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The asymptotic distribution of traces of Maass–Poincaré series

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Abstract

We establish an asymptotic formula with a power savings in the error term for traces of CM values of a family of Maass–Poincaré series which contains the modular j-function as a special case. By work of Borcherds (1998) [2], Zagier (2002) [31], and Bringmann and Ono (2007) [4], these traces are Fourier coefficients of half-integral weight weakly holomorphic modular forms and Maass forms.

Keywords: Asymptotic distribution; CM points; Maass–Poincaré series; Traces; Fourier coefficients

1. Introduction and statements of results

Let $j(z)$ be the classical modular j-function for $\mathrm{SL}_2(\mathbb{Z})$,

$$j(z) = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \cdots,$$

where $q = e^{2\pi i z}$. The values of $j(z)$ at CM points in the complex upper half-plane $\mathbb{H}$ are known as *singular moduli*. These are algebraic integers which play a fundamental role in number theory. For example, they generate class fields of imaginary quadratic fields, and parameterize isomorphism classes of elliptic curves with complex multiplication.

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In the influential paper [31], Zagier proved that traces of singular moduli are Fourier coefficients of a half-integral weight modular form and used this to give another proof of a famous theorem of Borcherds [2] on infinite product expansions of integer weight modular forms with Heegner divisor. Zagier's results have inspired a large number of works in recent years. An overview of some of these works can be found in the survey articles of Ono [26,27].

In this paper we will study the asymptotic distribution of "twisted traces" of CM values of a family of Maass–Poincaré series which contains the j-function as a special case. Let $m \in \mathbb{Z}$ with $m \neq 0$, and let $s \in \mathbb{C}$. Define the Maass–Poincaré series

$$F_m(z,s) := 2\pi |m|^{s-\frac{1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^{\frac{1}{2}} I_{s-\frac{1}{2}}\big(2\pi |m| \,\mathrm{Im}(\gamma z)\big) e\big(m\,\mathrm{Re}(\gamma z)\big), \quad \mathrm{Re}(s) > 1,$$

where $\Gamma_\infty < \mathrm{SL}_2(\mathbb{Z})$ is the subgroup of elements stabilizing the cusp at ∞, I_ν is the Bessel function of order ν, and $e(z) := e^{2\pi i z}$. This Poincaré series was first studied by Niebur in [25], and it has since appeared in many different contexts (see e.g. [4–7,12,13,24,18]).

Niebur [25] proved that $F_m(z,s)$ has an analytic continuation to $\mathrm{Re}(s) = 1$, and showed that for each $m \in \mathbb{Z}^-$,

$$F_m(z,1) = j_{|m|}(z) + 24\sigma\big(|m|\big), \tag{1.1}$$

where $j_{|m|}(z)$ is the unique modular function satisfying

$$j_{|m|}(z) = q^{-|m|} + O(q)$$

and $\sigma(|m|) := \sum_{\ell \mid |m|} \ell$ is the divisor function. In particular,

$$j_1(z) = j(z) - 744.$$

The twisted traces of CM values of $F_m(z,s)$ are defined as follows. Let $-D < 0$ be an odd fundamental discriminant. Let $\mathcal{Q}_D$ be the set of positive definite, primitive, integral binary quadratic forms $Q(X,Y) = [a,b,c] = aX^2 + bXY + cY^2$ of discriminant $b^2 - 4ac = -D$. Then $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ in the usual way. Let $\Lambda_D(1)$ be the set of Heegner points of discriminant $-D$ on the modular curve $X_0(1)$. There are exactly $h(-D)$ such points, where $h(-D)$ is the ideal class number of $K = \mathbb{Q}(\sqrt{-D})$. Furthermore, there is a bijection $\mathcal{Q}_D/\mathrm{SL}_2(\mathbb{Z}) \to \Lambda_D(1)$ given by $[Q] \mapsto z_Q$ where

$$z_Q = \frac{-b + \sqrt{-D}}{2a}$$

is the unique root in $\mathbb{H}$ of the dehomogenized form $Q(X,1) = aX^2 + bX + c$.

Let $d > 0$ be an odd fundamental discriminant (possibly 1) coprime to D. Define the genus character

$$\chi_d : \mathcal{Q}_{dD}/\mathrm{SL}_2(\mathbb{Z}) \to \{\pm 1\}$$

by

$$\chi_d(Q) = \chi_d(a,b,c) := \left(\frac{d}{n}\right)$$

where n is any integer coprime to d which is represented by $Q = [a,b,c] \in \mathcal{Q}_{dD}$. Here $(d/\cdot)$ is the usual Kronecker symbol. This definition is independent of the choice of n (see Section 4).

The "twisted trace" of $F_m(z,s)$ is defined by

$$\mathrm{Tr}_d\big(F_m(\cdot,s); D\big) := \sum_{z_Q \in \Lambda_{dD}(1)} \chi_d(Q) F_m(z_Q, s).$$

Zagier [31] proved that the traces $\mathrm{Tr}_d(j_1; D)$ are Fourier coefficients of a weight 3/2 weakly holomorphic modular form for $\Gamma_0(4)$. Generalizing many results in [31], Bringmann and Ono [4] proved that the traces $\mathrm{Tr}_d(F_m(\cdot,s); D)$ are Fourier coefficients of certain half-integral weight Maass forms (see also [12, Proposition 6]). For example, if $s \in \mathbb{Z}^+$ is an odd integer, one has

$$b_s(-d, D) = \frac{(-1)^{\frac{s+1}{2}} D^{\frac{s-1}{2}}}{d^{\frac{s}{2}}} \mathrm{Tr}_d\big(F_{-1}(\cdot,s); D\big),$$

where the numbers $b_s(-d, D)$ are Fourier coefficients of a weight $s + 1/2$ Maass form

$$P_s(-d; z) \in M^+_{s+\frac{1}{2}}\big(\Gamma_0(4)\big)$$

in Kohnen's plus-space with Fourier expansion at ∞ of the form

$$P_s(-d; z) = q^{-d} + \sum_{\substack{n \geqslant 0 \\ -n \equiv 0,1 \bmod 4}} b_s(-d,n) q^n.$$

In light of these results, it is natural to study the asymptotic distribution of the traces $\mathrm{Tr}_d(F_m(\cdot,s); D)$ as $D \to \infty$. For traces of singular moduli, this problem is closely related to the classical observation that the number

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.9999999999992\ldots$$

is nearly an integer (see [3,26]). In [8], Bruinier, Jenkins, and Ono conjectured that a certain "perturbed" average of the traces

$$\frac{\mathrm{Tr}_1(j_1; D) - G^{\mathrm{red}}(D) - G^{\mathrm{old}}(D)}{h(-D)} \to -24$$

as $D \to \infty$. See [26] for an explanation of why the number -24 appears in this limit. This conjecture was studied by Duke [10] from a somewhat different perspective. Using the equidistribution of Heegner points on $X_0(1)$ and a regularization of the pole at ∞ of $j(z)$, he established the following result which implies the conjecture of Bruinier, Jenkins, and Ono,

$$\frac{1}{h(-D)}\left(\mathrm{Tr}_1(j_{|m|}; D) - \frac{1}{2}\sum_{\substack{0<c<2\sqrt{D}\\ c\equiv 0 \ (\mathrm{mod}\ 4)}} S_D\big(|m|, c\big)\exp\left(\frac{4\pi|m|\sqrt{D}}{c}\right)\right) \to -24$$

as $D \to \infty$, where $S_D(|m|, c)$ is the exponential sum

$$S_D\big(|m|, c\big) := \sum_{x^2\equiv -D \ (\mathrm{mod}\ c)} e\left(\frac{2|m|x}{c}\right).$$

We now state our main results. For $n \in \mathbb{Z}^+$, define the twisted exponential sum

$$S_{D,d}(n, c) := \sum_{x^2\equiv -dD \ (\mathrm{mod}\ c)} \chi_d\left(\frac{c}{4}, x, \frac{x^2+dD}{c}\right) e\left(\frac{2nx}{c}\right),$$

and for $m \in \mathbb{Z}^-$ and $s \in \mathbb{Z}^+$, define the function

$$g_{m,s}^{D,d}(c) := |m|^{s-1}\sum_{j=0}^{s-1}\left(\frac{-\sqrt{dD}}{2\pi|m|c}\right)^j \frac{(s-1+j)!}{(s-1-j)!j!}.$$

Let $\mathcal{F}$ be the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. For $Y > 0$, define the truncated domain

$$\mathcal{F}_Y := \big\{z \in \mathcal{F}\colon \mathrm{Im}(z) \leqslant Y\big\},$$

and for $m \in \mathbb{Z}^-$, define the "regularized" integral

$$\int\limits_{reg} F_m(z, s)\, d\mu := \lim_{Y\to\infty} \int\limits_{\mathcal{F}_Y} F_m(z, s)\, d\mu,$$

where $d\mu = (3/\pi)\, dx\, dy/y^2$ is the normalized hyperbolic measure on the open modular curve $Y_0(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

We will establish the following asymptotic formula with a power savings in the error term for the traces $\mathrm{Tr}_d(F_m(\cdot, s); D)$ as $dD \to \infty$. An outline of the proof is given in Section 2.

Theorem 1.1. *Let $m \in \mathbb{Z}^-$, $s \in \mathbb{Z}^+$, and let $-D < 0$ and $d > 0$ be odd, coprime fundamental discriminants. Then there exists an effective constant $a > 0$ such that for all $\delta < 1/16$ and $0 < b < \delta/a$, the following asymptotic formulas hold*:

(1) *If $d = 1$ then*

$$\frac{1}{h(-D)}\left(\mathrm{Tr}_1\big(F_m(\cdot, s); D\big) - \frac{1}{2}\sum_{\substack{0<c<\frac{2\sqrt{D}}{1+D^{-b}}\\ c\equiv 0 \ (\mathrm{mod}\ 4)}} S_{D,1}\big(|m|, c\big) g_{m,s}^{D,1}(c)\exp\left(\frac{4\pi|m|\sqrt{D}}{c}\right)\right)$$

$$= \int\limits_{reg} F_m(z, s)\, d\mu + O\big(D^{-(\delta-ab)}\big) + O\big(D^{-b}\big)$$

as $D \to \infty$, where

$$\int\limits_{reg} F_m(z,s)\,d\mu = \begin{cases} 0, & \text{if } s = 1, \\ -24\sigma(|m|)|m|^{s-1}, & \text{if } s \geqslant 2. \end{cases}$$

(2) *If $d > 1$ then*

$$\frac{1}{h(-dD)}\left(\mathrm{Tr}_d\big(F_m(\cdot,s);D\big) - \frac{1}{2}\sum_{\substack{0<c<\frac{2\sqrt{dD}}{1+(dD)^{-b}}\\ c\equiv 0 \ (\mathrm{mod}\ 4)}} S_{D,d}\big(|m|,c\big) g_{m,s}^{D,d}(c)\exp\left(\frac{4\pi|m|\sqrt{dD}}{c}\right)\right)$$
$$= O\big((dD)^{-(\delta-ab)}\big) + O\big((dD)^{-b}\big)$$

as $dD \to \infty$.

If we let $s = 1$ in Theorem 1.1 and use the relation (1.1), we obtain the following asymptotic formula with a power savings in the error term for the traces $\mathrm{Tr}_d(j_{|m|};D)$ as $dD \to \infty$.

Corollary 1.2. *Let $m \in \mathbb{Z}^-$, and let $-D < 0$ and $d > 0$ be odd, coprime fundamental discriminants. Then there exists an effective constant $a > 0$ such that for all $\delta < 1/16$ and $0 < b < \delta/a$, the following asymptotic formulas hold*:

(1) *If $d = 1$ then*

$$\frac{1}{h(-D)}\left(\mathrm{Tr}_1(j_{|m|};D) - \frac{1}{2}\sum_{\substack{0<c<\frac{2\sqrt{D}}{1+D^{-b}}\\ c\equiv 0 \ (\mathrm{mod}\ 4)}} S_{D,1}\big(|m|,c\big)\exp\left(\frac{4\pi|m|\sqrt{D}}{c}\right)\right)$$
$$= -24\sigma\big(|m|\big) + O\big(D^{-(\delta-ab)}\big) + O\big(D^{-b}\big)$$

as $D \to \infty$.

(2) *If $d > 1$ then*

$$\frac{1}{h(-dD)}\left(\mathrm{Tr}_d(j_{|m|};D) - \frac{1}{2}\sum_{\substack{0<c<\frac{2\sqrt{dD}}{1+(dD)^{-b}}\\ c\equiv 0 \ (\mathrm{mod}\ 4)}} S_{D,d}\big(|m|,c\big)\exp\left(\frac{4\pi|m|\sqrt{dD}}{c}\right)\right)$$
$$= O\big((dD)^{-(\delta-ab)}\big) + O\big((dD)^{-b}\big)$$

as $dD \to \infty$.

We conclude by noting that in [14], we used techniques related to those in the proof of Theorem 1.1 to study the asymptotic distribution of the *partition function* $p(n)$, which counts the number of partitions of a positive integer n. Let $m \in \mathbb{Z}$ with $m \neq 0$, and let $s \in \mathbb{C}$. For $N \in \mathbb{Z}^+$, define the $\Gamma_0(N)$-invariant Maass–Poincaré series

$$F_m^N(z,s) := 2\pi |m|^{s-\frac{1}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \operatorname{Im}(\gamma z)^{\frac{1}{2}} I_{s-\frac{1}{2}}\big(2\pi |m| \operatorname{Im}(\gamma z)\big) e\big(m \operatorname{Re}(\gamma z)\big), \quad \operatorname{Re}(s) > 1$$

where $\Gamma_\infty < \Gamma_0(N)$ is the subgroup of elements stabilizing the cusp at ∞ of $X_0(N)$.

In [5], Bringmann and Ono established the following arithmetic reformulation of Rademacher's [28] exact formula for $p(n)$,

$$p(n) = \frac{1}{D_n} \sum_{z_Q \in \Lambda_{D_n}(6)} \chi_{12}(Q) F_{-1}^6(z_Q, 2),$$

where $D_n := 24n - 1$ is square-free, $\chi_{12} := (12/\cdot)$ is the Legendre symbol, and $\Lambda_{D_n}(6)$ is the set of Heegner points of discriminant $-D_n$ on $X_0(6)$ (see Section 3). In [14], we used this formula and the equidistribution of Galois orbits of Heegner points on the modular curve $X_0(6)$ to obtain a new asymptotic formula for $p(n)$ with an effective error term which sharpens the classical bounds of Hardy and Ramanujan, Rademacher, and Lehmer on the error term in Rademacher's exact formula for $p(n)$. The estimates in Lemmas 9.1 and 10.1 of this paper are crucial to the proofs in [14], so this paper can in some respects be viewed as a companion to [14].

2. Outline of the proof of Theorem 1.1

In this section we outline the proof of Theorem 1.1.

In Section 5 we establish an asymptotic formula for twisted traces of smooth, $\Gamma_0(N)$-invariant functions which are allowed to grow moderately in the cusps of $X_0(N)$ (see Theorem 5.1). For example, if $N = 1$ suppose that $F : \mathbb{H} \to \mathbb{C}$ is a smooth, $\mathrm{SL}_2(\mathbb{Z})$-invariant function such that for all $n \in \mathbb{Z}_{\geqslant 0}$,

$$\Delta^n\big(F(z) - c \cdot y^\alpha\big) = O\big(e^{-c_1 \cdot y}\big) \quad \text{as } y \to \infty \tag{2.1}$$

for some $c \in \mathbb{C}$, $c_1 > 0$ and $\alpha \leqslant 1/2$. Here Δ^n is the n-th iterate of the hyperbolic Laplacian

$$\Delta = -y^2\big(\partial_x^2 + \partial_y^2\big).$$

We show there exists an integer $n_0 > 0$ such that for all $\delta < 1/16$,

$$\frac{1}{h(-dD)} \operatorname{Tr}_d(F; D) = \delta_{d,1} \int_{\mathcal{F}} F(z)\, d\mu + O_\delta\big(\big(\big\|\Delta^{n_0} F_{T_0}\big\|_2 + 1\big)(dD)^{-\delta}\big) \tag{2.2}$$

as $dD \to \infty$. Here $\delta_{d,1}$ is the Kronecker diagonal symbol, $T_0 > 1$ is a fixed cutoff parameter, and $F_{T_0} := F - \eta_{T_0}$ where $\eta_{T_0} : \mathbb{H} \to \mathbb{C}$ is a smooth, $\mathrm{SL}_2(\mathbb{Z})$-invariant function with growth coinciding precisely with that of F for $y \gg T_0$.

The Fourier expansion of the Maass–Poincaré series $F_m(z,s)$ is of the form

$$F_m(z,s) = O\big(e^{c_2 \cdot y}\big) + O\big(y^{1-s}\big) + O\big(e^{-c_3 \cdot y}\big) \quad \text{as } y = \operatorname{Im}(z) \to \infty$$

where c_2, c_3 are positive constants. In Section 6 we use a variant of an argument of Duke [10] to construct a smooth, $\mathrm{SL}_2(\mathbb{Z})$-invariant Poincaré series $\mathscr{P}_{m,\varepsilon}$ for each $\varepsilon > 0$ such that the regularized function

$$F_{m,s,\varepsilon}(z) := F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)$$

satisfies the growth condition (2.1) with $\alpha := 1 - s$.

In Section 7 we substitute $F_{m,s,\varepsilon}$ into (2.2) to obtain the "preliminary" asymptotic formula

$$\frac{1}{h(-dD)}\big(\mathrm{Tr}_d\big(F_{m,s}(\cdot,s);D\big) - \mathrm{Tr}_d\big(\mathscr{P}_{m,\varepsilon}(\cdot,s);D\big)\big)$$
$$= \delta_{d,1}\int\limits_{\mathcal{F}} F_{m,s,\varepsilon}(z)\,d\mu + O_\delta\big(\big(\big\|\Delta^{n_0}F_{m,s,\varepsilon,T_0}\big\|_2 + 1\big)(dD)^{-\delta}\big)$$

as $dD \to \infty$. We then show that $\mathrm{Tr}_d(\mathscr{P}_{m,\varepsilon}(\cdot,s);D)$ equals the main term in Theorem 1.1 plus an explicit error term depending on ε and dD. See Eq. (7.3), and Sections 9 and 11.

In Section 8 we show that for all $\varepsilon > 0$,

$$\int\limits_{\mathcal{F}} F_{m,s,\varepsilon}(z)\,d\mu = \int\limits_{reg} F_m(z,s)\,d\mu,$$

and in Section 12 we use a Borcherds-type integration to evaluate this regularized integral.

In Section 10.1 we establish the upper bound

$$\big\|\Delta^{n_0}F_{m,s,\varepsilon,T_0}\big\|_2 \ll \varepsilon^{-A}$$

as $\varepsilon \to 0$ for some fixed positive integer A depending on n_0.

Finally, by choosing ε to be a sufficiently small negative power of dD, we obtain the final asymptotic formula with a power savings in the error term.

3. Heegner points on $X_0(N)$

In this section we review some facts concerning Heegner points on the modular curve $X_0(N)$.

Let N be a positive integer, and let $-D < 0$ be an odd fundamental discriminant coprime to N such that every prime divisor p of N is split in $K = \mathbb{Q}(\sqrt{-D})$. Let $\mathcal{Q}_{D,N}$ be the set of positive definite, primitive, integral binary quadratic forms $Q(X,Y) = [a,b,c] = aX^2 + bXY + cY^2$ of discriminant $b^2 - 4ac = -D$ with $N|a$. The set $\mathcal{Q}_{D,N}$ is stable under the action of $\Gamma_0(N)$.

Let $\Lambda_D(N)$ be the set of Heegner points of discriminant $-D$ on $X_0(N)$. There is a bijection

$$\mathcal{Q}_{D,N}/\Gamma_0(N) \to \Lambda_D(N)$$

given by $[Q] \mapsto z_Q$ where

$$z_Q = \frac{-b + \sqrt{-D}}{2Na}$$

is the unique root in $\mathbb{H}$ of the dehomogenized form $Q(X,1) = NaX^2 + bX + c$.

Fix a solution $r \bmod 2N$ of $r^2 \equiv -D \bmod 4N$. Note that there are exactly $2^{t(N)}$ such solutions r, where $t(N)$ is the number of distinct prime divisors of N. Define the subset of forms

$$\mathcal{Q}_{D,N,r} = \{Q = [a,b,c] \in \mathcal{Q}_{D,N}\colon\ b \equiv r \bmod 2N\}.$$

The set $\mathcal{Q}_{D,N,r}$ is also stable under the action of $\Gamma_0(N)$. There is a decomposition (see [15, p. 507])

$$\mathcal{Q}_{D,N}/\Gamma_0(N) = \bigcup_{\substack{r \bmod 2N \\ r^2 \equiv -D \bmod 4N}} \mathcal{Q}_{D,N,r}/\Gamma_0(N). \tag{3.1}$$

The natural map

$$\mathcal{Q}_{D,N,r}/\Gamma_0(N) \to \mathcal{Q}_{D,1}/\mathrm{SL}_2(\mathbb{Z})$$

is a bijection which makes the set $\mathcal{Q}_{D,N,r}/\Gamma_0(N)$ into a group of order $h(-D)$ via the Gauss law of composition on $\mathcal{Q}_{D,1}/\mathrm{SL}_2(\mathbb{Z})$. Here, $h(-D)$ is the ideal class number of $K = \mathbb{Q}(\sqrt{-D})$. Moreover, by class field theory

$$\mathcal{Q}_{D,N,r}/\Gamma_0(N) \cong \mathrm{CL}_K \cong \mathrm{Gal}(H/K)$$

where CL_K is the ideal class group of K and H is the Hilbert class field of K. For details concerning these facts, see [9].

The set $\Lambda_D(N)$ is divided into $2^{t(N)}$ simple, transitive $\mathrm{Gal}(H/K)$-orbits of size $h(-D)$ (see [16, pp. 235–236]). Define the Galois orbit

$$\mathcal{O}_r := \{z_{Q_r}^{\sigma}\colon\ \sigma \in \mathrm{Gal}(H/K)\},$$

where $[Q_r]$ is any class in $\mathcal{Q}_{D,N,r}/\Gamma_0(N)$. Then one has a bijection

$$\mathcal{Q}_{D,N,r}/\Gamma_0(N) \to \mathcal{O}_r. \tag{3.2}$$

4. Generalized genus characters

In this section we briefly review the definition of generalized genus characters. For more details concerning these facts, see [15, pp. 508–510].

Let N be a positive integer. Let $-D < 0$ and $d > 0$ be odd, coprime fundamental discriminants such that both D and d are squares modulo $4N$. For $Q = [Na, b, c] \in \mathcal{Q}_{dD,N}$ set

$$\chi_d(Q) := \left(\frac{d}{n}\right),$$

where n is an integer coprime to d represented by the form $[N_1 a, b, N_2 c]$ for some decomposition $N = N_1 N_2$, $N_i > 0$. Such an n exists, and the value of $(\frac{d}{n})$ is independent of the choice of N_1, N_2 and n. The function χ_d is $\Gamma_0(N)$-invariant and thus defines a function

$$\chi_d\colon \mathcal{Q}_{dD,N}/\Gamma_0(N) \to \{\pm 1\}.$$

Furthermore, χ_d restricts to a real ideal class group character on $\mathcal{Q}_{dD,N,r}/\Gamma_0(N)$ for each $r \bmod 2N$ such that $r^2 \equiv -dD \bmod 4N$.

5. Asymptotics for twisted traces

Let $F:\mathbb{H} \to \mathbb{C}$ be a C^∞, $\Gamma_0(N)$-invariant function. We say that F has *cuspidal growth of power* α for some $\alpha \in \mathbb{R}$ if for every cusp $\mathfrak{a}$ of $\Gamma_0(N)$ there exists a constant $c_{\mathfrak{a}} \in \mathbb{C}$ (possibly equal to 0) such that for each integer $a \geqslant 0$,

$$\Delta^a\big(F(\sigma_{\mathfrak{a}} z) - c_{\mathfrak{a}} y^\alpha\big) = O\big(e^{-cy}\big) \quad \text{as } y = \operatorname{Im}(z) \to \infty$$

for some $c > 0$. Here $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ is a scaling matrix such that $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$.

Let $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ be the open modular curve, and let $d\mu$ be the normalized hyperbolic measure on $Y_0(N)$.

Theorem 5.1. *Let N be a fixed positive integer. Let $-D < 0$ and $d > 0$ be odd, coprime fundamental discriminants such that D and d are squares modulo $4N$, dD is coprime to N, and every prime divisor p of N is split in $K = \mathbb{Q}(\sqrt{-dD})$. Let $F:\mathbb{H} \to \mathbb{C}$ be a C^∞, $\Gamma_0(N)$-invariant function with cuspidal growth of power α for some $\alpha \leqslant 1/2$. Then there exists an integer $a_0 > 0$ such that for all $\delta < 1/16$,*

$$\frac{1}{2^{t(N)} h(-dD)} \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) F(z_Q)$$
$$= \delta_{d,1} \int\limits_{Y_0(N)} F(z)\, d\mu + O_{N,\delta}\big(\big(\big\|\Delta^{a_0} F_{T_0}\big\|_2 + 1\big)(dD)^{-\delta}\big)$$

as $dD \to \infty$. Here $\chi_d : \mathcal{Q}_{dD,N}/\Gamma_0(N) \to \{\pm 1\}$ is a generalized genus character as defined in Section 4, and F_{T_0} is a regularized version of F where $T_0 \gg 1$ is a fixed cutoff parameter (see Eq. (5.9)).

Remark 5.2. Theorem 5.1 should be viewed as consisting of two cases: $d = 1$ and $d > 1$. What separates these cases is that when the genus character χ_d is trivial, the integral in the formula survives, while if χ_d is nontrivial, the integral is killed by orthogonality. For the proof it is more convenient to combine these cases, so we have used the diagonal symbol $\delta_{d,1}$.

Proof of Theorem 5.1. We begin by constructing a C^∞, $\Gamma_0(N)$-invariant cutoff function with growth coinciding with that of F in the cusps $\mathfrak{a}$ of $X_0(N)$, and which vanishes on the Heegner points $\Lambda_{dD}(N)$ for each sufficiently large cutoff parameter (see also [21]).

Lemma 5.3. *Let $T > 1$. There exists a C^∞, $\Gamma_0(N)$-invariant function $\eta_T : \mathbb{H} \to \mathbb{C}$ such that*

$$\eta_T(\sigma_{\mathfrak{a}} z) = \begin{cases} 0, & 1 < y < T, \\ c_{\mathfrak{a}} y^\alpha \chi(y/T), & T \leqslant y \leqslant 2T, \\ c_{\mathfrak{a}} y^\alpha, & y > 2T, \end{cases}$$

for each cusp $\mathfrak{a}$ *of* $X_0(N)$, *where* $\chi : \mathbb{R}^+ \to [0, 1]$ *is a* C^∞ *function such that*

$$\chi(t) = \begin{cases} 0, & t < 1, \\ 1, & t > 2. \end{cases}$$

Proof. Let χ be as in the statement of the lemma, and define $\psi_T \in C_0^\infty(\mathbb{R}^+)$ by

$$\psi_T(t) := t^\alpha \chi\left(\frac{t}{T}\right).$$

Then define η_T by (see [19, (3.10)])

$$\eta_T(z) := \sum_{\mathfrak{a}} c_{\mathfrak{a}} \cdot E_{\mathfrak{a}}(z|\psi_T), \tag{5.1}$$

where

$$E_{\mathfrak{a}}(z|\psi_T) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \psi_T\big(\mathrm{Im}\big(\sigma_{\mathfrak{a}}^{-1}\gamma z\big)\big).$$

Now, by [19, (3.17)] with $m = 0$, one has the Fourier expansion

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z|\psi_T) = \delta_{\mathfrak{a}\mathfrak{b}}\psi_T(y) + \sum_{n \in \mathbb{Z}} e(nx) \sum_{c>0} S_{\mathfrak{a}\mathfrak{b}}(0, n; c) \int_{\mathbb{R}} \psi_T\left(\frac{c^{-2}y}{t^2 + y^2}\right) e(-nt)\, dt, \tag{5.2}$$

where $\delta_{\mathfrak{a}\mathfrak{b}}$ is the Kronecker diagonal symbol and $S_{\mathfrak{a}\mathfrak{b}}(0, n; c)$ is the exponential sum

$$S_{\mathfrak{a}\mathfrak{b}}(0, n; c) = \sum_{\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in B \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma_0(N) \sigma_{\mathfrak{b}} / B} e\left(\frac{na}{c}\right),$$

where B denotes the group of integral translations

$$B = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}$$

(see [19, (2.23)]). This, combined with the inequality

$$\min\left\{ c > 0 \colon \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma_0(N) \sigma_{\mathfrak{b}} \right\} \geqslant 1 \tag{5.3}$$

for all cusps $\mathfrak{a}$, $\mathfrak{b}$ of $X_0(N)$ (see [19, (2.28)–(2.31)]), shows that η_T has the properties stated in the lemma. □

Lemma 5.4. *For* $T \gg \sqrt{dD}$, *the function* $\eta_T(z)$ *defined by* (5.1) *vanishes on the Heegner points* $\Lambda_{dD}(N)$.

Proof. By Minkowski's theorem, every ideal class of $K=\mathbb{Q}(\sqrt{-dD})$ contains a primitive integral ideal $\mathfrak{A}$ of norm

$$N_{K/\mathbb{Q}}(\mathfrak{A}) \ll \sqrt{dD}. \tag{5.4}$$

Fix a solution $r \bmod 2N$ of $r^2 \equiv -dD \bmod 4N$, and write

$$\mathfrak{A}=\left[a, \frac{b+\sqrt{-dD}}{2}\right], \quad a=N_{K/\mathbb{Q}}(\mathfrak{A}),\ b \in \mathbb{Z},$$

with

$$b \equiv r \bmod 2N, \qquad b^2 \equiv -dD \bmod 4Na.$$

Then

$$z_{\mathfrak{A}}=\frac{-b+\sqrt{-dD}}{2Na}$$

is the Heegner point corresponding to $Q(X,Y)=NaX^2+bXY+cY^2$.

For $\gamma \in \Gamma_0(N)$ and a cusp $\mathfrak{a}$ of $X_0(N)$, write

$$\sigma_{\mathfrak{a}}^{-1}\gamma=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Then

$$\mathrm{Im}\big(\sigma_{\mathfrak{a}}^{-1}\gamma z_{\mathfrak{A}}\big)=\frac{\mathrm{Im}(z_{\mathfrak{A}})}{|c'z_{\mathfrak{A}}+d'|^2} \leqslant \frac{1}{|c'z_{\mathfrak{A}}+d'|^2}\frac{\sqrt{dD}}{2N}. \tag{5.5}$$

If $c'=0$, then $d'=1$ (see [19, (2.15)–(2.17)]) and we have

$$\mathrm{Im}\big(\sigma_{\mathfrak{a}}^{-1}\gamma z_{\mathfrak{A}}\big) \leqslant \frac{\sqrt{dD}}{2N}. \tag{5.6}$$

On the other hand, if $c' \neq 0$ we write

$$c'z_{\mathfrak{A}}+d'=\left(-\frac{bc'}{2Na}+d'\right)+i\left(\frac{\sqrt{dD}c'}{2Na}\right),$$

so that by (5.4),

$$\begin{aligned} \left|c'z_{\mathfrak{A}}+d'\right|^2 &= \left(-\frac{bc'}{2Na}+d'\right)^2+\left(\frac{\sqrt{dD}c'}{2Na}\right)^2 \\ &\geqslant \frac{(c')^2 dD}{4N^2a^2} \\ &\gg \frac{(c')^2}{4N^2}. \end{aligned} \tag{5.7}$$

It follows from (5.5) and (5.7) that

$$\operatorname{Im}\big(\sigma_{\mathfrak{a}}^{-1}\gamma z_{\mathfrak{A}}\big) \leqslant \frac{1}{|c' z_{\mathfrak{A}} + d'|^2}\frac{\sqrt{dD}}{2N} \ll \frac{4N^2}{(c')^2}\frac{\sqrt{dD}}{2N} \leqslant 2N\sqrt{dD}, \tag{5.8}$$

where for the last inequality we used (5.3).

Since $\psi_T(y) = 0$ for $y < T$, we see from the inequalities (5.6) and (5.8) that for $T \gg \sqrt{dD}$,

$$\psi_T\big(\operatorname{Im}\big(\sigma_{\mathfrak{a}}^{-1}\gamma z_{\mathfrak{A}}\big)\big) = 0$$

for all $\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)$. It follows from (5.1) that $\eta_T(z_{\mathfrak{A}}) = 0$. □

Now, define the "regularized" function

$$F_T(z) := F(z) - \eta_T(z).$$

Then by Lemma 5.4, to prove Theorem 5.1 it suffices to prove the following proposition.

Proposition 5.5. *For $T \gg \sqrt{dD}$, we have*

$$\frac{1}{2^{t(N)}h(-dD)} \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) F_T(z_Q) = \delta_{d,1} \int\limits_{Y_0(N)} F(z)\, d\mu + O_{N,\delta}\big(\big(\big\|\Delta^{a_0} F_{T_0}\big\|_2 + 1\big)(dD)^{-\delta}\big)$$

as $dD \to \infty$.

Proof. Let $T \geqslant T_0 \gg 1$. Here T_0 is a fixed cutoff parameter which is independent of dD. We introduce T_0 in order to decompose F_T into a sum of two functions so that we can isolate the contribution of η_T to the spectral decomposition. Consider the decomposition

$$F_T(z) = F_{T_0}(z) + \tilde{\eta}_T(z) \tag{5.9}$$

where

$$\tilde{\eta}_T(z) := \eta_{T_0}(z) - \eta_T(z).$$

We will first show that for all $\delta_1 < 1/16$,

$$\frac{1}{2^{t(N)}h(-dD)} \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) F_{T_0}(z_Q) = \delta_{d,1} \int\limits_{Y_0(N)} F(z)\, d\mu - \delta_{d,1} \int\limits_{Y_0(N)} \eta_{T_0}(z)\, d\mu + O\big(\big\|\Delta^{a_0} F_{T_0}\big\|_2 (dD)^{-\delta_1}\big)$$

as $dD \to \infty$.

By definition of η_{T_0} and our assumption that F has cuspidal growth of power α, we have

$$\Delta^a F_{T_0}(\sigma_{\mathfrak{a}} z) = O\left(e^{-cy}\right) \quad \text{as } y \to \infty$$

for each integer $a \geqslant 0$. The spectral decomposition of $L^2(Y_0(N))$ with respect to the hyperbolic Laplacian Δ yields the expansion

$$F_{T_0}(z) = \langle F_{T_0}, 1\rangle_2 + \sum_{n=1}^{\infty} \langle F_{T_0}, u_n\rangle_2 u_n(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2 E_{\mathfrak{a}}\left(z, \frac{1}{2} + it\right) dt$$

which converges pointwise absolutely and uniformly on compact subsets of $Y_0(N)$ since $\Delta^a F_{T_0}$ is C^∞ with exponential decay in each cusp $\mathfrak{a}$ of $Y_0(N)$. Here $u_0(z) = 1$ is the constant eigenfunction for Δ corresponding to the eigenvalue $\lambda_0 = 0$, $\{u_n(z)\}_{n=1}^{\infty}$ is an orthonormal basis of Maass cusp forms satisfying $\Delta u_n = \lambda_n u_n$ for $n \in \mathbb{Z}^+$ where the eigenvalues $\lambda_n = 1/4 + t_n^2$ are ordered so that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, and $E_{\mathfrak{a}}(z, s)$ is the real-analytic Eisenstein series

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \operatorname{Im}\left(\sigma_{\mathfrak{a}}^{-1} \gamma z\right)^s, \quad z \in \mathbb{H},\ \operatorname{Re}(s) > 1.$$

Summing the spectral expansion yields

$$\sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) F_{T_0}(z_Q) = \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) \int_{Y_0(N)} F_{T_0}(z)\, d\mu + \sum_{n=1}^{\infty} \langle F_{T_0}, u_n\rangle_2 W_n + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2 W_{\mathfrak{a}}(t)\, dt$$

where

$$\sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) = \begin{cases} 2^{t(N)} h(-dD), & d = 1, \\ 0, & d > 1, \end{cases}$$

and the twisted hyperbolic Weyl sums are defined by

$$W_n := \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) u_n(z_Q)$$

and

$$W_{\mathfrak{a}}(t) := \sum_{Q \in \mathcal{Q}_{dD,N}/\Gamma_0(N)} \chi_d(Q) E_{\mathfrak{a}}\left(z_Q, \frac{1}{2} + it\right).$$

Here $F_{T_0} \in L^1(Y_0(N))$ because $y^\alpha \in L^1([C, \infty), dy/y^2)$ for all $\alpha < 1$ and $C > 0$.

Using (3.1) we have

$$W_n = \sum_{\substack{r \bmod 2N \\ r^2 \equiv -dD \bmod 4N}} \sum_{Q \in \mathcal{Q}_{dD,N,r}/\Gamma_0(N)} \chi_d(Q) u_n(z_Q). \tag{5.10}$$

Furthermore, using the bijection (3.2) we have

$$\sum_{Q \in \mathcal{Q}_{dD,N,r}/\Gamma_0(N)} \chi_d(Q) u_n(z_Q) = \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi_d(\sigma) u_n\big(z_{Q_r}^{\sigma}\big).$$

As a consequence of deep work of Waldspurger [29] and Zhang [32], one has an identity of the form

$$\Bigg| \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi_d(\sigma) u_n\big(z_{Q_r}^{\sigma}\big) \Bigg|^2 = C_{u_n} \sqrt{dD} L\bigg(u_n \otimes \left(\frac{d}{\cdot}\right), \frac{1}{2}\bigg) L\bigg(u_n \otimes \left(\frac{-D}{\cdot}\right), \frac{1}{2}\bigg) \tag{5.11}$$

where

$$C_{u_n} \ll \big(1 + |t_n|^2\big)^{A_1}$$

for some fixed constant $A_1 > 0$, and if Δ denotes a fundamental discriminant (to be distinguished from our notation for the hyperbolic Laplacian), then $L(u_n \otimes (\frac{\Delta}{\cdot}), s)$ is the quadratic twist of the L-function of u_n by the Dirichlet character $(\frac{\Delta}{\cdot})$. Here we used the factorization of the Rankin–Selberg L-function

$$L(u_n \otimes \Theta_{\chi_d}, s) = L\bigg(u_n \otimes \left(\frac{d}{\cdot}\right), s\bigg) L\bigg(u_n \otimes \left(\frac{-D}{\cdot}\right), s\bigg)$$

where Θ_{χ_d} is the theta series associated to the genus character χ_d. This follows as a consequence of Kronecker's factorization of the class group L-function of $K = \mathbb{Q}(\sqrt{-dD})$ associated to the genus character χ_d,

$$L(\chi_d, s) = L\bigg(\left(\frac{d}{\cdot}\right), s\bigg) L\bigg(\left(\frac{-D}{\cdot}\right), s\bigg).$$

Blomer and Harcos [1] established the following deep subconvexity bound, valid for all $\delta_2 < 1/8$ and some fixed constant $A_2 > 0$,

$$L\bigg(u_n \otimes \left(\frac{\Delta}{\cdot}\right), \frac{1}{2}\bigg) \ll \big(1 + |t_n|\big)^{A_2} |\Delta|^{\frac{1}{2} - \delta_2}. \tag{5.12}$$

After combining (5.11) and (5.12) with Siegel's theorem,

$$h(-dD) \gg_{\epsilon} (dD)^{\frac{1}{2} - \epsilon}, \tag{5.13}$$

one finds using (5.10) that for all $\delta_3 < 1/16$,

$$\frac{1}{2^{t(N)}h(-dD)} W_n \ll \left(1+|t_n|\right)^{A_3} (dD)^{-\delta_3}. \tag{5.14}$$

To estimate $W_{\mathfrak{a}}(t)$ we proceed as above to obtain the identity

$$W_{\mathfrak{a}}(t) = \sum_{\substack{r \bmod 2N \\ r^2 \equiv -dD \bmod 4N}} \sum_{Q \in \mathcal{Q}_{dD,N,r}/\Gamma_0(N)} \chi_d(Q) E_{\mathfrak{a}}\left(z_Q, \frac{1}{2}+it\right). \tag{5.15}$$

Again, using the bijection (3.2) we have

$$\sum_{Q \in \mathcal{Q}_{dD,N,r}/\Gamma_0(N)} \chi_d(Q) E_{\mathfrak{a}}\left(z_Q, \frac{1}{2}+it\right) = \sum_{\sigma \in \mathrm{Gal}(H/K)} \chi_d(\sigma) E_{\mathfrak{a}}\left(z_{Q_r}^{\sigma}, \frac{1}{2}+it\right). \tag{5.16}$$

Following the argument in [17, Section 6], one can reduce the estimate of (5.16) to an analogous estimate for

$$\sum_{\sigma \in \mathrm{Gal}(H/K)} \chi_d(\sigma) E\left(z^{\sigma}, \frac{1}{2}+it\right)$$

where $E(z,s)$ is the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ and $\Lambda_{dD}(1) = \{z^{\sigma}\colon \sigma \in \mathrm{Gal}(H/K)\}$ is the set of Heegner points of discriminant $-dD$ on the modular curve $X_0(1)$. By a classical formula of Dirichlet one has an identity of the form (see [16, p. 248])

$$\left|\sum_{\sigma \in \mathrm{Gal}(H/K)} \chi_d(\sigma) E\left(z^{\sigma}, \frac{1}{2}+it\right)\right|^2 = \frac{\mathcal{O}_K^{\times}}{2} \frac{\sqrt{dD}}{2} \left|L\left(\left(\frac{d}{\cdot}\right), \frac{1}{2}+it\right)\right|^2 \left|L\left(\left(\frac{-D}{\cdot}\right), \frac{1}{2}+it\right)\right|^2, \tag{5.17}$$

where we have used the fact that $L(\Theta_{\chi_d}, s) = L(\chi_d, s)$ and employed Kronecker's factorization a second time. By Blomer and Harcos [1] one has the following subconvexity bound, valid for any $\delta_4 < 1/8$ and some fixed constant $A_4 > 0$,

$$\left|L\left(\left(\frac{\Delta}{\cdot}\right), \frac{1}{2}+it\right)\right|^2 \ll \left(1+|t|\right)^{A_4} |\Delta|^{\frac{1}{2}-\delta_4}. \tag{5.18}$$

After combining (5.17) and (5.18) with Siegel's theorem (5.13), one finds using (5.15) that for all $\delta_5 < 1/16$,

$$\frac{1}{2^{t(N)}h(-dD)} W_{\mathfrak{a}}(t) \ll \left(1+|t|\right)^{A_5} (dD)^{-\delta_5}. \tag{5.19}$$

Because $\Delta^a F_{T_0}$ is C^∞ with exponential decay in each cusp $\mathfrak{a}$ of $X_0(N)$, a repeated application of Stokes' theorem (see e.g. [19, Lemma 1.18]) yields the following identities, valid for each integer $a \geqslant 0$,

$$\langle F_{T_0}, u_n\rangle_2 = \left(1/4 + t_n^2\right)^{-a}\left\langle \Delta^a F_{T_0}, u_n\right\rangle_2$$

and

$$\left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2 = \left(1/4 + t^2\right)^{-a}\left\langle \Delta^a F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2.$$

Moreover, by the Parseval formula (see [20, (15.17)]),

$$\sum_{n=1}^{\infty}\left|\left\langle \Delta^a F_{T_0}, u_n\right\rangle_2\right|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbb{R}}\left|\left\langle \Delta^a F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2\right|^2 dt = \left\|\Delta^a F_{T_0}\right\|_2^2.$$

Then by the Cauchy–Schwartz inequality, one finds that for some sufficiently large integer $a_0 > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\langle F_{T_0}, u_n\rangle_2\right|\left(1 + |t_n|\right)^{A_3} &= \sum_{n=1}^{\infty}\left|\left\langle \Delta^{a_0} F_{T_0}, u_n\right\rangle_2\right|\frac{(1 + |t_n|)^{A_3}}{(1/4 + t_n^2)^{a_0}} \\
&\leqslant \sqrt{\sum_{n=1}^{\infty}\left|\left\langle \Delta^{a_0} F_{T_0}, u_n\right\rangle_2\right|^2} \cdot \sqrt{\sum_{n=1}^{\infty}\frac{(1 + |t_n|)^{2A_3}}{(1/4 + t_n^2)^{2a_0}}} \\
&\ll \left\|\Delta^{a_0} F_{T_0}\right\|_2,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbb{R}}\left|\left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2\right|\left(1 + |t|\right)^{A_5} dt \\
&\quad = \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbb{R}}\left|\left\langle \Delta^{a_0} F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2\right|\frac{(1 + |t|)^{A_5}}{(1/4 + t^2)^{a_0}} dt \\
&\quad \leqslant \sqrt{\sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbb{R}}\left|\left\langle \Delta^{a_0} F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right)\right\rangle_2\right|^2 dt} \cdot \sqrt{\int_{\mathbb{R}}\frac{(1 + |t|)^{2A_5}}{(1/4 + t^2)^{2a_0}} dt} \\
&\quad \ll \left\|\Delta^{a_0} F_{T_0}\right\|_2.
\end{aligned}$$

By combining these estimates with (5.14) and (5.19), we find that for all $\delta_6 < 1/16$,

$$\frac{1}{2^{t(N)} h(-dD)}\sum_{n=1}^{\infty}\left|\langle F_{T_0}, u_n\rangle_2 W_n\right| \ll \left\|\Delta^{a_0} F_{T_0}\right\|_2 (dD)^{-\delta_6}$$

and

$$\frac{1}{2^{t(N)}h(-dD)}\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{\mathbb{R}}\left|\left\langle F_{T_0}, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2}+it\right)\right\rangle_2 W_{\mathfrak{a}}(t)\right| dt$$
$$\ll \left\|\Delta^{a_0}F_{T_0}\right\|_2 (dD)^{-\delta_6}. \tag{5.20}$$

We conclude from these estimates that for all $\delta_7 < 1/16$,

$$\frac{1}{2^{t(N)}h(-dD)}\sum_{Q\in\mathcal{Q}_{dD,N}/\Gamma_0(N)}\chi_d(Q)F_{T_0}(z_Q)$$
$$= \delta_{d,1}\int_{Y_0(N)} F(z)\,d\mu - \delta_{d,1}\int_{Y_0(N)}\eta_{T_0}(z)\,d\mu + O\left(\left\|\Delta^{a_0}F_{T_0}\right\|_2 (dD)^{-\delta_7}\right)$$

as $dD \to \infty$.

To finish the proof of Proposition 5.5, it suffices to show that for all $\delta_8 < 1/16$,

$$\frac{1}{2^{t(N)}h(-dD)}\sum_{Q\in\mathcal{Q}_{dD,N}/\Gamma_0(N)}\chi_d(Q)\tilde{\eta}_T(z_Q) = \delta_{d,1}\int_{Y_0(N)}\eta_{T_0}(z)\,d\mu$$
$$+ O\left((dD)^{-\delta_8}\right) + O\left((dD)^{\frac{\alpha-1}{2}}\right)$$

as $T \gg \sqrt{dD}$ and $dD \to \infty$.

By combining the definition of η_T in (5.1) with [19, (7.12)], [19, Theorem 11.3] and [19, (7.13)], one has

$$\tilde{\eta}_T(z) = \langle\tilde{\eta}_T, 1\rangle_2 + \frac{1}{2\pi}\sum_{\mathfrak{a}} c_{\mathfrak{a}}\int_{\mathbb{R}}\left(\hat{\psi}_{T_0}\left(\frac{1}{2}+it\right) - \hat{\psi}_T\left(\frac{1}{2}+it\right)\right)E_{\mathfrak{a}}\left(z, \frac{1}{2}+it\right)dt \tag{5.21}$$

where (see [19, (3.13)])

$$\hat{\psi}(s) := \int_0^\infty \psi(y)y^{-(s+1)}\,dy.$$

Summing over (5.21) yields

$$\sum_{Q\in\mathcal{Q}_{dD,N}/\Gamma_0(N)}\chi_d(Q)\tilde{\eta}_T(z_Q)$$
$$= \sum_{Q\in\mathcal{Q}_{dD,N}/\Gamma_0(N)}\chi_d(Q)\langle\tilde{\eta}_T, 1\rangle_2$$
$$+ \frac{1}{2\pi}\sum_{\mathfrak{a}} c_{\mathfrak{a}}\int_{\mathbb{R}}\left(\hat{\psi}_{T_0}\left(\frac{1}{2}+it\right) - \hat{\psi}_T\left(\frac{1}{2}+it\right)\right)W_{\mathfrak{a}}(t)\,dt. \tag{5.22}$$

Let $\sqrt{dD} \ll T \ll dD$. Then using Lemma 5.6 (proved below) and an argument similar to that in the proof of (5.20), we obtain the following estimate, valid for any $\delta_9 < 1/16$,

$$\frac{1}{2^{t(N)}h(-dD)}\frac{1}{2\pi}\sum_{\mathfrak{a}}|c_{\mathfrak{a}}|\int_{\mathbb{R}}\left|\hat{\psi}_{T_0}\left(\frac{1}{2}+it\right)-\hat{\psi}_T\left(\frac{1}{2}+it\right)W_{\mathfrak{a}}(t)\right|dt \ll \log(dD)(dD)^{-\delta_9}.$$

Moreover, for $T \gg \sqrt{dD}$ a straightforward estimate yields

$$\langle \eta_T, 1\rangle_2 = O\big((dD)^{\frac{\alpha-1}{2}}\big).$$

Finally, by combining these estimates with (5.22), we conclude that for $\sqrt{dD} \ll T \ll D$ and any $\delta_{10} < 1/16$,

$$\frac{1}{2^{t(N)}h(-dD)}\sum_{Q\in\mathcal{Q}_{dD,N}/\Gamma_0(N)}\chi_d(Q)\tilde{\eta}_T(z_Q) = \delta_{d,1}\int_{Y_0(N)}\eta_{T_0}(z)\,d\mu + O\big((dD)^{-\delta_{10}}\big) + O\big((dD)^{\frac{\alpha-1}{2}}\big)$$

as $dD \to \infty$. □

Lemma 5.6. *For each $B > 0$ we have*

$$\int_{\mathbb{R}}\left|\hat{\psi}_{T_0}\left(\frac{1}{2}+it\right)-\hat{\psi}_T\left(\frac{1}{2}+it\right)\right|\big(1+|t|\big)^B\,dt \ll \log(T).$$

Proof. Because $\chi(y/T_0)-\chi(y/T)$ is supported in $(T_0, 2T)$, we have the identity

$$f_T(t) := \hat{\psi}_{T_0}\left(\frac{1}{2}-it\right)-\hat{\psi}_T\left(\frac{1}{2}-it\right) = \int_{T_0}^{2T}\big(\chi(y/T_0)-\chi(y/T)\big)y^{it+\alpha-\frac{3}{2}}\,dy.$$

First assume that $|t| \geqslant 1$. Because the k-th derivative $\chi^{(k)}(y)$ is supported in $(1,2)$, integrating by parts k-times yields

$$(-1)^k\prod_{j=0}^{k-1}\left(it+\frac{1}{2}+(\alpha-1)+j\right)f_T(t)$$

$$= \big(T_0^{it+\frac{1}{2}+(\alpha-1)} - T^{it+\frac{1}{2}+(\alpha-1)}\big)\int_1^2 \chi^{(k)}(y)y^{it+\alpha-\frac{3}{2}+k}\,dy.$$

Now, we have the estimate

$$\left|(-1)^k \prod_{j=0}^{k-1}\left(it+\frac{1}{2}+(\alpha-1)+j\right)\right| \leqslant \prod_{j=0}^{k-1}\left(\frac{1}{2}+(1-\alpha)+j+1\right)|t|^k,$$

and the estimate

$$\left|T_0^{it+\frac{1}{2}+(\alpha-1)} - T^{it+\frac{1}{2}+(\alpha-1)}\right| \leqslant 2T_0^{\alpha-\frac{1}{2}}.$$

By combining the preceding facts we obtain

$$\left|f_T(t)\right| \leqslant 2T_0^{\alpha-\frac{1}{2}} \frac{\max_{1\leqslant y\leqslant 2}|\chi^{(k)}(y)|}{\prod_{j=0}^{k-1}(\frac{1}{2}+(1-\alpha)+j+1)} \frac{|2^{k+\alpha-\frac{1}{2}}|}{|k+\alpha-\frac{1}{2}|}|t|^{-k}.$$

Because $B>0$ is fixed and $k\geqslant 1$ is arbitrary, it follows that

$$\int\limits_{|t|\geqslant 1} \left|f_T(t)\right|\left(1+|t|\right)^B dt \ll T_0^{\alpha-\frac{1}{2}}.$$

Next assume that $|t|<1$. Since $\alpha\leqslant 1/2$ we have the estimate

$$\left|f_T(t)\right| \leqslant 2\sup_{y\in\mathbb{R}^+}\left|\chi(y)\right| \int\limits_{T_0}^{2T} y^{-1}\,dy \ll \log(T).$$

Then because $(1+|t|)^B \ll 1$ for $|t|<1$, it follows that

$$\int\limits_{|t|<1} \left|f_T(t)\right|\left(1+|t|\right)^B dt \ll \log(T). \qquad \square$$

6. Poincaré series

The following proposition gives the Fourier expansion of $F_m(z,s)$ in the cusp at ∞ of $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ (see [25]).

Proposition 6.1. *The Poincaré series $F_m(z,s)$ has a Fourier expansion in the cusp at ∞ of Γ of the form*

$$\begin{aligned} F_m(z,s) &= 2\pi|m|^{s-\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}\left(2\pi|m|y\right)e(mx)+c_{m,s}y^{1-s} \\ &\quad + 4\pi|m|^{s-\frac{1}{2}}\sum_{n\neq 0} b(m,n;s)y^{\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi|n|y\right)e(nx), \end{aligned}$$

where

$$c_{m,s} := \frac{4\pi^{1+s}\sigma_{2s-1}(|m|)}{(2s-1)\Gamma(s)\zeta(2s)},$$

and

$$b(m,n;s) = \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} \cdot \begin{cases} I_{2s-1}(\frac{4\pi\sqrt{|mn|}}{c}), & mn < 0, \\ J_{2s-1}(\frac{4\pi\sqrt{|mn|}}{c}), & mn > 0, \end{cases}$$

where $S(m,n;c)$ *is the Kloosterman sum, and* I_ν, J_ν *and* K_ν *are the Bessel functions of order* ν.

We now construct a family of Poincaré series which regularize $F_m(z,s)$ in the cusp. Let $\lambda : \mathbb{R} \to [0,1]$ be a C^∞ function such that

$$\lambda(t) = \begin{cases} 0, & t \leqslant 0, \\ 1, & t \geqslant 1. \end{cases}$$

Let $m \in \mathbb{Z}^-$, $s \in \mathbb{Z}^+$, and for $\varepsilon > 0$ define the function

$$\psi_{m,s,\varepsilon}(t) := \lambda\left(\frac{t-1}{\varepsilon}\right) \sum_{j=0}^{s-1} \kappa_{m,j}(s) t^{-j}$$

where

$$\kappa_{m,j}(s) := \frac{(-1)^j |m|^{s-j-1}(s-1+j)!}{(4\pi)^j (s-1-j)!j!}.$$

Then $\psi_{m,s,\varepsilon} : \mathbb{R} \to [0,1]$ is C^∞ and satisfies

$$\psi_{m,s,\varepsilon}(t) = \begin{cases} 0, & t \leqslant 1, \\ \sum_{j=0}^{s-1} \kappa_{m,j}(s) t^{-j}, & t > 1+\varepsilon. \end{cases}$$

Finally, define the Poincaré series

$$\mathscr{P}_{m,\varepsilon}(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi_{m,s,\varepsilon}(\operatorname{Im} \gamma z) e(m\gamma z).$$

Proposition 6.2. *For* $y > 1+\varepsilon$ *we have*

$$\begin{aligned} &F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s) \\ &\quad = c_{m,s} y^{1-s} + e^{-2\pi|m|y} e(mx)(-1)^s \sum_{j=0}^{s-1} (-1)^j y^{-j} \kappa_{m,j}(s) \\ &\qquad + 2\pi |m|^{s-\frac{1}{2}} \sum_{n \neq 0} e^{-2\pi|n|y} e(nx) b(m,n;s) \sum_{j=0}^{s-1} \frac{(s-1+j)!|n|^{-j-\frac{1}{2}} y^{-j}}{j!(s-1-j)!(4\pi)^j}. \end{aligned}$$

Proof. The Poincaré series $\mathscr{P}_{m,\varepsilon}(z,s)$ has a Fourier expansion of the form (see [19, p. 60])

$$\mathscr{P}_{m,\varepsilon}(z,s) = \psi_{m,s,\varepsilon}(y)e(mz) + \sum_{n\in\mathbb{Z}} e(nx) \sum_{c=1}^{\infty} S(m,n;c) \int_{\mathbb{R}} \psi_{m,s,\varepsilon}\left(\frac{c^{-2}y}{t^2+y^2}\right) e\left(\frac{-mc^{-2}}{t+iy} - nt\right) dt. \tag{6.1}$$

If $y \geqslant 1$ then

$$\frac{c^{-2}y}{t^2+y^2} \leqslant 1$$

for all $c \in \mathbb{Z}^+$ and $t \in \mathbb{R}$. It follows from the definition of $\psi_{m,s,\varepsilon}$ that if $y \geqslant 1$,

$$\mathscr{P}_{m,\varepsilon}(z,s) = \psi_{m,s,\varepsilon}(y)e(mz),$$

and if $y > 1+\varepsilon$,

$$\mathscr{P}_{m,\varepsilon}(z,s) = \sum_{j=0}^{s-1} \kappa_{m,j}(s) y^{-j} e(mz). \tag{6.2}$$

Now, the Bessel functions $I_{n+\frac{1}{2}}$ and $K_{n+\frac{1}{2}}$ of half-integral order have expansions given as follows (see [30, Section 3.71]): for non-negative integers n we have

$$y^{1/2} I_{n+\frac{1}{2}}(y) = \frac{1}{\sqrt{2\pi}} \left(e^y \sum_{j=0}^{n} \frac{(-1)^j (n+j)!}{j!(n-j)!(2y)^j} + (-1)^{n+1} e^{-y} \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!(2y)^j} \right)$$

and

$$y^{1/2} K_{n+\frac{1}{2}}(y) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-y} \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!(2y)^j}.$$

Substitute these expansions into the Fourier expansion of $F_m(z,s)$ given in Proposition 6.1 and use the fact that $m \in \mathbb{Z}^-$ and $s \in \mathbb{Z}^+$ to obtain

$$F_m(z,s) = \sum_{j=0}^{s-1} \kappa_{m,j}(s) y^{-j} e(mz) + c_{m,s} y^{1-s} + e^{-2\pi|m|y} e(mx)(-1)^s \sum_{j=0}^{s-1} (-1)^j y^{-j} \kappa_{m,j}(s) + 2\pi|m|^{s-\frac{1}{2}} \sum_{n\neq 0} e^{-2\pi|n|y} e(nx) b(m,n;s) \sum_{j=0}^{s-1} \frac{(s-1+j)!|n|^{-j-\frac{1}{2}} y^{-j}}{j!(s-1-j)!(4\pi)^j}. \tag{6.3}$$

Finally, by combining (6.3) and (6.2), we find that for $y > 1+\varepsilon$,

$$F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)$$
$$= c_{m,s} y^{1-s} + e^{-2\pi|m|y} e(mx)(-1)^s \sum_{j=0}^{s-1} (-1)^j y^{-j} \kappa_{m,j}(s)$$
$$+ 2\pi|m|^{s-\frac{1}{2}} \sum_{n\neq 0} e^{-2\pi|n|y} e(nx) b(m,n;s) \sum_{j=0}^{s-1} \frac{(s-1+j)!|n|^{-j-\frac{1}{2}} y^{-j}}{j!(s-1-j)!(4\pi)^j}. \qquad \square$$

7. Proof of Theorem 1.1

We will deduce Theorem 1.1 from the following theorem, whose proof will require further technical results from Sections 8–10.

Theorem 7.1. *Let $m \in \mathbb{Z}^-$, $s \in \mathbb{Z}^+$, and let $-D < 0$ and $d > 0$ be odd, coprime fundamental discriminants. Then there exists an effective constant $c > 0$ such that for all $\delta < 1/16$ and $0 < b < \delta/c$,*

$$\frac{1}{h(-dD)} \left(\sum_{z_Q \in \Lambda_{dD}(1)} \chi_d(Q) F_m(z_Q, s) - \sum_{j=0}^{s-1} \kappa_{m,j}(s) \sum_{\operatorname{Im}(z_Q) > 1 + (dD)^{-b}} \chi_d(Q) \operatorname{Im}(z_Q)^{-j} e(m z_Q) \right)$$
$$= \delta_{d,1} \int_{reg} F_m(z,s)\, d\mu + O\big((dD)^{-(\delta - bc)}\big) + O\big((dD)^{-b}\big)$$

as $dD \to \infty$.

Proof. Let $\varepsilon < 1/4$, and define the function

$$F_{m,s,\varepsilon}(z) := F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)$$

where $\mathscr{P}_{m,\varepsilon}(z,s)$ is the Poincaré series defined in Section 6. Then $F_{m,s,\varepsilon}(z)$ is C^∞ and Γ-invariant, and by Proposition 6.2 we see that $F_{m,s,\varepsilon}(z)$ has cuspidal growth of power $\alpha = 1 - s$. Here we emphasize that this growth is *uniform* in ε for $y > 5/4$, and thus the same choice of cutoff parameter $T_0 = 2$ and corresponding cutoff function η_2 given by

$$\eta_2(z) = \begin{cases} 0, & 1 < y \leqslant 2, \\ c_{m,s} y^{1-s} \chi(y/2), & 2 < y < 4, \\ c_{m,s} y^{1-s}, & y \geqslant 4, \end{cases}$$

works for each $F_{m,s,\varepsilon}(z)$. We now substitute $F_{m,s,\varepsilon}(z)$ into Theorem 5.1 with the choices $N = 1$ and $T_0 = 2$, and find that for all $\delta < 1/16$,

$$\frac{1}{h(-dD)} \sum_{z_Q \in \Lambda_{dD}(1)} \chi_d(Q)\big(F_m(z_Q,s) - \mathscr{P}_{m,\varepsilon}(z_Q,s)\big)$$
$$= \delta_{d,1} \int_{Y_0(1)} \big(F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)\big)\, d\mu + O\big(\big(\big\|\Delta^{a_0} F_{m,s,\varepsilon,2}\big\|_2 + 1\big)(dD)^{-\delta}\big) \qquad (7.1)$$

as $dD \to \infty$, where

$$F_{m,s,\varepsilon,2}(z,s) := \bigl(F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)\bigr) - \eta_2(z).$$

Because $\psi_{m,s,\varepsilon}(y) = 0$ for $y \leqslant 1$, it follows from the definitions of $\mathscr{P}_{m,\varepsilon}(z,s)$ and $\psi_{m,s,\varepsilon}$ that for dD sufficiently large,

$$\begin{aligned}\sum_{z_Q \in \Lambda_{dD}(1)} \chi_d(Q)\mathscr{P}_{m,\varepsilon}(z_Q,s) &= \sum_{\operatorname{Im}(z_Q)>1} \chi_d(Q)\psi_{m,s,\varepsilon}(\operatorname{Im} z_Q)e(mz_Q) \\ &= \sum_{1<\operatorname{Im}(z_Q)\leqslant 1+\varepsilon} \chi_d(Q)\psi_{m,s,\varepsilon}(\operatorname{Im} z_Q)e(mz_Q) \\ &\quad + \sum_{j=0}^{s-1}\kappa_{m,j}(s) \sum_{\operatorname{Im}(z_Q)>1+\varepsilon} \chi_d(Q)\operatorname{Im}(z_Q)^{-j}e(mz_Q).\end{aligned}$$

For notational convenience define

$$\begin{aligned}R_\varepsilon(dD) := \frac{1}{h(-dD)}\Biggl(&\sum_{z_Q\in\Lambda_{dD}(1)} \chi_d(Q)F_m(z_Q,s) \\ &- \sum_{j=0}^{s-1}\kappa_{m,j}(s) \sum_{\operatorname{Im}(z_Q)>1+\varepsilon} \chi_d(Q)\operatorname{Im}(z_Q)^{-j}e(mz_Q)\Biggr).\end{aligned}$$

Then we can write (7.1) in the equivalent form

$$\begin{aligned}R_\varepsilon(dD) = {}& \delta_{d,1}\int_{Y_0(1)} \bigl(F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)\bigr)\,d\mu + O\bigl(\bigl(\bigl\|\Delta^{a_0}F_{m,s,\varepsilon,2}\bigr\|_2 + 1\bigr)(dD)^{-\delta}\bigr) \\ &+ \frac{1}{h(-dD)}\sum_{1<\operatorname{Im}(z_Q)\leqslant 1+\varepsilon} \chi_d(Q)\psi_{m,s,\varepsilon}(\operatorname{Im} z_Q)e(mz_Q).\end{aligned}$$

By Proposition 8.2 and Lemma 10.1, we have

$$\begin{aligned}R_\varepsilon(dD) = {}& \delta_{d,1}\int_{reg} F_m(z,s)\,d\mu + O\bigl(\varepsilon^{-a_1}(dD)^{-\delta}\bigr) + O\bigl((dD)^{-\delta}\bigr) \\ &+ \frac{1}{h(-dD)}\sum_{1<\operatorname{Im}(z_Q)\leqslant 1+\varepsilon} \chi_d(Q)\psi_{m,s,\varepsilon}(\operatorname{Im} z_Q)e(mz_Q)\end{aligned}$$

for some sufficiently large integer $a_1 > 0$ (depending on a_0).

Now, we have the estimate

$$\left|\sum_{1<\operatorname{Im}(z_Q)\leqslant 1+\varepsilon} \chi_d(Q)\psi_{m,s,\varepsilon}(\operatorname{Im} z_Q)e(mz_Q)\right| \leqslant \sup_{t\in\mathbb{R}^+}\bigl|\psi_{m,s,\varepsilon}(t)\bigr|e^{2\pi m(1+\varepsilon)}\#\Lambda_\varepsilon(dD) \qquad (7.2)$$

where

$$\Lambda_\varepsilon(dD) := \{z_Q \in \Lambda_{dD}(1)\colon\ 1 < \operatorname{Im}(z_Q) \leqslant 1+\varepsilon\}.$$

By combining (7.2) with Lemma 9.1 and Siegel's theorem (5.13), we find that

$$\frac{1}{h(-dD)} \sum_{1<\operatorname{Im}(z_Q)\leqslant 1+\varepsilon} \chi_d(Q)\psi_{m,s,\varepsilon}(\operatorname{Im} z_Q)e(mz_Q) = O(\varepsilon) + O\big(\varepsilon^{-a_2}(dD)^{-\delta}\big) \tag{7.3}$$

for some sufficiently large integer $a_2 > 0$.

It follows from the preceding analysis that for all $\delta < 1/16$,

$$R_\varepsilon(dD) = \delta_{d,1} \int_{reg} F_m(z,s)\,d\mu + O\big(\varepsilon^{-a_1}(dD)^{-\delta}\big) + O\big((dD)^{-\delta}\big) + O(\varepsilon) + O\big(\varepsilon^{-a_2}(dD)^{-\delta}\big).$$

Let $c = \max\{a_1, a_2\}$ and choose $b > 0$ such that $b < \delta/c$. Because ε is independent of dD in the preceding estimates, we can set $\varepsilon = (dD)^{-b}$. Then

$$R_\varepsilon(dD) = \delta_{d,1} \int_{reg} F_m(z,s)\,d\mu + O\big((dD)^{-(\delta-bc)}\big) + O\big((dD)^{-b}\big)$$

as $dD \to \infty$. □

Proof of Theorem 1.1. Theorem 1.1 now follows by combining Theorem 7.1 with Propositions 11.1 and 12.1. □

8. Proof of Proposition 8.2

In this section we prove Proposition 8.2 (see also [11]).

For $\varepsilon > 0$ define the cutoff function

$$\psi^Y_{m,s,\varepsilon}(t) := \begin{cases} \psi_{m,s,\varepsilon}(t), & t \leqslant Y, \\ 0, & t > Y, \end{cases}$$

and the associated cutoff Poincaré series

$$\mathscr{P}_{m,\varepsilon,Y}(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi^Y_{m,s,\varepsilon}(\operatorname{Im}\gamma z)e(m\gamma z).$$

We will need the following lemma.

Lemma 8.1. *For $\varepsilon > 0$, we have*

$$\mathscr{P}_{m,\varepsilon}(z,s) = \mathscr{P}_{m,\varepsilon,Y}(z,s)$$

for all $z \in \mathcal{F}_Y$.

Proof. We need to show that for $\mathrm{Im}(z) \leqslant Y$,

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi^{Y}_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z). \tag{8.1}$$

By definition of $\psi^{Y}_{m,s,\varepsilon}$ we have

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi^{Y}_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z) = \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \mathrm{Im}(\gamma z)\leqslant Y}} \psi_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z).$$

Moreover, since $\mathrm{Im}(z) \leqslant Y$, which forces the identity matrix I into the first summand inside the brackets in (8.2), we have

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z) = \Bigg(\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \mathrm{Im}(\gamma z)\leqslant Y}} + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \mathrm{Im}(\gamma z)>Y\\ \gamma\neq I}}\Bigg)\psi_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z). \tag{8.2}$$

Let

$$S_Y := \big\{\gamma \in \Gamma_\infty\backslash\Gamma\colon\ \mathrm{Im}(\gamma z) > Y,\ \gamma \neq I\big\}.$$

Then

$$\Bigg|\sum_{\gamma\in S_Y} \psi_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z)\Bigg| \leqslant \sup_{t\in\mathbb{R}^+}\big|\psi_{m,s,\varepsilon}(t)\big| \max_{\gamma\in S_Y} e^{2\pi m\,\mathrm{Im}(\gamma z)}\#S_Y.$$

Suppose that $Y \geqslant 10$. Then by [19, Lemma 2.10], $\#S_Y < 10/Y \leqslant 1$, which implies that $\#S_Y = 0$. Thus

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \mathrm{Im}(\gamma z)>Y\\ \gamma\neq I}} \psi_{m,s,\varepsilon}(\mathrm{Im}\,\gamma z)e(m\gamma z) = 0,$$

and (8.1) holds for $Y \geqslant 10$. However, since $\mathcal{F}_Y \subset \mathcal{F}_{10}$ for $Y < 10$, the lemma follows. □

Proposition 8.2. *For $\varepsilon > 0$ we have*

$$\int\limits_{Y_0(1)} \big(F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)\big)\,d\mu = \lim_{Y\to\infty}\int\limits_{\mathcal{F}_Y} F_m(z,s)\,d\mu.$$

Proof. By Lemma 8.1 and an unfolding argument, we find that for all $Y > 0$,

$$\int_{\mathcal{F}_Y} \mathscr{P}_{m,\varepsilon}(z,s)\,d\mu = \int_{\mathcal{F}} \mathscr{P}_{m,\varepsilon,Y}(z,s)\,d\mu$$

$$= \frac{3}{\pi}\int_0^Y \psi_{m,s,\varepsilon}(y)e^{2\pi my}\frac{dy}{y^2}\cdot\int_0^1 e^{2\pi imx}\,dx = 0.$$

Then because $F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s) \in L^1(Y_0(1))$, it follows that

$$\lim_{Y\to\infty}\int_{\mathcal{F}_Y} F_m(z,s)\,d\mu = \lim_{Y\to\infty}\Bigl(\int_{\mathcal{F}_Y} F_m(z,s)\,d\mu - \int_{\mathcal{F}_Y} \mathscr{P}_{m,\varepsilon}(z,s)\,d\mu\Bigr)$$

$$= \lim_{Y\to\infty}\int_{\mathcal{F}_Y}\bigl(F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)\bigr)\,d\mu$$

$$= \int_{Y_0(1)}\bigl(F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)\bigr)\,d\mu. \qquad \square$$

Note that the regularized integral in Proposition 8.2 will be evaluated in Section 12.

9. Proof of Lemma 9.1

In this section we prove Lemma 9.1 (see also [10, pp. 248–249]).

Lemma 9.1. *For $\varepsilon < 1/4$ we have*

$$\#\Lambda_\varepsilon(dD) \leqslant 4\varepsilon h(-dD) + O_\epsilon\bigl(\varepsilon^{-a_2}(dD)^{\frac{7}{16}+\epsilon}\bigr)$$

for some sufficiently large integer $a_2 > 0$ and all $\epsilon > 0$.

Proof. For each $\varepsilon > 0$, one can construct a C^∞ function $\phi_\varepsilon : \mathbb{R} \to [0,1]$ which is supported in the interval $(1-\varepsilon, 1+2\varepsilon)$, which equals 1 on the interval $[1, 1+\varepsilon]$, and which satisfies the bound

$$\max_{t\in(1-\varepsilon,1+2\varepsilon)}\left|\frac{d^a}{dt^a}\phi_\varepsilon(t)\right| \ll \varepsilon^{-a}$$

for all $a \in \mathbb{Z}^+$.

Define the incomplete Eisenstein series

$$g_\varepsilon(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \phi_\varepsilon\bigl(\mathrm{Im}(\gamma z)\bigr).$$

Then using that $\phi_\varepsilon(y) = 1$ for $1 \leqslant y \leqslant 1+\varepsilon$, we obtain the decomposition

$$\sum_{z_Q \in \Lambda_{dD}(1)} g_\varepsilon(z_Q) = \#\Lambda_\varepsilon(dD) + \sum_{z_Q \in \Lambda_{dD}(1) \setminus \Lambda_\varepsilon(dD)} \phi_\varepsilon\big(\mathrm{Im}(z_Q)\big) + \sum_{z_Q \in \Lambda_{dD}(1)} \sum_{\substack{\gamma \in \Gamma_\infty \setminus \Gamma \\ \gamma \neq I}} \phi_\varepsilon\big(\mathrm{Im}(\gamma z_Q)\big).$$

Since $\phi_\varepsilon \geqslant 0$ it follows that

$$\#\Lambda_\varepsilon(dD) \leqslant \sum_{z_Q \in \Lambda_{dD}(1)} g_\varepsilon(z_Q).$$

The real-analytic Eisenstein series

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathrm{Im}(\gamma z)^s, \quad \mathrm{Re}(s) > 1$$

has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue $3/\pi$. Therefore we obtain from [19, (7.12)] the expansion

$$g_\varepsilon(z) = \frac{3}{\pi}\hat{\phi}_\varepsilon(1) + \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}_\varepsilon\left(\frac{1}{2} + it\right) E\left(z, \frac{1}{2} + it\right) dt$$

where

$$\hat{\phi}_\varepsilon(s) := \int_0^\infty \phi_\varepsilon(y) y^{-(s+1)}\, dy.$$

Summing the expansion yields

$$\sum_{z_Q \in \Lambda_{dD}(1)} g_\varepsilon(z_Q) = \frac{3}{\pi}\hat{\phi}_\varepsilon(1) h(-dD) + \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}_\varepsilon\left(\frac{1}{2} + it\right) \sum_{z_Q \in \Lambda_{dD}(1)} E\left(z_Q, \frac{1}{2} + it\right) dt.$$

Since ϕ_ε is supported in $(1 - \varepsilon, 1 + 2\varepsilon)$ and $\phi_\varepsilon \leqslant 1$, we find that for $\varepsilon < 1/4$,

$$\begin{aligned}
\hat{\phi}_\varepsilon(1) &= \int_0^\infty \phi_\varepsilon(y) y^{-2}\, dy \\
&\leqslant \left(\int_{1-\varepsilon}^{1} + \int_1^{1+\varepsilon} + \int_{1+\varepsilon}^{1+2\varepsilon} \right) y^{-2}\, dy \\
&\leqslant \frac{\varepsilon}{(1-\varepsilon)^2} + \varepsilon + \frac{\varepsilon}{(1+\varepsilon)^2} \\
&< 2\varepsilon + \varepsilon + \varepsilon = 4\varepsilon.
\end{aligned}$$

Therefore

$$\frac{3}{\pi}\hat{\phi}_\varepsilon(1)h(-dD) \leqslant 4\varepsilon h(-dD).$$

By a variant of the argument used to prove (5.19), one finds that

$$\sum_{z_Q\in\Lambda_{dD}(1)} E\left(z_Q, \frac{1}{2}+it\right) \ll_\epsilon \left(1/4+t^2\right)^{A_6}(dD)^{\frac{7}{16}+\epsilon}$$

for some fixed constant $A_6 > 0$ and all $\epsilon > 0$. Thus

$$\int_{\mathbb{R}} \left|\hat{\phi}_\varepsilon\left(\frac{1}{2}+it\right) \sum_{z_Q\in\Lambda_{dD}(1)} E\left(z_Q, \frac{1}{2}+it\right)\right| dt \ll_\epsilon (dD)^{\frac{7}{16}+\epsilon} \int_{\mathbb{R}} \left|\hat{\phi}_\varepsilon\left(\frac{1}{2}+it\right)\right| \left(1/4+t^2\right)^{A_6} dt.$$

Let $a \in \mathbb{Z}^+$. Then integrating by parts a-times and using the fact that the support of $\phi_\varepsilon^{(a)}$ is contained in $(1-\varepsilon, 1+2\varepsilon) \subset [3/4, 3/2]$ for $\varepsilon < 1/4$, we obtain

$$\begin{aligned}\hat{\phi}_\varepsilon\left(\frac{1}{2}+it\right) &= \int_0^\infty \hat{\phi}_\varepsilon(y)y^{-(\frac{3}{2}+it)}\,dy \\ &= (-1)^a \prod_{j=1}^{a}\left(-\left(\frac{3}{2}+it\right)+j\right)^{-1} \int_0^\infty \phi_\varepsilon^{(a)}(y)y^{-(\frac{3}{2}+it)+a}\,dy\end{aligned}$$

for all $t \in \mathbb{R}$. Thus for $\varepsilon < 1/4$,

$$\begin{aligned}\left|\hat{\phi}_\varepsilon\left(\frac{1}{2}+it\right)\right| &\leqslant \prod_{j=1}^{a}\left|\frac{3}{2}+it-j\right|^{-1} \int_{1-\varepsilon}^{1+2\varepsilon} \left|\phi_\varepsilon^{(a)}(y)\right| y^{-\frac{3}{2}+a}\,dy \\ &\leqslant \prod_{j=1}^{a}\left|\frac{3}{2}+it-j\right|^{-1} \max_{y\in(1-\varepsilon,1+2\varepsilon)}\left|\phi_\varepsilon^{(a)}(y)\right| \int_{1-\varepsilon}^{1+2\varepsilon} y^{-\frac{3}{2}+a}\,dy \\ &\ll \prod_{j=1}^{a}\left|\frac{3}{2}+it-j\right|^{-1} \varepsilon^{-a} \int_{3/4}^{3/2} y^{-\frac{3}{2}+a}\,dy \\ &\ll_a \varepsilon^{-a} \prod_{j=1}^{a}\left|\frac{3}{2}+it-j\right|^{-1}.\end{aligned}$$

We now see that for some sufficiently large integer $a_1 > 0$ (depending on A_6), we have

$$\int_{\mathbb{R}} \left| \hat{\phi}_\varepsilon\left(\frac{1}{2}+it\right) \sum_{z_Q \in \Lambda_{dD}(1)} E\left(z_Q, \frac{1}{2}+it\right) \right| dt$$
$$\ll_{\epsilon,a_1} \varepsilon^{-a_1} (dD)^{\frac{7}{16}+\epsilon} \int_{\mathbb{R}} \prod_{j=1}^{a_1} \left| \frac{3}{2}+it-j \right|^{-1} \left(1/4+t^2\right)^{A_6} dt$$
$$\ll_{\epsilon,a_1} \varepsilon^{-a_1} (dD)^{\frac{7}{16}+\epsilon}. \qquad \square$$

10. Proof of Lemma 10.1

Lemma 10.1. *For $\varepsilon < 1/4$ we have*

$$\left\| \Delta^{a_0} F_{m,s,\varepsilon,2} \right\|_2 = O_{a_0}\left(\varepsilon^{-a_2}\right)$$

for some sufficiently large integer $a_2 > 0$ depending on a_0.

Proof. Recall that

$$F_{m,s,\varepsilon,2}(z) := \left(F_m(z,s) - \mathscr{P}_{m,\varepsilon}(z,s)\right) - \eta_2(z),$$

where the cutoff function η_2 is given by

$$\eta_2(z) = \begin{cases} 0, & 1 < y \leqslant 2, \\ c_{m,s} y^{1-s} \chi(y/2), & 2 < y < 4, \\ c_{m,s} y^{1-s}, & y \geqslant 4. \end{cases}$$

For notational convenience we write the Fourier expansion (6.1) as

$$\mathscr{P}_{m,\varepsilon}(z,s) = \psi_{m,s,\varepsilon}(y) e(mz) + f_{m,s,\varepsilon}(x,y),$$

where

$$f_{m,s,\varepsilon}(x,y) := \sum_{n \in \mathbb{Z}} e(nx) \sum_{c=1}^{\infty} S(m,n;c) \int_{\mathbb{R}} \psi_{m,s,\varepsilon}\left(\frac{c^{-2} y}{t^2+y^2}\right) e\left(\frac{-mc^{-2}}{t+iy} - nt\right) dt.$$

Clearly we have

$$\left\| \Delta^{a_0} F_{m,s,\varepsilon,2} \right\|_2^2 \leqslant \int_{\sqrt{3}/2}^{\infty} \int_0^1 \left| \Delta^{a_0} F_{m,s,\varepsilon,2}(z) \right|^2 dx \, \frac{dy}{y^2}.$$

If we let $T = 2$ in the Fourier expansion (5.2), we see that

$$\psi_2\left(\frac{c^{-2}y}{t^2+y^2}\right)=0$$

for $y \geqslant \sqrt{3}/2$, so in fact $\eta_2(z) = 0$ in the larger range $\sqrt{3}/2 \leqslant y \leqslant 2$. Then using the definitions of $\psi_{m,s,\varepsilon}$, η_2, and Proposition 6.2, we find that

$$F_{m,s,\varepsilon,2}(z)=\begin{cases} F_m(z,s)-f_{m,s,\varepsilon}(x,y), & \sqrt{3}/2 \leqslant y < 1,\\ F_m(z,s)-\psi_{m,s,\varepsilon}(y)e(mz), & 1 \leqslant y \leqslant 1+\varepsilon,\\ c_{m,s}y^{1-s}+O(e^{-cy}), & 1+\varepsilon < y \leqslant 2,\\ c_{m,s}y^{1-s}(1-\chi(y/2))+O(e^{-cy}), & 2 < y < 4,\\ O(e^{-cy}), & y \geqslant 4. \end{cases}$$

Recalling the explicit form of the $O(e^{-cy})$ terms in Proposition 6.2, it follows that

$$\begin{aligned} &\int_{\sqrt{3}/2}^{\infty}\int_0^1 \left|\Delta^{a_0}F_{m,s,\varepsilon,2}(z)\right|^2 dx\,\frac{dy}{y^2}\\ &\quad=\int_{\sqrt{3}/2}^{1}\int_0^1 \left|\Delta^{a_0}\big(F_m(z,s)-f_{m,s,\varepsilon}(x,y)\big)\right|^2 dx\,\frac{dy}{y^2}\\ &\qquad+\int_1^{1+\varepsilon}\int_0^1 \left|\Delta^{a_0}\big(F_m(z,s)-\psi_{m,s,\varepsilon}(y)e(mz)\big)\right|^2 dx\,\frac{dy}{y^2}+O(1). \end{aligned}$$

By linearity of Δ^{a_0} and the triangle inequality, we have

$$\begin{aligned} I_1 &:= \int_1^{1+\varepsilon}\int_0^1 \left|\Delta^{a_0}\big(F_m(z,s)-\psi_{m,s,\varepsilon}(y)e(mz)\big)\right|^2 dx\,\frac{dy}{y^2}\\ &\leqslant \int_1^{1+\varepsilon}\int_0^1 \left|\Delta^{a_0}F_m(z,s)\right|^2 dx\,\frac{dy}{y^2}\\ &\quad+2\int_1^{1+\varepsilon}\int_0^1 \left|\Delta^{a_0}F_m(z,s)\right|\cdot\left|\Delta^{a_0}\psi_{m,s,\varepsilon}(y)\right| dx\,\frac{dy}{y^2}\\ &\quad+\int_1^{1+\varepsilon}\left|\Delta^{a_0}\psi_{m,s,\varepsilon}(y)\right|^2\frac{dy}{y^2}. \end{aligned}$$

Recall that

$$\psi_{m,s,\varepsilon}(t) := \lambda\left(\frac{t-1}{\varepsilon}\right)\sum_{j=0}^{s-1}\kappa_{m,j}(s)t^{-j},$$

where $\lambda:\mathbb{R}\to[0,1]$ is a C^∞ function such that

$$\lambda(t) = \begin{cases} 0, & t \leqslant 0, \\ 1, & t \geqslant 1. \end{cases}$$

Then it is clear that

$$\max_{y\in[1,1+\varepsilon]}\left|\frac{d^{2a_0}}{dy^{2a_0}}\psi_{m,s,\varepsilon}(y)\right| = O\big(\varepsilon^{-2a_0}\big).$$

Therefore (using that $\varepsilon < 1/4$),

$$I_1 = O(1) + O\big(\varepsilon^{-2a_0}\big) + O\big(\varepsilon^{-4a_0}\big).$$

Similarly, we have

$$\begin{aligned} I_2 &:= \int_{\sqrt{3}/2}^{1}\int_{0}^{1}\left|\Delta^{a_0}\big(F_m(z,s) - f_{m,s,\varepsilon}(x,y)\big)\right|^2 dx\,\frac{dy}{y^2} \\ &\leqslant \int_{\sqrt{3}/2}^{1}\int_{0}^{1}\left|\Delta^{a_0}F_m(z,s)\right|^2 dx\,\frac{dy}{y^2} \\ &\quad + 2\int_{\sqrt{3}/2}^{1}\int_{0}^{1}\left|\Delta^{a_0}F_m(z,s)\right|\cdot\left|\Delta^{a_0}f_{m,s,\varepsilon}(x,y)\right| dx\,\frac{dy}{y^2} \\ &\quad + \int_{\sqrt{3}/2}^{1}\int_{0}^{1}\left|\Delta^{a_0}f_{m,s,\varepsilon}(x,y)\right|^2 dx\,\frac{dy}{y^2}. \end{aligned}$$

Observe that for $y \geqslant \sqrt{3}/2$ and $c \geqslant 2$, we have

$$\frac{y}{c^2(t^2+y^2)} \leqslant \frac{2}{c^2\sqrt{3}} \leqslant 1.$$

Since $\psi_{m,s,\varepsilon}(u) = 0$ for $u \leqslant 1$, it follows that for $y \geqslant \sqrt{3}/2$ and $c \geqslant 2$,

$$\psi_{m,s,\varepsilon}\left(\frac{c^{-2}y}{t^2+y^2}\right) = 0.$$

Therefore, for $y \geqslant \sqrt{3}/2$ the function $f_{m,s,\varepsilon}(x,y)$ simplifies to

$$f_{m,s,\varepsilon}(x,y) = \sum_{n\in\mathbb{Z}} e(nx)S(m,n;1)\int_{\mathbb{R}} \psi_{m,s,\varepsilon}\left(\frac{y}{t^2+y^2}\right)e\left(\frac{-m}{t+iy} - nt\right)dt,$$

i.e., *only* the term with $c = 1$ remains.

Define the function

$$g_{m,y,\varepsilon}(t) := \psi_{m,s,\varepsilon}\left(\frac{y}{t^2+y^2}\right)e\left(\frac{-m}{t+iy}\right).$$

Then by definition of $\psi_{m,s,\varepsilon}$ we have

$$\lim_{|t|\to\infty} \frac{d^k}{dt^k} g_{m,y,\varepsilon}(t) = 0$$

for all integers $k \geqslant 0$. Integrating by parts $(2a_0+2)$-times yields

$$\int_{\mathbb{R}} g_{m,y,\varepsilon}(t)e(-nt)\,dt = \frac{1}{(2\pi in)^{2a_0+2}}\int_{\mathbb{R}} g_{m,y,\varepsilon}^{(2a_0+2)}(t)e(-nt)\,dt.$$

In fact, since

$$\frac{y}{t^2+y^2} \leqslant 1 \quad \Leftrightarrow \quad |t| \geqslant \sqrt{y(1-y)},$$

we have

$$g_{m,y,\varepsilon}^{(2a_0+2)}(t) = 0$$

for $|t| \geqslant \sqrt{y(1-y)}$. Thus in the range $\sqrt{3}/2 \leqslant y \leqslant 1$ we obtain

$$f_{m,s,\varepsilon}(x,y) = \sum_{n\in\mathbb{Z}} \frac{e(nx)}{(2\pi in)^{2a_0+2}} S(m,n;1)\int_{-1}^{1} g_{m,y,\varepsilon}^{(2a_0+2)}(t)e(-nt)\,dt.$$

We now analyze

$$\Delta^{a_0} f_{m,s,\varepsilon}(x,y)$$

and show that

$$I_2 = O(1) + O\left(\varepsilon^{-(2a_0+2)}\right) + O\left(\varepsilon^{-2(2a_0+2)}\right).$$

For clarity we first give the argument for $a_0 = 1$. We find that

$$-y^2\left(\partial_x^2+\partial_y^2\right)f_{m,s,\varepsilon}(x,y)=-y^2\sum_{n\in\mathbb{Z}}\frac{e(nx)}{(2\pi in)^2}S(m,n;1)\int_{-1}^{1}g^{(4)}_{m,y,\varepsilon}(t)e(-nt)\,dt$$
$$-y^2\sum_{n\in\mathbb{Z}}\frac{e(nx)}{(2\pi in)^4}S(m,n;1)\int_{-1}^{1}\frac{\partial^2}{\partial y^2}g^{(4)}_{m,y,\varepsilon}(t)e(-nt)\,dt.$$

Since (see [19, (2.37)])

$$S(m,n;1)\ll 1,$$

it follows that

$$\Delta f_{m,s,\varepsilon}(x,y)\ll y^2\max_{t\in[-1,1]}\left|g^{(4)}_{m,y,\varepsilon}(t)\right|+y^2\max_{t\in[-1,1]}\left|\frac{\partial^2}{\partial y^2}g^{(4)}_{m,y,\varepsilon}(t)\right|.$$

It is clear from the definition of $g_{m,y,\varepsilon}$ that

$$\max_{t\in[-1,1]}\left|g^{(4)}_{m,y,\varepsilon}(t)\right|=O_y\left(\varepsilon^{-4}\right)$$

and

$$\max_{t\in[-1,1]}\left|\frac{\partial^2}{\partial y^2}g^{(4)}_{m,y,\varepsilon}(t)\right|=O_y\left(\varepsilon^{-4}\right).$$

Hence

$$\Delta f_{m,s,\varepsilon}(x,y)=O_y\left(\varepsilon^{-4}\right).$$

Finally, it follows that

$$I_2=O(1)+O\left(\varepsilon^{-4}\right)+O\left(\varepsilon^{-8}\right).$$

The preceding argument generalizes in a straightforward way to higher derivatives, and thus we conclude that

$$I_2=O(1)+O\left(\varepsilon^{-(2a_0+2)}\right)+O\left(\varepsilon^{-2(2a_0+2)}\right). \qquad \square$$

11. Exponential sums

In this section we express the main term in Theorem 7.1 as a twisted exponential sum.

Let $n,c\in\mathbb{Z}^+$, and $\Delta,\delta\in\mathbb{Z}$. Following [22], we define the twisted exponential sum

$$S_{\Delta,\delta}(n,c):=\sum_{x^2\equiv-\delta\Delta\ (\mathrm{mod}\ c)}\chi_d\left(\frac{c}{4},x,\frac{x^2+\delta\Delta}{c}\right)e\left(\frac{2nx}{c}\right).$$

Proposition 11.1. *Let $m \in \mathbb{Z}^-$, let $s, c \in \mathbb{Z}^+$, and let $-D < 0$ and $d > 0$ be odd, coprime fundamental discriminants. Then*

$$\sum_{j=0}^{s-1} \kappa_{m,j}(s) \sum_{\mathrm{Im}(z_Q)>1+(dD)^{-b}} \chi_d(Q)\,\mathrm{Im}(z_Q)^{-j} e(mz_Q) = \frac{1}{2} \sum_{\substack{0<c<\frac{2\sqrt{dD}}{1+(dD)^{-b}} \\ c\equiv 0 \pmod 4}} S_{D,d}\big(|m|, c\big) g_{m,s}^{D,d}(c) \exp\left(\frac{4\pi|m|\sqrt{dD}}{c}\right),$$

where

$$g_{m,s}^{D,d}(c) := |m|^{s-1} \sum_{j=0}^{s-1} \left(\frac{-\sqrt{dD}}{2\pi|m|c}\right)^j \frac{(s-1+j)!}{(s-1-j)!\,j!}.$$

Proof. Each $Q \in \mathcal{Q}_{dD}$ is of the form $cX^2 + xXY + aY^2 \in \mathbb{Z}[X, Y]$ for integers c, x and a. A classical parameterization due to Gauss implies that those z_Q with $\mathrm{Im}(z_Q) > 1 + (dD)^{-b}$ are given by integer pairs (c, x) with $0 < c < \sqrt{dD}/2(1 + (dD)^{-b})$ and $x^2 \equiv -dD \pmod{4c}$. Letting $4c \to c$, we find that

$$\begin{aligned}
&\sum_{j=0}^{s-1} \kappa_{m,j}(s) \sum_{\mathrm{Im}(z_Q)>1+(dD)^{-b}} \chi_d(Q)\,\mathrm{Im}(z_Q)^{-j} e(mz_Q) \\
&\quad= \frac{1}{2} \sum_{j=0}^{s-1} \kappa_{m,j}(s) \sum_{\substack{0<c<\frac{2\sqrt{dD}}{1+(dD)^{-b}} \\ c\equiv 0 \pmod 4}} \exp\left(\frac{4\pi|m|\sqrt{dD}}{c}\right) \\
&\qquad\times \sum_{x^2\equiv -dD \pmod c} \chi_d\left(\frac{c}{4}, x, \frac{x^2+dD}{c}\right) \left(\frac{2\sqrt{dD}}{c}\right)^{-j} e\big(2|m|x/c\big) \\
&\quad= \frac{1}{2} \sum_{\substack{0<c<\frac{2\sqrt{dD}}{1+(dD)^{-b}} \\ c\equiv 0 \pmod 4}} g_{m,s}^{D,d}(c) S_{D,d}\big(|m|, c\big) \exp\left(\frac{4\pi|m|\sqrt{dD}}{c}\right). \qquad \square
\end{aligned}$$

12. Evaluation of the regularized integral

In this section we evaluate the regularized integral appearing in Theorem 7.1.

Proposition 12.1. *Let $m \in \mathbb{Z}^-$ and $s \in \mathbb{Z}^+$. Then*

$$\lim_{Y\to\infty} \int_{\mathcal{F}_Y} F_m(z, s)\, d\mu = \begin{cases} 0, & s = 1, \\ -24\sigma(|m|)|m|^{s-1}, & s \geqslant 2. \end{cases}$$

Proof. We proceed using the method of Lerche, Schellekens, and Warner [23] (see also [2,11]). First observe that one has the identity

$$\frac{\partial(E_2(z)-\frac{3}{\pi\,\mathrm{Im}(z)})}{\partial\bar{z}}=\frac{3i}{2\pi\,\mathrm{Im}(z)^2},$$

where

$$E_2(z):=1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n,\quad q:=e(z)$$

and $z=x+iy$. Because $d\bar{z}\,dz=2i\,dx\,dy$ and $F_m(z,s)$ is Γ-invariant, it follows from Stokes' theorem that

$$\lim_{Y\to\infty}\int_{\mathcal{F}_Y}F_m(z,s)\,d\mu=\lim_{Y\to\infty}\int_{-\frac{1}{2}+iY}^{\frac{1}{2}+iY}F_m(x+iY,s)\left(E_2(x+iY)-\frac{3}{\pi Y}\right)dx. \tag{12.1}$$

Using the expansion given in Proposition 6.1 for the forms $F_m(x+iY,s)$, we find after multiplying that

$$\begin{aligned}&F_m(x+iY,s)\left(E_2(x+iY)-\frac{3}{\pi Y}\right)\\&\quad=-24\sigma\big(|m|\big)2\pi|m|^{s-\frac{1}{2}}e^{-2\pi|m|Y}Y^{\frac{1}{2}}I_{s-\frac{1}{2}}\big(2\pi|m|Y\big)+\frac{4\pi^{1+s}\sigma_{2s-1}(|m|)Y^{1-s}}{(2s-1)\Gamma(s)\zeta(2s)}\left(1-\frac{3}{\pi Y}\right)\\&\qquad+\sum_{\substack{k\in\mathbb{Z}\\k\neq0}}a_k(m,Y,s)e(kx)+O\left(\frac{1}{Y}\right).\end{aligned}$$

Next, using the expansion for the Bessel function given in the proof of Proposition 6.1, we find that the integral (12.1) equals

$$\lim_{Y\to\infty}-24\sigma\big(|m|\big)|m|^{s-1}\left(1+\sum_{j=1}^{\infty}O\big(Y^{-j}\big)\right)+\frac{4\pi^{1+s}\sigma_{2s-1}(|m|)Y^{1-s}}{(2s-1)\Gamma(s)\zeta(2s)}\left(1-\frac{3}{\pi Y}\right)+O\left(\frac{1}{Y}\right).$$

The result follows upon taking the limit as $Y\to\infty$, where if $s=1$ we use the fact that $\zeta(2)=\pi^2/6$. □

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