ell)) \left(1 - \frac{1}{e(w + z_m + z_\ell)}\right),\end{aligned}$$

$$\begin{aligned}(4.2)\quad \Pi_n^\dagger(\boldsymbol{z}) := {}& \prod_{j=1}^{N} (e(z_n) - e(z_j))^2 \left(1 - \frac{1}{e(z_n + z_j)}\right)^2 \\ &\times \prod_{\substack{\ell=N+1 \\ \ell \neq n}}^{k-N} (e(z_n) - e(z_\ell)) \left(1 - \frac{1}{e(z_n + z_\ell)}\right),\end{aligned}$$

where $w \in \mathbb{R}$, and here and throughout, $e(x) := e^{2\pi i x}$. (As usual, we take the empty product to equal 1.) Using the products defined in (4.1) and (4.2), we define, assuming the notation above, with $s \in \mathbb{R}$, the following limiting differences of higher Appell functions, and their corresponding “non-holomorphic parts", respectively, by

$$F^+_{m,s}(\boldsymbol{z};\tau) := \lim_{w\to 0} \frac{e(-z_m)}{e(w)-e(-w)} \times \left(e^{s\pi i w}\frac{A_3(-w+z_m,-2\tau;\tau)}{\Pi_m(\boldsymbol{z},-w)} - e^{-s\pi i w}\frac{A_3(w+z_m,-2\tau;\tau)}{\Pi_m(\boldsymbol{z},w)}\right),$$

$$F^-_{m,s}(\boldsymbol{z};\tau) := \lim_{w\to 0} \frac{e(-z_m)}{e(w)-e(-w)} \times \left(e^{s\pi i w}\frac{\mathcal{R}_3(-w+z_m,-2\tau;\tau)}{\Pi_m(\boldsymbol{z},-w)} - e^{-s\pi i w}\frac{\mathcal{R}_3(w+z_m,-2\tau;\tau)}{\Pi_m(\boldsymbol{z},w)}\right),$$

where

$$\mathcal{R}_3(u,v;\tau) := \frac{i}{2}\sum_{j=0}^{2} e(ju)\vartheta(v+j\tau+1;3\tau)R(3u-v-j\tau-1;3\tau).$$

Using $F^-_{m,s}(\boldsymbol{z};\tau)$, $\mathcal{R}_3(u,v;\tau)$, $\Pi_m(\boldsymbol{z};w)$ and $\Pi^\dagger_n(\boldsymbol{z})$, we also define the "non-holomorphic" function

$$(4.3)\qquad B^-_k(\boldsymbol{\zeta}_k;q) := \frac{1}{(q)_\infty}\sum_{i=1}^{N}(\zeta_{2\beta_i}^{-\alpha_i}F^-_{i,1}(\boldsymbol{\alpha}_k,w;\tau) - \zeta_{2\beta_i}^{-3\alpha_i}F^-_{i,3}(\boldsymbol{\alpha}_k,w;\tau))$$
$$+\frac{1}{(q)_\infty}\sum_{i=N+1}^{k-N}(\zeta_{2\beta_i}^{-3\alpha_i}-\zeta_{2\beta_i}^{-\alpha_i})\frac{\mathcal{R}_3(\alpha_i/\beta_i,-2\tau;\tau)}{\Pi^\dagger_i(\boldsymbol{\alpha}_k)}$$
$$+\frac{1}{(q)_\infty}\sum_{j=1}^{N}\zeta_{2\beta_j}^{-\alpha_j}\frac{\zeta_{\beta_j}^{-\alpha_j}}{2}\left(\frac{3}{\Pi_j(\boldsymbol{\alpha}_k,0)}+\frac{\frac{d}{dw}\Pi_j(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{\pi i(\Pi_j(\boldsymbol{\alpha}_k,0))^2}\right)\mathcal{R}_3\left(\frac{\alpha_j}{\beta_j},-2\tau;\tau\right)$$
$$-\frac{1}{(q)_\infty}\sum_{j=1}^{N}\zeta_{2\beta_j}^{-3\alpha_j}\frac{\zeta_{\beta_j}^{-\alpha_j}}{2}\left(\frac{1}{\Pi_j(\boldsymbol{\alpha}_k,0)}+\frac{\frac{d}{dw}\Pi_j(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{\pi i(\Pi_j(\boldsymbol{\alpha}_k,0))^2}\right)\mathcal{R}_3\left(\frac{\alpha_j}{\beta_j},-2\tau;\tau\right),$$

and using $A_3(u,v;\tau)$ also the holomorphic function

$$(4.4)\qquad b_k(\boldsymbol{\zeta}_k;q) :=$$
$$\frac{1}{(q)_\infty}\sum_{j=1}^{N}\zeta_{2\beta_j}^{-\alpha_j}\frac{\zeta_{\beta_j}^{-\alpha_j}}{2}\left(\frac{3}{\Pi_j(\boldsymbol{\alpha}_k,0)}+\frac{\frac{d}{dw}\Pi_j(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{\pi i(\Pi_j(\boldsymbol{\alpha}_k,0))^2}\right)A_3\left(\frac{\alpha_j}{\beta_j},-2\tau;\tau\right)$$
$$-\frac{1}{(q)_\infty}\sum_{j=1}^{N}\zeta_{2\beta_j}^{-3\alpha_j}\frac{\zeta_{\beta_j}^{-\alpha_j}}{2}\left(\frac{1}{\Pi_j(\boldsymbol{\alpha}_k,0)}+\frac{\frac{d}{dw}\Pi_j(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{\pi i(\Pi_j(\boldsymbol{\alpha}_k,0))^2}\right)A_3\left(\frac{\alpha_j}{\beta_j},-2\tau;\tau\right),$$

where the vectors $\boldsymbol{\alpha}_k$ are defined in (4.20). We will show in §4.2 and §4.3 how (4.3) and (4.4) may indeed be used to "complete" the combinatorial function $R_k(\boldsymbol{\zeta}_k; q)$. Before doing so, using $F^+_{m,s}$, $F^-_{m,s}$, and $\widehat{A}_3$, we define

$$\begin{aligned}
\widehat{F}_{m,s}(\boldsymbol{z};\tau) &:= F^+_{m,s}(\boldsymbol{z};\tau) + F^-_{m,s}(\boldsymbol{z};\tau),\\
\widehat{G}_{m,s}(\boldsymbol{\alpha}_k;\tau) &:= \frac{\zeta_{\beta_m}^{-\alpha_m}}{2}\left(\frac{4-s}{\Pi_m(\boldsymbol{\alpha}_k,0)} + \frac{\frac{d}{dw}\Pi_m(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{\pi i(\Pi_m(\boldsymbol{\alpha}_k,0))^2}\right)\\
&\qquad \times \widehat{A}_3\left(\frac{\alpha_m}{\beta_m}, -2\tau;\tau\right),
\end{aligned} \tag{4.5}$$

$$\widehat{H}_{m,s}(\boldsymbol{\alpha}_k;\tau) := \widehat{F}_{m,s}(\boldsymbol{\alpha}_k;\tau) + \widehat{G}_{m,s}(\boldsymbol{\alpha}_k;\tau). \tag{4.6}$$

We establish the following proposition which gives the modular transformation properties of the non-holomorphic functions $\widehat{F}_{m,s}, \widehat{G}_{m,s}$, and $\widehat{H}_{m,s}$, for a fixed integer m, $1 \le m \le N$.

PROPOSITION 4.1. *Let* $\gamma := \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2\beta_m^2) \cap \Gamma_1(2\beta_m)$. *We have*

$$\begin{aligned}
\widehat{F}_{m,s}(\boldsymbol{\alpha}_k;\gamma\tau) &= (c\tau+d)^2\widehat{F}_{m,s}(\boldsymbol{\alpha}_k;\tau)\\
&\qquad + ((c\tau+d)^2 - (c\tau+d))\widehat{G}_{m,s}(\boldsymbol{\alpha}_k;\tau),
\end{aligned} \tag{4.7}$$

$$\widehat{G}_{m,s}(\boldsymbol{\alpha}_k;\gamma\tau) = (c\tau+d)\widehat{G}_{m,s}(\boldsymbol{\alpha}_k;\tau), \tag{4.8}$$

$$\widehat{H}_{m,s}(\boldsymbol{\alpha}_k;\gamma\tau) = (c\tau+d)^2\widehat{H}_{m,s}(\boldsymbol{\alpha}_k;\tau). \tag{4.9}$$

Proof. We first prove (4.7). For ease of notation, we let $\alpha = \alpha_m$ and $\beta = \beta_m$, and define the supplementary functions

$$\begin{aligned}
\epsilon^{\pm}_{\alpha/\beta}(w) = \epsilon^{\pm}_{\alpha/\beta}(w;\tau,\gamma) &:= e^{\pi i c(-3(c\tau+d)(\alpha/\beta\pm w)^2 - 4(a\tau+b)(\alpha/\beta\pm w))},\\
\xi^{\pm}_{\alpha/\beta}(w) = \xi^{\pm}_{\alpha/\beta}(w;\tau,\gamma) &:= e^{-2\pi i(\alpha/\beta\pm w)(c\tau+d)(-2a+2)},\\
\psi^{\pm}_{\alpha/\beta}(w) = \psi^{\pm}_{\alpha/\beta}(w;\tau,\gamma) &:= e^{3\pi i(\frac{\alpha}{\beta}(c+d-1))} e^{\frac{6\pi i c\alpha}{\beta}(\alpha/\beta\pm w(c\tau+d))} q^{\frac{3c^2\alpha^2}{2\beta^2}+\frac{2c\alpha}{\beta}}.
\end{aligned}$$

We compute, using (3.3) and (3.4), that

$$\begin{aligned}
&\widehat{A}_3(\alpha/\beta \pm w, -2\gamma\tau;\gamma\tau)\\
&\qquad = (c\tau+d)\epsilon^{\pm}_{\alpha/\beta}(w)\xi^{\pm}_{\alpha/\beta}(w)\psi^{\pm}_{\alpha/\beta}(w)\widehat{A}_3(\alpha/\beta \pm w(c\tau+d), -2\tau;\tau),
\end{aligned} \tag{4.10}$$

where we have imposed the stated hypotheses on γ. Using (4.5) and (4.10), we rewrite

$$\widehat{F}_{m,s}(\boldsymbol{\alpha}_k;\gamma\tau) = (c\tau+d)(f^+_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau) + f^-_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau)), \tag{4.11}$$

where

$$f^+_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau) := \lim_{w\to 0}\frac{e(-\alpha/\beta)}{e(w)-e(-w)}$$
$$\times\Bigg(e^{s\pi iw}\frac{A_3(\alpha/\beta - w(c\tau+d), -2\tau;\tau)}{\Pi_m(\boldsymbol{\alpha}_k,-w)}\epsilon^-_{\alpha/\beta}(w)\xi^-_{\alpha/\beta}(w)\psi^-_{\alpha/\beta}(w)$$
$$- e^{-s\pi iw}\frac{A_3(\alpha/\beta + w(c\tau+d), -2\tau;\tau)}{\Pi_m(\boldsymbol{\alpha}_k,w)}\epsilon^+_{\alpha/\beta}(w)\xi^+_{\alpha/\beta}(w)\psi^+_{\alpha/\beta}(w)\Bigg),$$
$$f^-_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau) := \lim_{w\to 0}\frac{e(-\alpha/\beta)}{e(w)-e(-w)}$$
$$\times\Bigg(e^{s\pi iw}\frac{\mathcal{R}_3(\alpha/\beta - w(c\tau+d), -2\tau;\tau)}{\Pi_m(\boldsymbol{\alpha}_k,-w)}\epsilon^-_{\alpha/\beta}(w)\xi^-_{\alpha/\beta}(w)\psi^-_{\alpha/\beta}(w)$$
$$- e^{-s\pi iw}\frac{\mathcal{R}_3(\alpha/\beta + w(c\tau+d), -2\tau;\tau)}{\Pi_m(\boldsymbol{\alpha}_k,w)}\epsilon^+_{\alpha/\beta}(w)\xi^+_{\alpha/\beta}(w)\psi^+_{\alpha/\beta}(w)\Bigg).$$

We first consider $f^+_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau)$. Before rewriting this function, we define additional supplementary functions

$$h(w) = h_{s,n}(\alpha_m/\beta_m, w;\gamma,\tau) := \frac{e^{s\pi iw+3\pi i(-w+\alpha_m/\beta_m)}}{1-e(-w+\alpha_m/\beta_m)q^n},$$
$$\widetilde{h}(w) = \widetilde{h}_{s,n}(\alpha_m/\beta_m, w;\gamma,\tau) := h(w(c\tau+d))g(w),$$

where

$$g(w) = g_s(\alpha_m/\beta_m, w;\gamma,\tau)$$
$$:= e^{-s\pi iw(c\tau+d)+s\pi iw}\epsilon^-_{\alpha_m/\beta_m}(w)\xi^-_{\alpha_m/\beta_m}(w)\psi^-_{\alpha_m/\beta_m}(w).$$

In what follows, we will use the following formulas, which are not difficult to show:

$$\text{(4.12)}\qquad \begin{aligned}\frac{d}{dw}h(w)\bigg|_{w=0} &= -\frac{2\pi i e^{3\pi i\alpha/\beta}}{(1-\zeta^\alpha_\beta q^n)^2} + \frac{\pi i e^{3\pi i\alpha/\beta}(s-1)}{1-\zeta^\alpha_\beta q^n},\\ h(0) &= \frac{e^{3\pi i\alpha/\beta}}{1-\zeta^\alpha_\beta q^n},\end{aligned}$$

where again for ease of notation we have replaced α_m/β_m by α/β. After two lengthy yet straightforward calculations, using the hypotheses on γ, we find that

$$\text{(4.13)}\qquad \frac{d}{dw}\widetilde{h}(w)\bigg|_{w=0} = -\frac{2\pi i e^{3\pi i\alpha/\beta}(c\tau+d)}{(1-\zeta^\alpha_\beta q^n)^2} + \frac{\pi i e^{3\pi i\alpha/\beta}(3(c\tau+d)+(s-4))}{1-\zeta^\alpha_\beta q^n},$$

$$\widetilde{h}(0) = \frac{e^{3\pi i\alpha/\beta}}{1-\zeta_\beta^\alpha q^n}. \tag{4.14}$$

Using (4.12)–(4.14), and L'Hôpital's rule, we find that

$$\begin{aligned}
&\lim_{w\to 0} \frac{1}{e(w)-e(-w)}\Bigg(\frac{e^{s\pi i w}e^{-3\pi i w(c\tau+d)}\epsilon^-_{\alpha/\beta}(w)\xi^-_{\alpha/\beta}(w)\psi^-_{\alpha/\beta}(w)}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,-w)(1-\zeta_\beta^\alpha e(-w(c\tau+d))q^n)} \\
&\qquad - \frac{e^{-s\pi i w}e^{3\pi i w(c\tau+d)}\epsilon^+_{\alpha/\beta}(w)\xi^+_{\alpha/\beta}(w)\psi^+_{\alpha/\beta}(w)}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)(1-\zeta_\beta^\alpha e(w(c\tau+d))q^n)}\Bigg) \\
&= \frac{e^{3\pi i\alpha/\beta}}{(\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0))^2}\Bigg(\frac{-(c\tau+d)}{(1-\zeta_\beta^\alpha q^n)^2} + \frac{3(c\tau+d)+(s-4)}{2(1-\zeta_\beta^\alpha q^n)} \\
&\qquad - \frac{d}{dw}\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)\Big|_{w=0}\frac{e^{3\pi i\alpha/\beta}}{2\pi i(1-\zeta_\beta^\alpha q^n)}\Bigg).
\end{aligned} \tag{4.15}$$

We point out that the hypotheses on α,β guarantee that $1-\zeta_\beta^\alpha q^n \neq 0$ for all $n\in\mathbb{Z}$. Using (4.15) we find that

$$\begin{aligned}
f^+_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau) &= -\frac{(c\tau+d)\zeta_{2\beta_m}^{\alpha_m}}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)}\sum_n \frac{(-1)^n q^{(3n^2+n)/2}}{(1-\zeta_{\beta_m}^{\alpha_m}q^n)^2} \\
&+\Bigg(\frac{\zeta_{\beta_m}^{-\alpha_m}(3(c\tau+d)+(s-4))}{2\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)} - \frac{\zeta_{\beta_m}^{-\alpha_m}\frac{d}{dw}\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{2\pi i(\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0))^2}\Bigg)A_3\Big(\frac{\alpha_m}{\beta_m},-2\tau;\tau\Big).
\end{aligned}$$

In a similar manner, we find that

$$\begin{aligned}
F^+_{m,s}(\boldsymbol{\alpha}_k;\tau) &= \frac{-\zeta_{2\beta_m}^{\alpha_m}}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)}\sum_n \frac{(-1)^n q^{(3n^2+n)/2}}{(1-\zeta_{\beta_m}^{\alpha_m}q^n)^2} \\
&\quad + \Bigg(\frac{\zeta_{\beta_m}^{-\alpha_m}(s-1)}{2\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)} - \frac{\zeta_{\beta_m}^{-\alpha_m}\frac{d}{dw}\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{2\pi i(\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0))^2}\Bigg)A_3\Big(\frac{\alpha_m}{\beta_m},-2\tau;\tau\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&f^+_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau) = (c\tau+d)F^+_{m,s}(\boldsymbol{\alpha}_k;\tau) \\
&+\frac{\zeta_{\beta_m}^{-\alpha_m}}{2}(c\tau+d-1)\Bigg(\frac{4-s}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)} + \frac{\frac{d}{dw}\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{\pi i(\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0))^2}\Bigg)A_3\Big(\frac{\alpha_m}{\beta_m},-2\tau;\tau\Big).
\end{aligned} \tag{4.16}$$

Next we treat $f^-_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau)$. For ease of notation, we will write $\mathcal{R}_3(u,v;\tau) = \mathcal{R}_3(u)$. We compute that

$$\begin{aligned}(4.17)\quad &\lim_{w\to 0}\frac{\zeta_{\beta_m}^{-\alpha_m}}{e(w)-e(-w)}\\ &\times\Bigg(e^{s\pi i w}\frac{\mathcal{R}_3(\alpha/\beta - w(c\tau+d))}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,-w)}\epsilon^-_{\alpha/\beta}(w)\xi^-_{\alpha/\beta}(w)\psi^-_{\alpha/\beta}(w)\\ &\quad - e^{-s\pi i w}\frac{\mathcal{R}_3(\alpha/\beta + w(c\tau+d))}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)}\epsilon^+_{\alpha/\beta}(w)\xi^+_{\alpha/\beta}(w)\psi^+_{\alpha/\beta}(w)\Bigg)\\ &= \zeta_\beta^{-\alpha}\left(\frac{s-4+4(c\tau+d)}{2\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)} - \frac{\frac{d}{dw}\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{2\pi i(\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0))^2}\right)\mathcal{R}_3\left(\frac{\alpha}{\beta}\right)\\ &\quad - \frac{\zeta_\beta^{-\alpha}(c\tau+d)}{2\pi i\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)}\frac{d}{du}\mathcal{R}_3(u)\Big|_{u=\alpha/\beta}\end{aligned}$$

where as before, (4.17) follows after a lengthy but straightforward calculation using the hypotheses on γ and L'Hôpital's rule. Similarly, we find that

$$\begin{aligned}(4.18)\quad F^-_{m,s}(\boldsymbol{\alpha}_k;\tau) &= \zeta_\beta^{-\alpha}\left(\frac{s}{2\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)} - \frac{\frac{d}{dw}\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{2\pi i(\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0))^2}\right)\mathcal{R}_3\left(\frac{\alpha}{\beta}\right)\\ &\quad - \frac{\zeta_\beta^{-\alpha}}{2\pi i\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)}\frac{d}{du}\mathcal{R}_3(u)\Big|_{u=\alpha/\beta.}\end{aligned}$$

Thus we have

$$\begin{aligned}(4.19)\quad f^-_{m,s}(\boldsymbol{\alpha}_k;\gamma,\tau) &= (c\tau+d)F^-_{m,s}(\boldsymbol{\alpha}_k;\tau)\\ &\quad + \frac{\zeta_\beta^{-\alpha}}{2}(c\tau+d-1)\left(\frac{4-s}{\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0)} + \frac{\frac{d}{dw}\mathit{\Pi}_m(\boldsymbol{\alpha}_k,w)\big|_{w=0}}{\pi i(\mathit{\Pi}_m(\boldsymbol{\alpha}_k,0))^2}\right)\mathcal{R}_3\left(\frac{\alpha}{\beta}\right).\end{aligned}$$

Using (4.11), (4.16), and (4.19) we find (4.7) as claimed. The transformation of (4.8) follows after a short calculation using (3.3) and (3.4), making use of the hypotheses imposed on the matrices γ. Finally, by way of (4.7), (4.8) and definition (4.6), we also have (4.9) as claimed. ∎

4.2. The singular series $R_k(\boldsymbol{\zeta}_k;q)$. In light of Proposition 4.1, in order to prove Theorem 1.1, we would like to relate the combinatorial series $R_k(\boldsymbol{\zeta}_k;q)$ to the auxiliary functions defined in §4.1. To this end, we establish the following proposition.

PROPOSITION 4.2. *Let k and N be fixed integers satisfying $0 \le N \le \lfloor k/2\rfloor$, and $k \ge 2$. Suppose that for $1 \le i \le k-N$, $\alpha_i \in \mathbb{Z}$ and $\beta_i \in \mathbb{N}$, where $\beta_i \nmid \alpha_i$, $\beta_i \nmid 2\alpha_i$, and that $\alpha_i/\beta_i \pm \alpha_j/\beta_j \notin \mathbb{Z}$ if $1 \le i \ne j \le k-N$. Let*

$$(4.20)\quad \boldsymbol{\alpha}_k = \boldsymbol{\alpha}_{k,N} := \Big(\underbrace{\frac{\alpha_1}{\beta_1}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_N}{\beta_N}, \frac{\alpha_N}{\beta_N}}_{2N}, \underbrace{\frac{\alpha_{N+1}}{\beta_{N+1}}, , \ldots, \frac{\alpha_{k-N}}{\beta_{k-N}}}_{k-2N}\Big) \in \mathbb{Q}^k,$$

$$(4.21)\quad \boldsymbol{\zeta}_k = \boldsymbol{\zeta}_{k,N} := \big(\underbrace{\zeta_{\beta_1}^{\alpha_1}, \zeta_{\beta_1}^{\alpha_1}, \ldots, \zeta_{\beta_N}^{\alpha_N}, \zeta_{\beta_N}^{\alpha_N}}_{2N}, \underbrace{\zeta_{\beta_{N+1}}^{\alpha_{N+1}}, \ldots, \zeta_{\beta_{k-N}}^{\alpha_{k-N}}}_{k-2N}\big) \in \mathbb{C}^k.$$

Then

$$(q)_\infty R_k(\boldsymbol{\zeta}_k; q) = \sum_{i=1}^{N} (\zeta_{2\beta_i}^{-\alpha_i} F_{i,1}^{+}(\boldsymbol{\alpha}_k; \tau) - \zeta_{2\beta_i}^{-3\alpha_i} F_{i,3}^{+}(\boldsymbol{\alpha}_k; \tau)) + \sum_{i=N+1}^{k-N} (\zeta_{2\beta_i}^{-3\alpha_i} - \zeta_{2\beta_i}^{-\alpha_i}) \frac{A_3(\alpha_i/\beta_i, -2\tau; \tau)}{\Pi_i^{\dagger}(\boldsymbol{\alpha}_k)}.$$

Remark 4. If $N = 0$, as usual we take the empty sum $\sum_{i=1}^{N}$ to equal 0.

Ideally, we would like to proceed as in the proof of Theorem 7 in [2], where Andrews makes use of the partial fraction identity

$$(4.22)\quad \frac{(1+w)(1-w)^2 w^{n-2}}{\prod_{j=1}^{n} (1 - x_j w)\big(1 - \frac{w}{x_j}\big)} = \sum_{s=1}^{n} \frac{x_s - 1}{\prod_{\substack{j=1 \\ j \neq s}}^{n} (x_s - x_j)\big(1 - \frac{1}{x_s x_j}\big)} \left(\frac{1}{1 - x_s w} - \frac{x_s^{-1}}{1 - \frac{w}{x_s}}\right).$$

Unfortunately (as remarked in [2]) identity (4.22) is only valid provided $x_s \neq x_t^{\pm 1}$ for any $1 \le s \neq t \le n$ (and of course also for $w \neq x_s^{\pm 1}$ for any $1 \le s \le n$). Here, given the definition of $R_k(x_1, \ldots, x_k; q)$, the vector $\boldsymbol{\zeta}_k$, and the hypotheses given in Proposition 4.2, we wish to take the x_s among the $\zeta_{\beta_i}^{\alpha_i}$, in particular, $x_1 = x_2 = \zeta_{\beta_1}^{\alpha_1}$, $x_3 = x_4 = \zeta_{\beta_2}^{\alpha_2}, \ldots, x_{2N-1} = x_{2N} = \zeta_{\beta_N}^{\alpha_N}$, which means we are not in a position to make use of (4.22). Instead, we establish the following generalization of (4.22) which allows for some of the $x_s = x_t$ by way of the introduction of an extra parameter u that tends to 1.

Proposition 4.3. *Let k and N be fixed integers satisfying $0 \le N \le \lfloor k/2 \rfloor$, and $k \ge 2$. Suppose for $1 \le i \le k - N$, $x_i \neq \pm 1$, and that $x_i \neq x_j^{\pm 1}$ if $1 \le i \neq j \le k - N$. Suppose also that $w \neq x_j^{\pm 1}$ for any $1 \le j \le k - N$. Then*

$$(4.23)\quad \frac{(1+w)(1-w)^2 w^{k-2}}{\prod_{j=1}^{N} (1 - x_j w)^2 \big(1 - \frac{w}{x_j}\big)^2 \prod_{\ell=N+1}^{k-N} (1 - x_\ell w)\big(1 - \frac{w}{x_\ell}\big)}$$

$$= \sum_{s=N+1}^{k-N} \frac{(x_s-1)\left(\frac{1}{1-x_sw}-\frac{1}{x_s-w}\right)}{\prod_{j=1}^{N}(x_s-x_j)^2\left(1-\frac{1}{x_sx_j}\right)^2 \prod_{\substack{\ell=N+1\\ \ell\neq s}}^{k-N}(x_s-x_\ell)\left(1-\frac{1}{x_sx_\ell}\right)} + \sum_{s=1}^{N}\lim_{u\to 1}\left(g_s(u)+g_s\left(\frac{1}{u}\right)\right),$$

where for $1 \le m \le N$,

$$g_m(u) = g_{m,k,N}(u,w,x_1,\ldots,x_{k-N}) := \frac{(ux_m-1)\left(\frac{1}{1-ux_mw}-\frac{1}{ux_m-w}\right)\left(ux_m-\frac{x_m}{u}\right)^{-1}\left(1-\frac{1}{x_m^2}\right)^{-1}}{\prod_{\substack{j=1\\ j\neq m}}^{N}(ux_m-x_j)^2\left(1-\frac{1}{ux_mx_j}\right)^2\prod_{\ell=N+1}^{k-N}(ux_m-x_\ell)\left(1-\frac{1}{ux_mx_\ell}\right)}.$$

Remark 5. If $N=0$ then we recover (4.22) from (4.23).

Proof of Proposition 4.3. For brevity we provide a detailed sketch. First we replace the product $\prod_{j=1}^{n}(1-x_jw)^2(1-w/x_j)^2$ appearing on the left hand side of (4.23) by

$$\prod_{j=1}^{N}(1-x_juw)\left(1-\frac{w}{x_ju}\right)(1-x_jw/u)\left(1-\frac{wu}{x_j}\right),$$

where u is a parameter that we will later let tend to 1. That is, for each j, $1\le j\le N$, we first rewrite $(1-x_jw)^2(1-w/x_j)^2 = p_1(x_j,w)p_2(x_j,w)$ where $p_1(x_j,w)=p_2(x_j,w)=(1-x_jw)(1-w/x_j)$. Then, we replace $p_1(x_j,w)$ by $p_1(x_ju,w)$, and $p_2(x_j,w)$ by $p_2(x_j/u,w)$, making the change of variable $x_j\mapsto x_ju$ in p_1, and the change of variable $x_j\mapsto x_j/u$ in p_2. With these changes of variables, assuming $u\neq\pm1$, we find that all k variables, $\{x_ju, x_j/u\}_{1\le j\le N}\cup\{x_j\}_{N+1\le j\le k-N}$ are now distinct. After an application of (4.22) and some simplification, we arrive at (4.23). ∎

Proof of Proposition 4.2. We begin with a result of Andrews, which gives the following series expansions for the generating function $R_k(\boldsymbol{x};q)$.

Theorem ([2, Theorem 3 and Corollary 6]). *For* $k\ge 2$,

$$R_k(\boldsymbol{x};q) = \frac{1}{(q)_\infty}\sum_{n\ge1}\frac{(-1)^{n-1}(1+q^n)(1-q^n)^2q^{3n(n-1)/2+kn}}{\prod_{j=1}^{k}(1-x_jq^n)(1-x_j^{-1}q^n)} = \frac{1}{2(q)_\infty}\sum_{n\neq0}\frac{(-1)^{n-1}(1+q^n)(1-q^n)^2q^{3n(n-1)/2+kn}}{\prod_{j=1}^{k}(1-x_jq^n)(1-x_j^{-1}q^n)}. \tag{4.24}$$

With $\boldsymbol{x} = \boldsymbol{\zeta}_k$, we use (4.24) followed by Proposition 4.3, and find after a short calculation

$$\begin{aligned} R_k(\boldsymbol{\zeta}_k; q) = & \lim_{w\to 0} \frac{1}{e(w) - e(-w)} \sum_{i=1}^{N} \left(\frac{R_1(\zeta_{\beta_i}^{\alpha_i} e(w))}{\zeta_{\beta_i}^{\alpha_i} \varPi_i(w)} - \frac{R_1(\zeta_{\beta_i}^{\alpha_i} e(-w))}{\zeta_{\beta_i}^{\alpha_i} \varPi_i(-w)} \right) \\ & + \sum_{i=N+1}^{k-N} \frac{R_1(\zeta_{\beta_i}^{\alpha_i}; q)}{\varPi_i^{\dagger}(\boldsymbol{\alpha}_k)} \\ = & \frac{1}{(q)_\infty} \sum_{i=1}^{N} (\zeta_{2\beta_i}^{-\alpha_i} F_{i,1}^{+}(\boldsymbol{\alpha}_k, w; \tau) - \zeta_{2\beta_i}^{-3\alpha_i} F_{i,3}^{+}(\boldsymbol{\alpha}_k, w; \tau)) \\ & + \frac{1}{(q)_\infty} \sum_{i=N+1}^{k-N} (\zeta_{2\beta_i}^{-3\alpha_i} - \zeta_{2\beta_i}^{-\alpha_i}) \frac{A_3(\alpha_i/\beta_i, -2\tau; \tau)}{\varPi_i^{\dagger}(\boldsymbol{\alpha}_k)}, \end{aligned}$$

where

$$R_1(x; q) := \frac{1-x}{(q)_\infty} \sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1 - xq^n}. \ \blacksquare$$

4.3. Proof of Theorem 1.1. We begin by defining

$$\begin{aligned} \widehat{\mathcal{H}}(\boldsymbol{\zeta}_k; q) &= \widehat{\mathcal{H}}_{k,N}(\boldsymbol{\zeta}_k; q) \\ &:= \frac{1}{\eta(\tau)} \sum_{i=1}^{N} (\zeta_{2\beta_i}^{-\alpha_i} \widehat{H}_{i,1}(\boldsymbol{\alpha}_k; \tau) - \zeta_{2\beta_i}^{-3\alpha_i} \widehat{H}_{i,3}(\boldsymbol{\alpha}_k; \tau)), \end{aligned} \tag{4.25}$$

$$\begin{aligned} \widehat{\mathcal{A}}(\boldsymbol{\zeta}_k; q) &= \widehat{\mathcal{A}}_{k,N}(\boldsymbol{\zeta}_k; q) \\ &:= \frac{1}{\eta(\tau)} \sum_{i=N+1}^{k-N} (\zeta_{2\beta_i}^{-3\alpha_i} - \zeta_{2\beta_i}^{-\alpha_i}) \frac{\widehat{A}_3(\alpha_i/\beta_i, -2\tau; \tau)}{\varPi_i^{\dagger}(\boldsymbol{\alpha}_k)}. \end{aligned} \tag{4.26}$$

From the definition of $\widehat{B}_k(\boldsymbol{\zeta}_k; q)$ given in (1.7), definition (4.3), definition (4.4), and Proposition 4.2, we find that

$$\widehat{B}_k(\boldsymbol{\zeta}_k; q) = \widehat{\mathcal{H}}(\boldsymbol{\zeta}_k; q) + \widehat{\mathcal{A}}(\boldsymbol{\zeta}_k; q). \tag{4.27}$$

Theorem 1.1 now follows after a short calculation using (4.27), Propositio 4.1, and transformation laws (3.3) and (3.4), using the fact that

$$\varGamma_{k,N} := \bigcap_{i=1}^{k-N} \varGamma_0(2\beta_i^2) \cap \varGamma_1(2\beta_i). \ \blacksquare \tag{4.28}$$

5. Proof of Theorem 1.2. The proof of Theorem 1.2 is similar, yet not identical, to that of Theorem 1.1 given the different natures of the associated combinatorial generating functions. In §5.1 we define auxiliary functions $\widehat{F}_r(\tau), \widehat{G}_r(\tau)$, and $\widehat{H}_r(\tau)$, and give their modular transformation properties.

In §5.2 we relate the singular combinatorial series $R_{2^r+1}(\boldsymbol{\zeta}'_{2^r};q)$ to these auxiliary functions. In §5.3 we prove Theorem 1.2.

5.1. Auxiliary functions II. Here we prove Theorem 1.2, which holds for

$$\boldsymbol{\zeta}'_{2^r} := (1, \zeta_{2^r}^0, \zeta_{2^r}^1, \ldots, \zeta_{2^r}^{2^r-1}). \tag{5.1}$$

We define

$$F_r^+(\tau) := \lim_{w\to 0} \frac{4^{-r}}{e(w)-e(-w)} \sum_{j=1}^{2^r-1} e^{-3\pi i j/2^r}\bigg(A_3\Big(\frac{j}{2^r}-w,-2\tau;\tau\Big)e^{3\pi i w} - A_3\Big(\frac{j}{2^r}+w,-2\tau;\tau\Big)e^{-3\pi i w}\bigg),$$

$$F_r^-(\tau) := \lim_{w\to 0} \frac{4^{-r}}{e(w)-e(-w)} \sum_{j=1}^{2^r-1} e^{-3\pi i j/2^r}\bigg(\mathcal{R}_3\Big(\frac{j}{2^r}-w,-2\tau;\tau\Big)e^{3\pi i w} - \mathcal{R}_3\Big(\frac{j}{2^r}+w,-2\tau;\tau\Big)e^{-3\pi i w}\bigg),$$

and

$$\widehat{F}_r(\tau) := F_r^+(\tau) + F_r^-(\tau). \tag{5.2}$$

To prove Theorem 1.2, we also define

$$G_r^+(\tau) := 4^{-r}\sum_{j=1}^{2^r-1} A_3\Big(\frac{j}{2^r},-2\tau;\tau\Big)e\Big(-\frac{3j}{2^{r+1}}\Big),$$

$$G_r^-(\tau) := 4^{-r}\sum_{j=1}^{2^r-1} \mathcal{R}_3\Big(\frac{j}{2^r},-2\tau;\tau\Big)e\Big(-\frac{3j}{2^{r+1}}\Big),$$

and $\widehat{G}_r(\tau) := G_r^+(\tau) + G_r^-(\tau)$. Using $\widehat{F}_r(\tau)$ and $\widehat{G}_r(\tau)$ we define

$$\widehat{H}_r(\tau) := \widehat{F}_r(\tau) + \tfrac{1}{2}\widehat{G}_r(\tau),$$

and

$$c_{2^r+1}(\boldsymbol{\zeta}'_{2^r};q^{24}) := \frac{1}{(q^{24};q^{24})_\infty}\bigg(\frac{G^+(24\tau)}{2} + \frac{E_2(24\tau)-1}{8\cdot 4^r} + \frac{1}{12}\bigg), \tag{5.3}$$

$$C^-_{2^r+1}(\boldsymbol{\zeta}'_{2^r};q^{24}) := \frac{\frac{1}{2}G_r^-(24\tau) + F_r^-(24\tau)}{\eta(24\tau)} - \frac{i}{4^{r+1}\pi\sqrt{2}}\int_{-\overline{\tau}}^{i\infty} \frac{\eta(24z)}{(-i(z+\tau))^{3/2}}\,dz, \tag{5.4}$$

where the quasimodular Eisenstein series $E_2(\tau)$ is defined in (3.1). We will show in §5.2 and §5.3 the roles played by the functions c_{2^r+1} and $C^-_{2^r+1}$. Before doing so, we first establish the following proposition.

PROPOSITION 5.1. *Let* $\gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2^{2r+1}) \cap \Gamma_1(2^{r+1})$. *Then*

$$\widehat{F}_r(\gamma\tau) = (c\tau+d)^2\widehat{F}_r(\tau) + \tfrac{1}{2}((c\tau+d)^2 - (c\tau+d))\widehat{G}_r(\tau), \tag{5.5}$$

$$\widehat{G}_r(\gamma\tau) = (c\tau+d)\widehat{G}_r(\tau), \tag{5.6}$$

$$\widehat{H}_r(\gamma\tau) = (c\tau+d)^2\widehat{H}_r(\tau). \tag{5.7}$$

Proof. We first prove (5.5). Using the transformations (3.3) and (3.4), we find that

$$\begin{aligned}(5.8)\quad &\widehat{A}_3(j/2^r \pm w, -2\gamma\tau; \gamma\tau) \\ &\quad = (c\tau+d)\epsilon^{\pm}_{j/2^r}(w)\xi^{\pm}_{j/2^r}(w)\psi^{\pm}_{j/2^r}(w)\widehat{A}_3(j/2^r \pm w(c\tau+d), -2\tau; \tau),\end{aligned}$$

where we have imposed the stated hypotheses on γ. Using (5.2) and (5.8), we can thus rewrite

$$\widehat{F}_r(\gamma\tau) = (c\tau+d)(f_r^+(\gamma;\tau) + f_r^-(\gamma;\tau)), \tag{5.9}$$

where

$$\begin{aligned} f_r^+(\gamma;\tau) := \lim_{w\to 0} &\frac{4^{-r}}{e(w)-e(-w)} \sum_{j=1}^{2^r-1} e^{-3\pi i j/2^r} \\ &\times \big(A_3(j/2^r - w(c\tau+d), -2\tau;\tau)e^{3\pi i w}\epsilon^-_{j/2^r}(w)\xi^-_{j/2^r}(w)\psi^-_{j/2^r}(w) \\ &- A_3(j/2^r + w(c\tau+d), -2\tau;\tau)e^{-3\pi i w}\epsilon^+_{j/2^r}(w)\xi^+_{j/2^r}(w)\psi^+_{j/2^r}(w)\big), \\ f_r^-(\gamma;\tau) := \lim_{w\to 0} &\frac{4^{-r}}{e(w)-e(-w)} \sum_{j=1}^{2^r-1} e^{-3\pi i j/2^r} \\ &\times \big(\mathcal{R}_3(j/2^r - w(c\tau+d), -2\tau;\tau)e^{3\pi i w}\epsilon^-_{j/2^r}(w)\xi^-_{j/2^r}(w)\psi^-_{j/2^r}(w) \\ &- \mathcal{R}_3(j/2^r + w(c\tau+d), -2\tau;\tau)e^{-3\pi i w}\epsilon^+_{j/2^r}(w)\xi^+_{j/2^r}(w)\psi^+_{j/2^r}(w)\big). \end{aligned}$$

We first consider $f_r^+(\gamma;\tau)$. We compute that

$$\begin{aligned} \lim_{w\to 0} \frac{1}{e(w)-e(-w)} &\bigg(\frac{e^{3\pi i w(1-(c\tau+d))}\epsilon^-_{j/2^r}(w)\xi^-_{j/2^r}(w)\psi^-_{j/2^r}(w)}{1-\zeta_{2^r}^j e(-w(c\tau+d))q^n} \\ &\qquad - \frac{e^{-3\pi i w(1-(c\tau+d))}\epsilon^+_{j/2^r}(w)\xi^+_{j/2^r}(w)\psi^+_{j/2^r}(w)}{1-\zeta_{2^r}^j e(w(c\tau+d))q^n}\bigg) \\ &= \frac{1}{2}\bigg(\frac{c\tau+d-1}{1-\zeta_{2^r}^j q^n} - \frac{2(c\tau+d)\zeta_{2^r}^j q^n}{(1-\zeta_{2^r}^j q^n)^2}\bigg), \end{aligned}$$

which follows after a lengthy but straightforward calculation using the hypotheses on γ and L'Hôpital's rule. We point out that because $1 \le j \le 2^r - 1$, $1 - \zeta_{2^r}^j q^n \neq 0$ for all $n \in \mathbb{Z}$. In a similar manner, we find that

$$F_r^+(\tau) = 4^{-r} \sum_{j=1}^{2^r-1} \sum_n \frac{(-1)^{n-1} q^{(3n^2+n)/2} \zeta_{2^r}^j}{(1 - q^n \zeta_{2^r}^j)^2}.$$

Thus,

$$\begin{aligned} f_r^+(\gamma;\tau) &= \frac{c\tau + d}{4^r} \sum_{j=1}^{2^r-1} \sum_{n\in\mathbb{Z}} \frac{(-1)^{n-1} \zeta_{2^r}^j q^{(3n^2+n)/2}}{(1 - \zeta_{2^r}^j q^n)^2} \\ &\quad + \frac{c\tau + d - 1}{2 \cdot 4^r} \sum_{j=1}^{2^r-1} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(3n^2-n)/2}}{1 - \zeta_{2^r}^j q^n} \\ &= (c\tau + d) F_r^+(\tau) + \frac{c\tau + d - 1}{2} G_r^+(\tau). \end{aligned} \tag{5.10}$$

Next we treat $f_r^-(\gamma;\tau)$. For ease of notation, we will write $\mathcal{R}_3(u, v;\tau) = \mathcal{R}_3(u)$. We compute that

$$\begin{aligned} &\lim_{w\to 0} \frac{1}{e(w) - e(-w)} \\ &\quad \times \big(\mathcal{R}_3(j/2^r - w(c\tau + d)) e^{3\pi i w} \epsilon_{j/2^r}^-(w) \xi_{j/2^r}^-(w) \psi_{j/2^r}^-(w) \\ &\quad - \mathcal{R}_3(j/2^r + w(c\tau + d)) e^{-3\pi i w} \epsilon_{j/2^r}^+(w) \xi_{j/2^r}^+(w) \psi_{j/2^r}^+(w)\big) \\ &= \frac{4(c\tau + d) - 1}{2} \mathcal{R}_3\left(\frac{j}{2^r}\right) - \frac{c\tau + d}{2\pi i} \frac{d}{du} \mathcal{R}_3(u) \bigg|_{u=j/2^r}, \end{aligned} \tag{5.11}$$

where as before, (5.11) follows after a lengthy but straightforward calculation using the hypotheses on γ and L'Hôpital's rule. Similarly, we find that

$$F_r^-(\tau) = \frac{3}{2 \cdot 4^r} \sum_{j=1}^{2^r-1} e^{-3\pi i j/2^r} \mathcal{R}_3\left(\frac{j}{2^r}\right) - \frac{1}{2\pi i \cdot 4^r} \frac{d}{du} \mathcal{R}_3(u) \bigg|_{u=j/2^r}.$$

Thus,

$$f_r^-(\gamma;\tau) = (c\tau + d) F_r^-(\tau) + \frac{c\tau + d - 1}{2} G_r^-(\tau). \tag{5.12}$$

From (5.9), (5.10), and (5.12), equation (5.5) now follows. Identity (5.6) follows after a short calculation using (3.3) and (3.4), making use of the hypotheses imposed on the matrix γ. Finally, (5.7) follows immediately from (5.5) and (5.6). ∎

5.2. The singular series $R_{2^r+1}(\boldsymbol{\zeta}'_{2^r};q)$. Here we relate the combinatorial series $R_{2^r+1}(\boldsymbol{\zeta}'_{2^r};q)$ to the form $F_r^+(\tau)$.

PROPOSITION 5.2. *For any integer $r \geq 1$,*

$$(q)_\infty R_{2^r+1}(\boldsymbol{\zeta}'_{2^r};q) = 4^{-r}(q)_\infty R_2(1,1;q) + F_r^+(\tau) + \frac{1-4^r}{3\cdot 4^{r+1}}.$$

Proof. Using the definition of $F_r^+(\tau)$, we find that

$$F_r^+(\tau) = 4^{-r}\sum_{j=1}^{2^r-1}\sum_{n\in\mathbb{Z}} \frac{(-1)^{n-1}q^{(3n^2+n)/2}\zeta_{2^r}^j}{(1-\zeta_{2^r}^j q^n)^2}, \tag{5.13}$$

where we have used the easily verifiable fact that for $1 \leq j \leq 2^r-1$, and any $n \in \mathbb{Z}$,

$$\lim_{w\to 0}\frac{(1-q^n e^{-2\pi i w}\zeta_{2^r}^j)^{-1} - (1-q^n e^{2\pi i w}\zeta_{2^r}^j)^{-1}}{e^{2\pi i w}-e^{-2\pi i w}} = -\frac{\zeta_{2^r}^j q^n}{(1-\zeta_{2^r}^j q^n)^2}.$$

We rewrite (5.13) as

$$\begin{aligned} -4^{-r}\sum_{j=1}^{2^r-1}\frac{\zeta_{2^r}^j}{(1-\zeta_{2^r}^j)^2} &+ 4^{-r}\sum_{j=1}^{2^r-1}\sum_{n\neq 0}\frac{(-1)^{n-1}q^{(3n^2+n)/2}\zeta_{2^r}^j}{(1-\zeta_{2^r}^j q^n)^2} \\ &= \frac{4^r-1}{3\cdot 4^{r+1}} + \sum_{n\neq 0}(-1)^{n-1}q^{(3n^2+n)/2}\left(-\frac{1}{4^r(1-q^n)^2} + \frac{q^{(2^r-1)n}}{(1-q^{2^r n})^2}\right), \end{aligned}$$

where we have again used the facts that for integers $r \geq 1$ and $n \geq 0$,

$$\sum_{j=1}^{2^r-1}\frac{\zeta_{2^r}^j}{(1-\zeta_{2^r}^j q^n)^2} = \frac{-1}{(1-q^n)^2} + \frac{4^n q^{(2^r-1)n}}{(1-q^{2^r n})^2},$$

and for integers $r \geq 1$,

$$\sum_{j=1}^{2^r-1}\frac{\zeta_{2^r}^j}{(1-\zeta_{2^r}^j)^2} = \frac{1-4^r}{12},$$

which are not difficult to verify. Proposition 5.2 now follows using the facts that (see (4.24))

$$(q)_\infty R_2(1,1;q) = \sum_{n\neq 0}\frac{(-1)^{n-1}q^{(3n^2+3n)/2}}{(1-q^n)^2} = \sum_{n\neq 0}\frac{(-1)^{n-1}q^{(3n^2+n)/2}}{(1-q^n)^2},$$

$$R_{2^r+1}(\boldsymbol{\zeta}'_{2^r};q) = \frac{1}{(q)_\infty}\sum_{n\neq 0}\frac{(-1)^{n-1}q^{3n^2/2+(2^{r+1}-1)n/2}}{(1-q^{2^r n})^2}. \quad \blacksquare$$

5.3. Proof of Theorem 1.2. Theorem 1.2 now follows in a similar manner to Theorem 1.1 after a short calculation, making use of Proposition 5.1, Proposition 5.2, and Theorem 1.1 of [4], using the fact that

$$\Gamma'_r := \Gamma_1(144 \cdot 4^r). \tag{5.14}$$

6. Proof of Theorem 1.3. Here we look specifically at the function $R_3(1,1,-1;q) =: R_3(q)$ as defined in (1.11). This series is obtained by specializing $r = 1$ in our family R_{2^r+1} considered in Theorem 1.2. To obtain the asymptotic formula for its coefficients $a_3(n)$ as given in Theorem 1.3, we must first consider its transformation under matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We note that we rewrite

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h & -(hh'+1)/k \\ k & -h' \end{pmatrix}$$

for some $h, k \in \mathbb{Z}$, where $(h,k) = 1$, and $hh' \equiv -1 \pmod{k}$. (Note. We abuse notation and reuse the variable k, as is standard in applications of the Circle Method, which we use in what follows.)

6.1. A transformation law for $R_3(q)$

6.1.1. *Definitions.* In order to determine the asymptotic expansion given in (1.13) of Theorem 1.3 for $R_3(q)$, we cannot proceed by applying the standard technique of Poisson summation (see [1], [22]) because the sum defining $R_3(q)$ given in (1.11) is taken over the incomplete lattice $\mathbb{Z}\setminus\{0\}$. To deal with the missing lattice point, one might first try to add an additional summand to $R_3(q)$, proceed by Poisson summation, and then later remove the extra summand. However, adding the missing summand (at $n = 0$) to $R_3(q)$ would result in the addition of a term with a double pole, so we cannot proceed in this way. Instead, using a technique of Bringmann [4], we rewrite $R_3(q)$ as the derivative of another (full lattice) sum, defined with an additional parameter w that we let tend to 0. Namely, we define the series

$$\widetilde{R}_3(w;q) := \sum_{n\in\mathbb{Z}} \frac{(-1)^{n-1} q^{n(3n-1)/2}}{1 - e^{2\pi i w} q^{2n}}, \tag{6.1}$$

and the differential operator

$$L(g(w)) := \frac{1}{2\pi i}\left[\frac{d}{dw} g\right]_{w=0}.$$

It is not difficult to show that

$$R_3(q) = \frac{1}{(q)_\infty} L(\widetilde{R}'_3(w;q)), \tag{6.2}$$

where $\widetilde{R}'_3(w;q)$ denotes the function obtained from $\widetilde{R}_3(w;q)$ by summing only over integers $n \neq 0$. Bringmann used this technique in [4] (as mentioned above) in order to study $R_2(q) := R_2(1,1;q)$ by defining

$$\widetilde{R}_2(w;q) := \sum_{n\in\mathbb{Z}} \frac{(-1)^{n-1}q^{n(3n+1)/2}}{1-e^{2\pi iw}q^n}. \tag{6.3}$$

Despite the similarities between (6.1) and (6.3), we cannot proceed directly as in [4] due to the denominators $1-e^{2\pi iw}q^{2n}$ appearing in (6.1), which differ in a subtle, yet bothersome way from the denominators $1-e^{2\pi iw}q^n$ appearing in (6.3). Instead, we rewrite $\widetilde{R}_3(w;q)$ as a sum of *universal mock theta functions* and then proceed to study these functions. Gordon and McIntosh [16] show that all of Ramanujan's original mock theta functions can be written in terms of two "universal" mock theta functions $g_2(x;q)$ and $g_3(x;q)$ upon suitable specialization of the parameters x and q, one of which is given by

$$g_3(x;q) := \sum_{n\geq 0} \frac{q^{n(n+1)}}{(x;q)_{n+1}(x^{-1}q;q)_{n+1}}.$$

(The universal mock theta function $g_2(x;q)$ is defined similarly.)

We first rewrite

$$\widetilde{R}_3(w;q) = \frac{1}{2e^{\pi iw}}(R_3^-(w;q) - R_3^+(w;q)), \tag{6.4}$$

where

$$R_3^\pm(w;q) := \sum_{n\in\mathbb{Z}} \frac{(-1)^{n-1}q^{3n(n-1)/2}}{1\pm e^{\pi iw}q^n} = \mp e^{-\pi iw}(q)_\infty g_3(\mp e^{-\pi iw};q), \tag{6.5}$$

which can be deduced from (3.3) of [16]. The modular transformation properties of $g_3(x;q)$, which are needed for the proof of Theorem 1.3, have been explored when x is replaced by q^t for $t\in\mathbb{Q}$ (see [8], [9], [15], [16]), a setting which is useful for studying Ramanujan's original mock theta functions. Here we wish to study the universal mock theta functions $g_3(\mp e^{\pi iw};q)$ for $w\in\mathbb{C}$. We give a transformation law for these universal mock theta functions in Corollary 6.4 in a form which does not appear to have been previously recorded in the literature, and readily lends itself to the proof of Theorem 1.3.

To state the modular transformation laws for $\widetilde{R}_3(w;q)$, we need to define the following q-series:

$$b^e(q) := \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^{n-1}q^{3n(n+1)/2}}{(1+q^n)^2}, \qquad b^o(w;q) := \sum_{n\geq 0} \frac{(-1)^n q^{3n(n+1)/2}}{1-q^{n+1/2}e^{\pi w/z}},$$

and the integrals, with $\nu, \delta \in \mathbb{Z}$,

$$I_k^{\pm}(\nu, \delta; w, z) := \int_{-\infty}^{\infty} \frac{e^{-(3\pi z/k)(x^2+x)}}{1 \pm e^{\pi i w} e^{2\pi i \nu/k - (2\pi z/k)(x + (i/6z)(2\delta+1))}}\, dx,$$

$$J_k^{-}(\nu, \delta; z) := \int_{-\infty}^{\infty} \frac{e^{-3\pi z x^2/k + 3\pi z x/k}}{\sinh^2\left(\frac{\pi i \nu}{k} - \frac{\pi z}{k}\left(x + \frac{(2\delta+1)i}{6z}\right)\right)}\, dx,$$

$$J_k^{+}(\nu, \delta; z) := \int_{-\infty}^{\infty} \frac{e^{-3\pi z x^2/k + 3\pi z x/k}}{\cosh^2\left(\frac{\pi i \nu}{k} - \frac{\pi z}{k}\left(x + \frac{(2\delta+1)i}{6z}\right)\right)}\, dx.$$

6.1.2. *Transformation law.* To determine the modular transformation laws for $\widetilde{R}_3(w;q)$, we need to consider two cases: when k is even and when k is odd. We obtain the following results:

THEOREM 6.1. *Let $(h,k) = 1$, with $k > 0$ and k even, and let $q := e((h+iz)/k)$, $q_1 := e((h' + iz^{-1})/k)$, $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, where h' is defined by $hh' \equiv -1 \pmod{2k}$. Then $\widetilde{R}_3(w;q)$ equals*

$$\frac{e^{(3\pi w/2)(i + wk/(2z)) - \pi w/(2z)}}{2ize^{\pi i w}} \left(-\widetilde{R}_2\left(\frac{iw}{2z}; q_1\right) + ie^{(\pi i h'/2)(1-k/2)} \widetilde{R}_2\left(\frac{iw}{2z} + \frac{1}{2}; q_1\right)\right)$$
$$- \frac{(q_1)_\infty}{2ke^{\pi i w}} \sum_{\nu \,(\mathrm{mod}\, k)} (-1)^\nu e^{-(3\pi i \nu/k)(1 + h'\nu)} e^{-\pi/(12kz)}$$
$$\times \sum_{\delta=-1}^{0} e^{(\pi i/k)(2\delta+1)(1/2 + h'\nu)} (I_k^-(\nu, \delta; w, z) - I_k^+(\nu, \delta; w, z)).$$

THEOREM 6.2. *Let $(h,k) = 1$, with $k > 0$ and k odd, and let $q := e((h+iz)/k)$, $q_1 := e((h' + iz^{-1})/k)$, $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, where h' is defined to be an even solution to $hh' \equiv -1 \pmod{k}$. Then $\widetilde{R}_3(w;q)$ equals*

$$\frac{e^{(3\pi w/2)(i + wk/(2z))}}{2ize^{\pi i w}} \Bigg(- e^{-\pi w/(2z)} \widetilde{R}_2\left(\frac{iw}{2z}; q_1\right)$$
$$- i(-1)^{(k+1)/2} q_1^{3/8} e^{(\pi i h'/2)(1-k/2)} (b^o(-w; q_1) + b^o(w; q_1))\Bigg)$$
$$- \frac{(q_1)_\infty}{2ke^{\pi i w}} \sum_{\nu \,(\mathrm{mod}\, k)} (-1)^\nu e^{-(3\pi i \nu/k)(1 + h'\nu)} e^{-\pi/(12kz)}$$
$$\times \sum_{\delta=-1}^{0} e^{(\pi i/k)(2\delta+1)(1/2 + h'\nu)} (I_k^-(\nu, \delta; w, z) - I_k^+(\nu, \delta; w, z)).$$

From Theorems 6.1 and 6.2, we obtain the modular transformation laws for $R_3(q)$:

Corollary 6.3. *Assume the hypotheses above.*

(1) *If k is even, $R_3(q)$ equals*

$$\frac{\omega_{h,k}z^{1/2}e^{(\pi/(12k))(z^{-1}-z)}}{(q_1)_\infty}\left(\frac{1}{96z^2}-\frac{3k}{16\pi z}-\frac{5}{96}\right) - \frac{\omega_{h,k}e^{(\pi/(12k))(z^{-1}-z)}}{4z^{3/2}}R_2(q_1)+\frac{\omega_{h,k}e^{(\pi i h'/2)(1-k/2)}}{4z^{1/2}e^{(\pi/(12k))(z-z^{-1})}}b^e(q_1) + \frac{1}{4k}\omega_{h,k}z^{1/2}e^{-\pi z/(12k)}\sum_{\nu \pmod k}(-1)^\nu e^{-3\pi i\nu/k-3\pi i h'\nu^2/k} \times \sum_{\delta=-1}^{0} e^{(\pi i/k)(2\delta+1)(1/2+h'\nu)} \times\left(I_k^-(\nu,\delta;0,z)-I_k^+(\nu,\delta;0,z)-\frac{1}{4}(J_k^-(\nu,\delta;z)+J_k^+(\nu,\delta;z))\right).$$

(2) *If k is odd, $R_3(q)$ equals*

$$\frac{\omega_{h,k}z^{1/2}e^{(\pi/(12k))(z^{-1}-z)}}{(q_1)_\infty}\left(\frac{1}{96z^2}-\frac{3k}{16\pi z}-\frac{5}{96}\right) - \frac{\omega_{h,k}e^{(\pi/(12k))(z^{-1}-z)}}{4z^{3/2}}R_2(q_1) - \frac{(-1)^{(k+1)/2}\omega_{h,k}e^{(\pi i h'/2)(1-k/2)}}{4z^{1/2}e^{(\pi/(12k))(z-z^{-1})}(q_1)_\infty}q_1^{3/8}b^o(0;q_1) + \frac{1}{4k}\omega_{h,k}z^{1/2}e^{-\pi z/(12k)}\sum_{\nu \pmod k}(-1)^\nu e^{-3\pi i\nu/k-3\pi i h'\nu^2/k} \times \sum_{\delta=-1}^{0} e^{(\pi i/k)(2\delta+1)(1/2+h'\nu)} \times\left(I_k^-(\nu,\delta;0,z)-I_k^+(\nu,\delta;0,z)-\frac{1}{4}(J_k^-(\nu,\delta;z)+J_k^+(\nu,\delta;z))\right).$$

Here, $\omega_{h,k}$ is as defined in Lemma 3.2.

We now prove Theorem 6.1, in which k is even.

Proof of Theorem 6.1. We first observe that, for $w \in \mathbb{C}$ with $\mathrm{Re}(w) \neq 0$ sufficiently small, $\widetilde{R}_3(w;q)$ is a holomorphic function of z. Similarly, $R_3^\pm(w;q)$ are holomorphic functions in z. Recall that we have $q = e((h+iz)/k)$ and $q_1 = e((h'+iz^{-1})/k)$ with $hh' \equiv -1 \pmod{2k}$. Now, looking at $R_3^\pm(w;q)$, we write $n = \nu + km$ with $m \in \mathbb{Z}$ and ν taken modulo k, and rewrite $R_3^\pm(w;q)$, which after Poisson summation (and the change of variable $x \mapsto \nu + kx$) becomes

(6.6)
$$-\frac{1}{k}\sum_{\nu\,(\mathrm{mod}\,k)}(-1)^{\nu}e^{(3\pi i h/k)(\nu^2-\nu)}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\frac{e^{(\pi i/k)(2n+1)(x-\nu)}e^{-(3\pi z/k)(x^2-x)}}{1\pm e^{\pi i w}e^{2\pi i h\nu/k-2\pi z x/k}}\,dx.$$

We find that the integrand has at most simple poles at the points
$$x_m = x_{m,k}(w,z) := \frac{i}{z}\left(m+\frac{wk}{2}\right),$$
(which in the case of $R_3^+(w;q)$ follows from the fact that k is even) and $m \equiv h\nu - \varrho(k) \pmod{k}$, where $\varrho(k) := k/2$ in $R_3^+(w;q)$ and $\varrho(k) := 0$ in $R_3^-(w;q)$. We now replace ν with ν_m defined by
$$\nu_m = \nu_{m,h,k} := -mh' - \varrho(k)h'.$$

Using the residue theorem, we shift the path of integration through
$$w_n = w_n(z) := \frac{(2n+1)i}{6z}.$$
Assuming w is sufficiently small, for $n \geq 0$, the residue sum will involve a sum over those m such that $0 \leq 3m \leq n$, and for $n < 0$, those m such that $n+1 \leq 3m < 0$, in both $R_3^+(w;q)$ and $R_3^-(w;q)$. Denoting the associated residues of each summand by $\lambda_{n,m}$ (omitting dependence on w, z, h, k for ease of notation), we have
$$R_3^{\pm}(w;q) = 2\pi i\left(\sum\nolimits_1^+ + \sum\nolimits_1^-\right) + \sum\nolimits_2,$$
where

(6.7)
$$\sum\nolimits_1^+ := -\frac{1}{k}\sum_{\nu\,(\mathrm{mod}\,k)}(-1)^{\nu}e^{(3\pi i h/k)(\nu^2-\nu)}\sum_{n\geq 0}\sum_{0\leq 3m\leq n}\lambda_{n,m},$$
$$\sum\nolimits_1^- := \frac{1}{k}\sum_{\nu\,(\mathrm{mod}\,k)}(-1)^{\nu}e^{(3\pi i h/k)(\nu^2-\nu)}\sum_{n<0}\sum_{n+1\leq 3m<0}\lambda_{n,m},$$
$$\sum\nolimits_2 := -\frac{1}{k}\sum_{\nu\,(\mathrm{mod}\,k)}(-1)^{\nu}e^{(3\pi i h/k)(\nu^2-\nu)}$$
$$\times\sum_{n\in\mathbb{Z}}\int_{-\infty+w_n}^{\infty+w_n}\frac{e^{(\pi i/k)(2n+1)(x-\nu)}e^{-(3\pi z/k)(x^2-x)}}{1\pm e^{\pi i w}e^{2\pi i h\nu/k-2\pi z x/k}}\,dx.$$

We first consider $\sum_1 := 2\pi i(\sum_1^+ + \sum_1^-)$. It is not difficult to show that

(6.8)
$$\lambda_{n,m} = \frac{k}{2\pi z}e^{(\pi i/k)(2n+1)(x_m-\nu_m)}e^{-(3\pi z/k)(x_m^2-x_m)},$$
and thus

(6.9)
$$\lambda_{n+l,m} = e^{-\pi l w/z}q_1^{lm}\alpha^l\lambda_{n,m},$$
where $\alpha := 1$ in the case of $R_3^-(w;q)$ and $\alpha := -1$ in the case of $R_3^+(w;q)$.

Using (6.9), we see that

$$\sum_{n\geq 0}\sum_{0\leq 3m\leq n}\lambda_{n,m}=\sum_{m\geq 0}\sum_{n\geq 3m}\lambda_{n,m}=\sum_{m\geq 0}\frac{\lambda_{3m,m}}{1-q_1^m e^{-\pi w/z}\alpha},$$

$$\sum_{n<0}\sum_{n+1\leq 3m<0}\lambda_{n,m}=\sum_{m<0}\sum_{n\leq 3m-1}\lambda_{n,m}=-\sum_{m<0}\frac{\lambda_{3m,m}}{1-q_1^m e^{-\pi w/z}\alpha},$$

so that, in both cases,

$$\sum\nolimits_1=-\frac{2\pi i}{k}\sum_{m\in\mathbb{Z}}\frac{(-1)^{\nu_m}e^{(3\pi ih/k)(\nu_m^2-\nu_m)}\lambda_{3m,m}}{1-q_1^m e^{-\pi w/z}\alpha}. \tag{6.10}$$

Using (6.10) one can show that for $R_3^+(w;q)$,

$$\sum\nolimits_1=-\frac{i}{z}e^{(3\pi i/2)(w-1)-\pi w/(2z)+3\pi w^2k/(4z)}e^{(\pi ih'/2)(1-k/2)}\times\sum_{m\in\mathbb{Z}}\frac{(-1)^m q_1^{m(3m+1)/2}}{1+q_1^m e^{-\pi w/z}}, \tag{6.11}$$

and for $R_3^-(w;q)$,

$$\sum\nolimits_1=-\frac{i}{z}e^{3\pi iw/2-\pi w/(2z)+3\pi w^2k/(4z)}\sum_{m\in\mathbb{Z}}\frac{(-1)^m q_1^{m(3m+1)/2}}{1-q_1^m e^{-\pi w/z}}. \tag{6.12}$$

We next turn to $\sum_2$. We make the change of variable $x\mapsto x+w_n+1/2$, and replace $n\in\mathbb{Z}$ by $3p+\delta$, where p runs over all integers and $\delta\in\{0,\pm 1\}$. Replacing ν by $-h'(\nu+p)$, a lengthy but straightforward calculation gives

$$\sum\nolimits_2=-\frac{1}{k}e^{3\pi z/(4k)}\sum_{\nu\,(\mathrm{mod}\,k)}e^{-\pi ih'\nu-3\pi i\nu/k+3\pi ik'(h'\nu^2+\nu)-3\pi ih'\nu^2/k}$$
$$\times\sum_{\delta=-1}^{1}e^{\pi i(2\delta+1)/(2k)}e^{(\pi ih'\nu/k)(2\delta+1)}e^{-\pi(2\delta+1)^2/(12kz)}$$
$$\times\int_{-\infty}^{\infty}\frac{e^{-3\pi zx^2/k}dx}{1\pm e^{\pi iw}e^{2\pi i\nu/k}e^{-(2\pi z/k)(x+(2\delta+1)i/(6z)+1/2)}}\sum_{p\in\mathbb{Z}}(-1)^p q_1^{(p/2)(3p+2\delta+1)}$$

where $hh'=-1+kk'$ and k' is even. The innermost sum on p equals 0 if $\delta=1$ and equals $(q_1)_\infty$ if $\delta=0$ or -1. Thus, after we let $x\mapsto x-1/2$, we find

$$\sum\nolimits_2=-(q_1)_\infty\frac{1}{k}\sum_{\nu\,(\mathrm{mod}\,k)}(-1)^\nu e^{-3\pi i\nu/k-3\pi ih'\nu^2/k}e^{-\pi/(12kz)}\times\sum_{\delta=-1}^{0}e^{\pi i(2\delta+1)/(2k)}e^{(\pi ih'\nu/k)(2\delta+1)}I_k^{\pm}(\nu,\delta;w,z). \tag{6.13}$$

Using (6.11)–(6.13), we deduce Theorem 6.1 from (6.4). ∎

Next, we proceed to the proof of Theorem 6.2, in which k is odd.

Proof of Theorem 6.2. The parity of k does not play a role in the derivation of (6.12) in the proof of Theorem 6.1 pertaining to $R_3^-(w;q)$. Thus, we turn our attention to $R_3^+(w;q)$.

As in the proof of Theorem 6.1, by taking $n = \nu + km$ and applying Poisson summation, we obtain (6.6) for $R_3^+(w;q_1)$. Now, with k odd, we see that the integrand of (6.6) has at most simple poles at

$$x_m = x_{m,k}(w;z) := \frac{i}{z}\left(m + \frac{wk}{2} + \frac{1}{2}\right),$$

with $m \equiv h\nu - \varrho(k) \pmod k$, where $\varrho(k) := (k+1)/2$ and, as before, we let

$$\nu = \nu_{m,h,k} := -mh' - \varrho(k)h'.$$

Again, shifting the path of integration through $w_n = (2n+1)i/(6z)$, we see that the residue sum for $n \geq 0$ is over those m such that $0 \leq 3m+1 \leq n$ and for $n < 0$, those m such that $n \leq 3m+1 < 0$. Thus, we have

$$R_3^+(w;q) = 2\pi i\Big(\sum\nolimits_1^+ + \sum\nolimits_1^-\Big) + \sum\nolimits_2$$

where

$$\sum\nolimits_1^+ := -\frac{1}{k}\sum_{\nu \,(\mathrm{mod}\, k)} e^{(3\pi ih/k)(\nu^2-\nu)} \sum_{n\geq 0}\sum_{0\leq 3m+1\leq n} \lambda_{n,m},$$

$$\sum\nolimits_1^- := \frac{1}{k}\sum_{\nu \,(\mathrm{mod}\, k)} e^{(3\pi ih/k)(\nu^2-\nu)} \sum_{n<0}\sum_{n\leq 3m+1<0} \lambda_{n,m},$$

and $\sum_2$ is as in (6.7).

We first consider $\sum_1 := 2\pi i(\sum_1^+ + \sum_1^-)$. The parity of k does not change the expression of $\lambda_{n,m}$ given in (6.8), and we again have (6.9), with $\alpha := q_1^{1/2}$. Therefore,

$$\sum\nolimits_1^+ := -\frac{1}{k}\sum_{m\geq 0} \frac{(-1)^{\nu_m} e^{(3\pi ih/k)(\nu_m^2-\nu_m)}\lambda_{3m+1,m}}{1 - q_1^{m+1/2}e^{-\pi w/z}},$$

$$\sum\nolimits_1^- := \frac{1}{k}\sum_{m<0} \frac{(-1)^{\nu_m} e^{(3\pi ih/k)(\nu_m^2-\nu_m)}\lambda_{3m+1,m}}{1 - q_1^{-m-1/2}e^{\pi w/z}}.$$

By a simple calculation, and letting $m \mapsto -m-1$ in $\sum_1^-$, we obtain, for $R_3^+(w;q)$ with k odd,

$$\sum\nolimits_1 = \frac{(-1)^{(k+1)/2}}{z} e^{3\pi iw/2 + 3\pi kw^2/(4z)} q_1^{3/8} e^{(\pi ih'/2)(1-k/2)} \times \left(b^o(-w;q_1) + b^o(w;q_1)\right).$$

Next we turn to $\sum_2$. The parity of k only plays a role in that we take h' to be an even solution of $hh' \equiv -1 \pmod k$, meaning k' must be odd, so we obtain (6.13) for both $R_3^+(w;q)$ and $R_3^-(w;q)$ exactly as in the proof of Theorem 6.1. ∎

From the proofs of Theorems 6.1 and 6.2 along with equations (6.5) and (6.14), we obtain the following transformation laws for the universal mock theta function $g_3(\pm e^{-\pi i w};q)$, a result which is of independent interest.

COROLLARY 6.4. *Assume the hypotheses above.*

(1) *We have that* $g_3(e^{-\pi i w};q)$ *equals*

$$-\frac{i\omega_{h,k}e^{(\pi/(12k))(z^{-1}-z)}}{z^{1/2}(q_1)_\infty}e^{3\pi i w/2+\pi i w-\pi w/(2z)+3\pi w^2k/(4z)}\sum_{m\in\mathbb{Z}}\frac{(-1)^m q_1^{m(3m+1)/2}}{1-q_1^m e^{-\pi w/z}} + \frac{\omega_{h,k}z^{1/2}e^{(\pi/(12k))(z^{-1}-z)}}{(q_1)_\infty}e^{\pi i w}\sum\nolimits_2$$

where $\sum_2$ *is as in* (6.13), *with the choice of integral* $I_k^-(\nu,\delta;w,z)$.

(2) *For* k *even,* $g_3(-e^{-\pi i w};q)$ *equals*

$$\frac{i\omega_{h,k}e^{(\pi/(12k))(z^{-1}-z)}}{z^{1/2}(q_1)_\infty}e^{(3\pi i/2)(w-1)+\pi i w-\pi w/(2z)+3\pi w^2k/(4z)} \times e^{(\pi i h'/2)(1-k/2)}\sum_{m\in\mathbb{Z}}\frac{(-1)^m q_1^{m(3m+1)/2}}{1+q_1^m e^{-(\pi w/z)}} - \frac{\omega_{h,k}z^{1/2}e^{(\pi/(12k))(z^{-1}-z)}}{(q_1)_\infty}e^{\pi i w}\sum\nolimits_2$$

where $\sum_2$ *is as in* (6.13), *with the choice of integral* $I_k^+(\nu,\delta;w,z)$.

(3) *For* k *odd,* $g_3(-e^{-\pi i w};q)$ *equals*

$$-\frac{(-1)^{(k+1)/2}\omega_{h,k}e^{(\pi/(12k))(z^{-1}-z)}}{z^{1/2}(q_1)_\infty}e^{3\pi i w/2+\pi i w+3\pi k w^2/(4z)}q_1^{3/8} \times e^{(\pi i h'/2)(1-k/2)}(b^o(-w;q_1)+b^o(w;q_1)) - \frac{\omega_{h,k}z^{1/2}e^{(\pi/(12k))(z^{-1}-z)}}{(q_1)_\infty}e^{\pi i w}\sum\nolimits_2$$

where $\sum_2$ *is as in* (6.13), *with the choice of integral* $I_k^+(\nu,\delta;w,z)$.

Returning to $R_3(q)$, from Theorems 6.1 and 6.2, we now conclude the modular transformation properties of $R_3(q)$.

Proof of Corollary 6.3. We wish to compute $L(\widetilde{R}_3'(w;q))$. Since $\widetilde{R}_3'(w;q)$ is a sum over only those terms in $\widetilde{R}_3(w;q)$ with $n\neq 0$, we have to subtract the term for $n=0$, which is taken into account in the first line of the following set of equations. It is not difficult to show that

$$L\bigg(\frac{1}{1-e^{2\pi i w}}+\frac{e^{3\pi i w/2+3\pi w^2k/(4z)-\pi w/(2z)}}{2ize^{\pi i w}(1-e^{-\pi w/z})}\bigg)=\frac{\pi-18kz-5\pi z^2}{96\pi z^2},$$

$$L\left(-\frac{e^{3\pi iw/2+3\pi w^2k/(4z)-\pi w/(2z)}}{2ize^{\pi iw}}\widetilde{R}_2'\left(\frac{iw}{2z};q\right)\right)=\frac{-1}{4z^2}(q_1)_\infty R_2(q_1),$$

$$L\left(\frac{e^{3\pi iw/2+3\pi w^2k/(4z)-\pi w/(2z)+(\pi ih'/2)(1-k/2)}}{2ze^{\pi iw}}\widetilde{R}_2\left(\frac{iw}{2z}+\frac{1}{2};q_1\right)\right)$$
$$=\frac{e^{(\pi ih'/2)(1-k/2)}}{4z}(q_1)_\infty b^e(q_1),$$

$$L\left(-\frac{(-1)^{(k+1)/2}e^{(3\pi w/2)(i+wk/(2z))}e^{(\pi ih'/2)(1-k/2)}}{2ze^{\pi iw}}q_1^{3/8}(b^o(-w;q_1)+b^o(w;q_1))\right)$$
$$=-\frac{(-1)^{(k+1)/2}e^{(\pi ih'/2)(1-k/2)}}{4z}q_1^{3/8}\ b^o(0;q_1),$$

and, using the notation as in the proofs of Theorems 6.1 and 6.2,

$$L\Big(\sum\nolimits_2\Big)=\frac{(q_1)_\infty}{4k}e^{-\pi/(12kz)}\sum_{\nu\,(\mathrm{mod}\,k)}(-1)^\nu e^{-3\pi i\nu/k-3\pi ih'\nu^2/k}$$
$$\times\sum_{\delta=-1}^{0}e^{(\pi i/k)(2\delta+1)(1/2+h'\nu)}$$
$$\times\left(I_k^-(\nu,\delta;0,z)-I_k^+(\nu,\delta;0,z)-\frac{1}{4}(J_k^-(\nu,\delta;z)+J_k^+(\nu,\delta;z))\right).$$

Corollary 6.3 now follows from (6.2) by using the fact that

$$(q_1)_\infty=\omega_{h,k}z^{1/2}e^{(\pi/(12k))(z^{-1}-z)}(q)_\infty, \tag{6.14}$$

which may be deduced from the transformation of η given in Lemma 3.2. ∎

6.2. Proof of Theorem 1.3. Here, we use the Circle Method to prove Theorem 1.3. First, we estimate the integrals $I_k^\pm(\nu,\delta;0,z)$ and $J_k^\pm(\nu,\delta;z)$, as well as certain Kloosterman sums. This well known technique has been used to provide asymptotic results in a number of works, including [1], [4], and [12].

LEMMA 6.5. *Let $n\in\mathbb{N}$ and $\nu\in\mathbb{Z}$ with $-\frac{1}{2}(k+1)\le\nu\le\frac{1}{2}(k+1)$ if k is odd, and $-\frac{1}{2}k\le\nu\le\frac{1}{2}k$ if k is even. Let $z:=k/n-ik\varphi$, $-1/(k(k+k_1))\le\varphi\le 1/(k(k+k_2))$, where $h_1/k_2<h/k<h_2/k_2$ are adjacent Farey fractions in the Farey sequence of order N with $N:=\lfloor n^{1/2}\rfloor$. Then*

$$z^{1/2}I_k^-(\nu,\delta;0,z)\ll\frac{kn^{1/4}}{|\nu-(2\delta+1)/6|},$$
$$z^{1/2}J_k^-(\nu,\delta;0,z)\ll\frac{k^2n^{1/4}}{|\nu-(2\delta+1)/6|^2}.$$

LEMMA 6.6. *Let $n \in \mathbb{N}$ and $\nu \in \mathbb{Z}$ with $0 \le \nu < k$. Let $z := k/n - ik\varphi$, $-1/(k(k+k_1)) \le \varphi \le 1/(k(k+k_2))$, where $h_1/k_2 < h/k < h_2/k_2$ are adjacent Farey fractions in the Farey sequence of order N with $N := \lfloor n^{1/2} \rfloor$. Then*

$$z^{1/2} I_k^+(\nu, \delta; 0, z) \ll \frac{kn^{1/4}}{|k - 2(\nu - (2\delta+1)/6)|},$$
$$z^{1/2} J_k^+(\nu, \delta; 0, z) \ll \frac{k^2 n^{1/4}}{|k - 2(\nu - (2\delta+1)/6)|^2}.$$

Proof of Lemma 6.5. For brevity, we only consider $I_k^-(\nu, \delta; 0, z)$; the proof of the estimate for $J_k^-(\nu, \delta; 0, z)$ follows similarly.

We first note that with $\tau := \pi z x/k$, the function $z^{1/2} I_k^-(\nu, \delta; 0, z)$ equals

$$\frac{k}{2\pi z^{1/2}} e^{(2\delta+1)\pi i/(6k) - \pi i\nu/k} \int_S \frac{e^{-3k\tau^2/(\pi z) + 4\tau}}{\sinh(-\pi i\nu/k + \tau + (2\delta+1)\pi i/(6k))}\, d\tau, \tag{6.15}$$

where S is the line through 0 and $Ce^{iA} := \pi z/k$.

We want to consider integration along the contour consisting of this line, the real line, and the arcs $\pm Re^{it}$ where $0 \le t \le A$. For $\tau = Re^{it}$ on these arcs, the integrand (6.15) goes to 0 as $R \to \infty$. The poles of this integrand occur only when τ is a non-zero, purely imaginary number. Thus, none of the poles are contained within our contour, so by the residue theorem, the function $z^{1/2} I_k^-(\nu, \delta; 0, z)$ equals

$$\frac{k}{2\pi z^{1/2}} e^{(2\delta+1)\pi i/(6k) - \pi i\nu/k} \int_{\mathbb{R}} \frac{e^{-3kt^2/(\pi z) + 4t}}{\sinh(-\pi i\nu/k + t + (2\delta+1)\pi i/(6k))}\, dt.$$

Now, for $-\pi/2 \le y \le \pi/2$,

$$|\sinh(x + iy)| \ge |\sin y| \ge |2y/\pi|. \tag{6.16}$$

Using this fact, we have

$$\begin{aligned} &z^{1/2} I_k^-(\nu, \delta; 0, z) \\ &\quad \ll \frac{k^2}{|z|^{1/2}} |-v + (2\delta+1)/6|^{-1} \int_{\mathbb{R}} e^{-3kt^2 \operatorname{Re}(z^{-1})/\pi + 4t}\, dt \\ &\quad \ll \frac{k^2}{|z|^{1/2}} |v - (2\delta+1)/6|^{-1} |e^{4\pi/(3k \operatorname{Re}(z^{-1}))}| \int_{\mathbb{R}} e^{-3kt^2 \operatorname{Re}(z^{-1})/\pi}\, dt. \end{aligned}$$

Since $z = k/n - k\varphi i$, $-1/(k(k+k_1)) \le \varphi \le 1/(k(k+k_2))$, and $1/(k+k_j) \le 1/(N+1)$ for $j = 1, 2$, we have $k \operatorname{Re}(z^{-1}) \ge 1/2$. We let $u = t\sqrt{3k \operatorname{Re}(z^{-1})/\pi}$, and find

$$(6.17)\quad z^{1/2} I_k^-(\nu, \delta; 0, z) \ll \frac{k^2}{|z|^{1/2}} |v - (2\delta+1)/6|^{-1} (3k \operatorname{Re}(z^{-1}))^{-1/2} \int_{\mathbb{R}} e^{-u^2}\, du.$$

Because the integral in (6.17) converges, and

$$|z|^{-1/2} \operatorname{Re}(z^{-1})^{-1/2} \ll n^{1/4} k^{-1/2},$$

we obtain the desired result. ∎

Proof of Lemma 6.6. Lemma 6.6 is proven in a manner nearly identical to Lemma 6.5, so for brevity we omit details. One must only replace the estimate (6.16) by the estimate $0 \le y \le \pi$, $|\cosh(x+iy)| \ge |\cos y| \ge |(\pi - 2y)/\pi|$. ∎

We next estimate certain Kloosterman sums.

Lemma 6.7. *Let $n, m \in \mathbb{Z}$, $0 \le \sigma_1 < \sigma_2 \le k$ and $D \in \mathbb{Z}$ with $(D,k) = 1$. Then:*

(1) *For all k,*

$$\sum_{\substack{h \,(\mathrm{mod}\, k)^\star \\ \sigma_1 \le Dh' \le \sigma_2}} \omega_{h,k} e^{(2\pi i/k)(hn+h'm)} \ll (24n+1,k)^{1/2} k^{1/2+\varepsilon}.$$

(2) *If k is even, then*

$$\sum_{\substack{h \,(\mathrm{mod}\, k)^\star \\ \sigma_1 \le Dh' \le \sigma_2}} \omega_{h,k} e^{(2\pi i/k)(hn+h'm)+(\pi i h'/2)(1-k/2)} \ll (24n+1,k)^{1/2} k^{1/2+\varepsilon}.$$

(3) *If k is odd and $h' \equiv 0 \pmod 2$, then*

$$\sum_{\substack{h \,(\mathrm{mod}\, k)^\star \\ \sigma_1 \le Dh' \le \sigma_2}} \omega_{h,k} e^{(2\pi i/k)(hn+h'm)+(\pi i h'/2)(1-k/2)+3\pi i h'/(4k)} \ll (24n+1,k)^{1/2} k^{1/2+\varepsilon}.$$

Proof. (1) is proved in [1]; (2) follows from (1) using

$$(\pi i h'/2)(1-k/2) = 2\pi i h' l/k$$

for some $l \in \mathbb{Z}$; and (3) follows in the same way, given the assumption that $h' \equiv 0 \pmod 2$. ∎

Proof of Theorem 1.3. By Cauchy's theorem, for $n > 0$, we have

$$a_3(n) = \frac{1}{2\pi i} \int_C \frac{R_3(q)}{q^{n+1}}\, dq,$$

where C is any path in the unit circle surrounding zero, traversed counterclockwise. Taking C to be the circle of radius $e^{-2\pi/n}$ and letting $q := e^{-2\pi/n+2\pi it}$ with $0 \le t \le 1$, we have

$$a_3(n) = \int_0^1 R_3(e^{-2\pi/n+2\pi it})e^{2\pi-2\pi int}\,dt.$$

Define

$$\vartheta'_{h,k} := \frac{1}{k(k+k_1)}, \qquad \vartheta''_{h,k} := \frac{1}{k(k+k_2)},$$

where $h_1/k_1 < h/k < h_2/k_2$ are adjacent Farey fractions in the Farey sequence of order $N = \lfloor n^{1/2}\rfloor$. We decompose the path of integration in the paths along the Farey arcs $-\vartheta'_{h,k} \le \varphi \le \vartheta''_{h,k}$, where $\varphi := t - h/k$. This gives

$$a_3(n) = \sum_{h,k} e^{-2\pi ihn/k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} R_3(e^{(2\pi i/k)(h+iz)})e^{2\pi inz/k}\,d\varphi, \tag{6.18}$$

where $z := k/n - ik\varphi$ and the sum in (6.18) is taken over k from 1 to N, and then over $h \pmod k$ with $(h,k)=1$. Corollary 6.3 gives

$$\begin{aligned} a_3(n) = {} & \sum_{h,k} e^{-2\pi ihn/k}\omega_{h,k} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi nz/k}e^{(\pi/(12k))(z^{-1}-z)}z^{1/2} \\ & \times \left(\frac{1}{(q_1)_\infty}\left(\frac{1}{96z^2} - \frac{3k}{16\pi z} - \frac{5}{96}\right) - \frac{1}{4z^2}R_2(q_1)\right)d\varphi \\ & + \sum_{\substack{h \\ k \text{ even}}} e^{-2\pi ihn/k}e^{(\pi ih'/2)(1-k/2)}\omega_{h,k} \\ & \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi nz/k}e^{(\pi/(12k))(z^{-1}-z)}z^{1/2}\left(\frac{1}{4z}b^e(q_1)\right)d\varphi \\ & + \sum_{\substack{h \\ k \text{ odd}}} e^{-2\pi ihn/k}e^{(\pi ih'/2)(1-k/2)}\omega_{h,k} \\ & \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi nz/k}e^{(\pi/(12k))(z^{-1}-z)}z^{1/2}\left(\frac{-(-1)^{(k+1)/2}q_1^{3/8}}{4z(q_1)_\infty}b^o(0;q_1)\right)d\varphi \\ & + \sum_{h,k}\frac{1}{4k}e^{-2\pi ihn/k}\omega_{h,k}\sum_\nu (-1)^\nu e^{-3\pi i\nu/k-3\pi ih'\nu^2/k} \end{aligned} \tag{6.19}$$

$$\times \sum_{\delta=-1}^{0} e^{(\pi i/k)(2\delta+1)(1/2+h'\nu)} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{1/2} e^{(2\pi z/k)(n-1/24)}$$
$$\times \left(I_k^-(\nu,\delta;0,z) - I_k^+(\nu,\delta;0,z) - \frac{1}{4}(J_k^-(\nu,\delta;z) + J_k^+(\nu,\delta;z)) \right) d\varphi.$$

We denote the first three summands in (6.19) by $\sum_{11}$, $\sum_{12}$ and $\sum_{13}$ respectively, and the last summand by $\sum_2$. We will examine $\sum_{11}$ first and start with the contribution coming from

$$\frac{1}{96z^2(q_1)_\infty} =: \frac{1}{z^2}\left(\frac{1}{96} + \sum_{r>0} a(r) q_1^r\right).$$

We consider the constant term and the term arising from $r \geq 1$ separately because they contribute to the main term and the error term, respectively, and denote them as S_1 and S_2. We first estimate S_2. We will use the facts that $\mathrm{Re}(z) = k/n$, $\mathrm{Re}(z^{-1}) > k/2$, $|z|^{1/2} \geq k/n$, and that

$$\vartheta'_{h,k} + \vartheta''_{h,k} \leq 2/(k(N+1)).$$

Since $k_1, k_2 \leq N$, we can split the integral in S_2 as

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-1/(k(N+k))}^{1/(k(N+k))} + \int_{-1/(k(k+k_1))}^{-1/(k(N+k))} + \int_{1/(k(N+k))}^{1/(k(k+k_2))} =: S_{21} + S_{22} + S_{23}.$$

We first consider S_{21}. Using Lemma 6.7, and that $r \geq 1$, we find that

$$S_{21} \ll \Bigg| \sum_{r\geq 1} a(r) \sum_k \sum_h \omega_{h,k} e^{-2\pi i h n/k + 2\pi i r h'/k}$$
$$\times \int_{-1/(k(N+k))}^{1/(k(N+k))} z^{-3/2} e^{(2\pi z/k)(n-1/24)-(2\pi/(kz))(r-1/24)} \, d\varphi \Bigg|$$
$$\ll \sum_{r\geq 1} |a(r)| e^{-\pi r} n \sum_k (24n-1,k)^{1/2} k^{-2+\varepsilon} \ll n^{1+\varepsilon}.$$

Since S_{22} and S_{23} are estimated similarly, for brevity, we consider only S_{22}. Writing

$$\int_{-1/(k(k+k_1))}^{-1/(k(N+k))} = \sum_{l=k_1+k}^{N+k-1} \int_{-1/(kl)}^{-1/(k(l+1))},$$

we have

(6.20)

$$S_{22} \ll \Big| \sum_{r\ge 1} a(r) \sum_k \sum_{l=k_1+k}^{N+k-1} \int_{-1/(kl)}^{-1/(k(l+1))} z^{-3/2} e^{(2\pi z/k)(n-1/24)-(2\pi/(kz))(r-1/24)}\, d\varphi \times \sum_{\substack{h \\ N<k+k_1\le l}} \omega_{h,k} e^{-2\pi i hn/k+2\pi i rh'/k} \Big|.$$

From the theory of Farey fractions, we conclude that $k_1 \equiv -h' \pmod{k}$, $N-k < k_1 \le N$, and $k_2 \equiv h' \pmod{k}$, $N-k < k_2 \le N$. Thus, (6.20) can be estimated in a manner similar to S_{21}, using Lemma 6.7.

By the same argument as above, we can show that the terms with positive exponents in the Fourier expansions of $\sum_{11}$ and $\sum_{12}$, as well as all of $\sum_{13}$ introduce an error of $O(n^{1+\varepsilon})$. Thus, after some simplification, we find

$$\begin{aligned}\sum\nolimits_1 &:= \sum\nolimits_{11} + \sum\nolimits_{12} + \sum\nolimits_{13} \\ &= \sum_{h,k} e^{-2\pi i hn/k} \omega_{h,k} \\ &\quad \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi nz/k} e^{(\pi/(12k))(z^{-1}-z)} \left(\frac{1}{96 z^{3/2}} - \frac{3k}{16\pi z^{1/2}} - \frac{5z^{1/2}}{96} \right) d\varphi \\ &\quad + \sum_{\substack{h \\ k \text{ even}}} e^{-2\pi i hn/k} e^{(\pi i h'/2)(1-k/2)} \omega_{h,k} \\ &\qquad \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{2\pi nz/k} e^{(\pi/(12k))(z^{-1}-z)} \left(-\frac{1}{16 z^{1/2}} \right) d\varphi + O(n^{1+\varepsilon}).\end{aligned}$$

We symmetrize the path of integration by writing

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-1/(kN)}^{1/(kN)} - \int_{-1/(kN)}^{-1/(k(k+k_1))} - \int_{1/(k(k+k_2))}^{1/(kN)},$$

and we denote the associated sums by S_{11}, S_{12} and S_{13}, respectively. We will show that the sums S_{12} and S_{13} contribute to the error term. We will consider S_{12} only, noting that S_{13} can be estimated similarly. Again, we consider only the first summand since the other pieces can be shown to have error of at most $O(n^{1+\varepsilon})$. Writing

$$\int_{-1/(kN)}^{-1/(k(k+k_1))} = \sum_{l=N}^{k+k_1-1} \int_{-1/(kl)}^{-1/(k(l+1))},$$

and using the facts that $\mathrm{Re}(z) = k/n$, $\mathrm{Re}(z^{-1}) < 4k$ (for $-1/(kN) \le \varphi \le$

$-1/(k(k+k_1)))$, and $|z|^2 \geq k^2/n^2$, we can estimate the previous integral as we estimated S_2 (using Lemma 6.7) as $O(n^{1+\varepsilon})$. Thus

$$(6.21)\qquad \sum\nolimits_1 = \sum_{h,k} e^{-2\pi ihn/k}\omega_{h,k} \times \int_{-1/(kN)}^{1/(kN)} e^{2\pi nz/k} e^{(\pi/(12k))(z^{-1}-z)} \left(\frac{1}{96z^{3/2}} - \frac{3k}{16\pi z^{1/2}} - \frac{5z^{1/2}}{96}\right) d\varphi + \sum_{\substack{h \\ k \text{ even}}} e^{-2\pi ihn/k} e^{(\pi ih'/2)(1-k/2)}\omega_{h,k} \times \int_{-1/(kN)}^{1/(kN)} e^{2\pi nz/k} e^{(\pi/(12k))(z^{-1}-z)} \left(-\frac{1}{16z^{1/2}}\right) d\varphi + O(n^{1+\varepsilon}).$$

To finish our estimate of $\sum_1$, we consider integrals of the form

$$I_{k,r} := \int_{-1/(kN)}^{1/(kN)} z^r \cdot e^{2\pi nz/k + \pi/(12k)(z^{-1}-z)}\, d\varphi,$$

where $r \in \{-3/2, -1/2, 1/2\}$. As shown in [4],

$$(6.22)\qquad I_{k,r} = \frac{2\pi}{k}(24n-1)^{-(r+1)/2} I_{r+1}\left(\frac{\pi\sqrt{24n-1}}{6k}\right) + O(n^{1+\varepsilon}).$$

Using (6.22), (6.21) becomes

(6.23)

$$\sum\nolimits_1 = \sum_{k=1}^{\lfloor n^{1/2}\rfloor} A_k(n)\left[\frac{\pi(24n-1)^{1/4}}{48k} I_{-1/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) - \frac{3}{8(24n-1)^{1/4}} \times I_{1/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) - \frac{5\pi}{48k(24n-1)^{3/4}} I_{3/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right)\right] - \sum_{\substack{k=1 \\ k \text{ even}}}^{\lfloor n^{1/2}\rfloor} A_k^e(n) \frac{\pi}{8k(24n-1)^{1/4}} I_{1/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) + O(n^{1+\varepsilon}),$$

where $A_k(n)$ and $A_k^e(n)$ are defined by

$$(6.24)\qquad A_k(n) := \sum_{h \,(\mathrm{mod}\, k)^\star} \omega_{h,k} e^{-2\pi ihn/k},$$

$$(6.25)\qquad A_k^e(n) := \sum_{h \,(\mathrm{mod}\, k)^\star} \omega_{h,k} e^{-2\pi ihn/k + \pi ih'(1-k/2)/2}.$$

We next consider $\sum_2$ of (6.19). We first note that

$$\sum_{\nu}\sum_{\delta=-1}^{0}\frac{kn^{-1/4}}{|\nu-(2\delta+1)/6|}\ll k^2n^{-1/4},$$

$$\sum_{\nu}\sum_{\delta=-1}^{0}\frac{kn^{1/4}}{|k-2(\nu-(2\delta+1)/6)|}\ll k^2n^{-1/4},$$

and similarly that

$$\sum_{\nu}\sum_{\delta=-1}^{0}\frac{k^2n^{-1/4}}{|\nu-(2\delta+1)/6|^2}\ll k^2n^{-1/4},$$

$$\sum_{\nu}\sum_{\delta=-1}^{0}\frac{k^2n^{1/4}}{|k-2(\nu-(2\delta+1)/6)|^2}\ll k^2n^{-1/4}.$$

Thus, using Lemmas 6.5 and 6.6, we have

$$\sum\nolimits_2 \ll n^{-1/4}\sum_k(24n-1,k)^{1/2}k^{3/2+\varepsilon}\ll n^{3/4+\varepsilon}. \tag{6.26}$$

Combining the estimates for $\sum_1$ and $\sum_2$ given in (6.23) and (6.26) respectively, and replacing k by ℓ, gives Theorem 1.3. ∎

Proof of Corollary 1.4. The corollary follows from Theorem 1.3 and the fact that for $x\to\infty$,

$$I_a(x)\sim\frac{1}{\sqrt{2\pi x}}e^x. \ ∎$$

References

[1] G. E. Andrews, *On the theorems of Watson and Dragonette for Ramanujan's mock theta functions*, Amer. J. Math. 88 (1966), 454–490.

[2] G. E. Andrews, *Partitions, Durfee symbols, and the Atkin–Garvan moments of ranks*, Invent. Math. 169 (2007), 37–73.

[3] G. E. Andrews and K. Eriksson, *Integer Partitions*, Cambridge Univ. Press, 2004.

[4] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J. 144 (2008), 195–233.

[5] K. Bringmann, F. Garvan, and K. Mahlburg, *Partition statistics and quasiharmonic Maass forms*, Int. Math. Res. Notices 2009, 63–97.

[6] K. Bringmann and B. Kane, *Second order cusp forms and mixed modular forms*, Ramanujan J., special volume in honour of Ismail and Stanton, to appear.

[7] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. 165 (2006), 243–266.

[8] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. of Math. 171 (2010), 419–449.

[9] K. Bringmann, K. Ono, and R. Rhoades, *Eulerian series as modular forms*, J. Amer. Math. Soc. 21 (2008), 1085–1104.

[10] J. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. 125 (2004), 45–90.

[11] C. Conley and M. Raum, *Harmonic Maass–Jacobi forms of degree 1 with higher rank indices*, preprint.
[12] L. A. Dragonette, *Some asymptotic formulae for the mock theta series of Ramanujan*, Trans. Amer. Math. Soc. 72 (1952), 474–500.
[13] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progr. Math. 55, Birkhäuser Boston, Boston, MA, 1985.
[14] J. Funke and J. Millson, *The geometric theta correspondence for Hilbert modular surfaces*, preprint.
[15] B. Gordon and R. McIntosh, *Modular transformations of Ramanujan's fifth and seventh order mock theta functions*, Ramanujan J. 7 (2003), 193–222.
[16] B. Gordon and R. McIntosh, *A survey of classical mock theta functions*, in: Partitions, q-series and Modular Forms, Dev. Math. 23, Springer, New York, 2012, 95–144.
[17] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. 36 (1976), 57–113.
[18] M. Kaneko and D. Zaiger, *A generalized Jacobi theta function and quasimodular forms*, in: The Moduli Space of Curves (Texel Island, 1994), Progr. Math. 129, Birkhaüser Boston, MA 1995, 165–172.
[19] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, in: Current Developments in Mathematics (2008), Int. Press, Somerville, MA, 2009, 347–454.
[20] H. Rademacher, *Topics in Analytic Number Theory*, Grundlehren Math. Wiss. 169, Springer, Berlin, 1973.
[21] R.C. Rhoades, *Asymptotics for the number of strongly unimodal sequences*, Int. Math. Res. Notices, to appear.
[22] G. N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc. 11 (1936), 55–80.
[23] D. Zagier, *Nombres de classes et formes modulaires de poids* 3/2, C. R. Acad. Sci. Paris Sér. A-B 281 (1975), 883–886.
[24] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono–Bringmann)*, in: Séminaire Bourbaki Vol. 2007/2008, exp. 986, Astérisque 326 (2009), vii–viii, 143–164.
[25] D. Zagier, talk at the conference: Mock Theta Functions and Applications in Combinatorics, Algebraic Geometry and Mathematical Physics, Max Planck Inst. Math., Bonn, 2009.
[26] S. Zwegers, *Mock theta functions*, PhD thesis, Universiteit Utrecht, 2002.
[27] S. Zwegers, *Multivariable Appell functions*, preprint.

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Received on 19.6.2012
and in revised form on 12.10.2012 (7106)