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Duality involving the mock theta function $f(q)$

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Dedicated to Dorian Goldfeld on the occasion of his sixtieth birthday

Abstract

We show that the coefficients of Ramanujan's mock theta function $f(q)$ are the first non-trivial coefficients of a canonical sequence of modular forms. This fact follows from a duality which equates coefficients of the holomorphic projections of certain weight 1/2 Maass forms with coefficients of certain weight 3/2 modular forms. This work depends on the theory of Poincaré series, and a modification of an argument of Goldfeld and Sarnak on Kloosterman–Selberg zeta functions.

1. *Introduction and statement of results*

In his plenary address at the Millenial Number Theory Conference at the University of Illinois in 2000, Andrews [2] promoted a number of conjectures and problems occurring at the interface of the theory of q-series and the theory of modular forms. The enigma of Ramanujan's mock theta functions played a central role among these problems and conjectures. Thanks to recent works of Zwegers [19, 20], Ramanujan's mock theta functions are no longer a deep mystery. They are holomorphic projections of weight 1/2 harmonic weak Maass forms. As Andrews anticipated, these developments, and their generalizations, indeed play an important role in the resolution of open problems in the theory of partitions (for example, see [3, 5]).

Thanks to this new perspective, it is not difficult to obtain new relationships between mock theta functions, formerly little more than formal power series, and classical modular forms. Here, we obtain a new relationship between modular forms and the mock theta function

$$\begin{aligned} f(q) &:= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdot \ldots \cdot (1+q^n)^2} \\ &= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + 7q^7 - 6q^8 + \ldots + 486q^{47} - \ldots . \end{aligned} \tag{1.1}$$

To describe this phenomenon, in Section 3 we construct two canonical sequences of power series. For every positive integer m we use the theory of harmonic Maass forms to define a series

$$f_m(q) := 1 + \sum_{n=1}^{\infty} a_m(n) q^n. \tag{1.2}$$

Results in [3] imply that $f_1(q) = f(q)$, and numerics for $f_2(q), \ldots, f_4(q)$ suggest the following.

$$\begin{aligned} f_1(q) &= 1 + q - 2q^2 + 3q^3 - 3q^4 + \ldots + 486q^{47} + \ldots \\ f_2(q) &= 1 - 281q + 2859q^2 - 17902q^3 + \ldots \\ f_3(q) &= 1 + 3406q - 102061q^2 + \ldots \\ f_4(q) &= 1 - 23194q + \ldots \\ &\vdots \end{aligned} \tag{1.3}$$

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Each $q^{-1}f_m(q^{24})$ is, up to the principal part, which equals $-q^{23-24m}+q^{-1}$, the holomorphic part of a specific weight 1/2 harmonic weak Maass form on $\Gamma_0(144)$ with Nebentypus $\chi_{12} := \left(\frac{12}{\cdot}\right)$, where $q := e^{2\pi iz}$. (Note that for $m = 1$ the principal part vanishes.) For these reasons we choose to refer to the functions f_m as *mock theta functions*.

In a parallel calculation, we use the classical theory of modular forms to construct, for every positive integer m, a weight 3/2 weakly holomorphic modular form $-qg_m(q^{24})$ on $\Gamma_0(144)$ with Nebentypus χ_{12}, where $g_m(q)$ is of the form

$$g_m(q) := -q^{-m} + \sum_{n=1}^{\infty} b_m(n)q^n. \tag{1.4}$$

We let $g_m^*(q) := g_m(q) - \beta_{3/2,3/4}(0,-m)$, where $\beta_{\kappa,s}(u,v)$ is defined in Subsection 3.3. Numerics suggest that

$$\begin{aligned}
g_1^*(q) &= -q^{-1} + 1 - 281q + 3406q^2 - 23194q^3 + \dots \\
g_2^*(q) &= -q^{-2} - 2 + 2859q - 102061q^2 + \dots \\
g_3^*(q) &= -q^{-3} + 3 - 17902q + \dots \\
g_4^*(q) &= -q^{-4} - 3 + \dots \\
&\vdots \\
g_{47}^*(q) &= -q^{-47} + 486 + \dots \\
&\vdots
\end{aligned} \tag{1.5}$$

A brief inspection of (1.3) and (1.5) reveals a striking pattern. One notices that the coefficients of the g, when grouped by column, appear to be the coefficients of the mock theta functions $f_m(q)$ in order, where $b_m(0) := -\beta_{3/2,3/4}(0,-m)$. Here we prove that this is indeed the case.

Theorem 1.1. *Assume that the notation above holds. If m and n are positive integers, then*

$$a_m(n) = b_n(m-1).$$

Remarks. (1) Similar duality theorems for Poincaré series and modular forms appear in earlier works by Bruinier [**6**], Bringmann and Ono [**4**], and Zagier [**18**].

(2) It is likely that this phenomenon holds for most (if not all) of Ramanujan's original mock theta functions. As the methods of this paper will reveal, the main difficulty arises from the task of expressing these mock theta functions in terms of explicit Maass–Poincaré series. (For more on the history of the mock theta functions see [**1**].) For example, the results of [**3**] lead to a similar duality statement for Ramanujan's mock theta function

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} = \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \dots,$$

where $(a;q)_n := \prod_{m=0}^{n-1}(1-aq^m)$.

(3) It is conceivable that the Fourier coefficients $a_m(n), b_m(n)$ are always integers, and we would be interested in a proof or disproof of this claim. Although a weaker statement, one may be able to employ the Galois action on $M_{3/2}^!(144,\chi_{12})$ and the fact that $S_{3/2}(144,\chi_{12}) = 0$ to deduce rationality of the coefficients $b_m(n)$, and hence rationality of the $a_m(n)$ by Theorem 1.1.

In Section 3 we construct the relevant sequences of weight 1/2 harmonic weak Maass forms and weight 3/2 weakly holomorphic modular forms using the method of Poincaré series.

The numbers $a_m(n)$ and $b_m(n)$ are then explicitly given as complicated infinite sums of Kloosterman-type sums weighted by suitable Bessel functions. Armed with these formulas, Theorem 1.1 then follows from a duality statement for these Kloosterman-type sums. The connection to $f(q) = f_1(q)$ arises from earlier work by Bringmann and the second author [3], where $f(q)$ is proven to be the holomorphic part of such a Maass form. Unfortunately, there is a catch. The complicated formulas for the numbers $a_m(n)$ and $b_m(n)$ are not visibly convergent. Convergence in the case of $a_1(n)$ was established in [3] using a relationship between these Kloosterman-type sums and Salié-type sums, and the 'equidistribution' of CM points. Such arguments do not appear to apply for $a_m(n)$ when $m > 1$, and so we are forced to resort to other methods. To establish convergence, we modify an argument of Goldfeld and Sarnak [8, 15], which stems from earlier work of Selberg [16]. These issues are addressed in Section 2.

2. *A Kloosterman–Selberg zeta function*

In this section we address the convergence of certain Kloosterman–Selberg zeta functions at the point $s = 3/4$. This fact will be employed to justify the convergence of the formulas for the $a_m(n)$ derived in Section 3. Arguments of this type appear in works by Selberg [16], Goldfeld and Sarnak [8], Iwaniec [10], Iwaniec and Kowalski [11], and Pribitkin [13].

First we fix notation. Let $\kappa \in \mathbb{R}$, let $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index, and let Ψ be a multiplier system of weight κ on Γ satisfying the consistency conditions

$$\Psi(-I) = e(\kappa/2), \tag{2.1}$$

$$j_{\gamma\gamma'}(\tau)^\kappa \Psi(\gamma\gamma') = j_\gamma(\gamma'(\tau))^\kappa j_{\gamma'}(\tau)^\kappa \Psi(\gamma)\Psi(\gamma'), \tag{2.2}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\gamma' \in \Gamma$, where $j_\gamma(\tau) := c\tau + d$, $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e(t) := e^{2\pi i t}$, and $\tau \in \mathcal{H}$, the upper-half complex plane. Let $g > 0$ be the minimal positive integer such that $\left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, and fix $0 \leqslant \alpha < 1$ such that

$$\Psi\left(\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\right) = e(-\alpha).$$

For such Ψ, we define the generalized Kloosterman sum $S_k(n, m, \Psi)$ for arbitrary integer pairs $(n, m) \in \mathbb{Z}^2$, and $k \in \mathbb{N}$, by

$$S_k(n, m, \Psi) := \sum_{\substack{\gamma \in \Gamma \\ 0 \leqslant \overline{d},\, d \leqslant gk}} \overline{\Psi(\gamma)} e\left(\frac{(n-\alpha)d + (m-\alpha)\overline{d}}{gk}\right), \tag{2.3}$$

where $d\overline{d} \equiv 1 \bmod k$. (Note that κ differs from k).

Selberg's Kloosterman zeta function $Z_{n,m}(s, \Psi)$ is defined in terms of the generalized Kloosterman sums by

$$Z_{n,m}(s, \Psi) := \sum_{k>0} \frac{S_k(n, m, \Psi)}{k^{2s}}. \tag{2.4}$$

For our purposes, we assume that $\kappa \in \frac{1}{2} + \mathbb{Z}$, and $\Gamma = \Gamma_0(2)$ such that $g = 1$. Moreover, for those $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ with $c \geqslant 0$, we define the character χ by

$$\chi(\gamma) := \begin{cases} e\left(\dfrac{-b}{24}\right) & c = 0, \\ i^{-1/2}(-1)^{(1/2)(c+ad+1)} e\left(-\dfrac{a+d}{24c} - \dfrac{a}{4} + \dfrac{3dc}{8}\right) \omega_{-d,c}^{-1} & c > 0, \end{cases} \tag{2.5}$$

where

$$\omega_{d,k} := e^{\pi i s(d,k)},$$

and the Dedekind sum $s(d,k)$ is defined by

$$s(d,k) := \sum_{j \bmod k} \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{dj}{k}\right)\right),$$

with

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

In the notation above we are interested in the case $\Psi = \hat{\chi}$, where

$$\hat{\chi}(\gamma) := \begin{cases} ie\left(\dfrac{-b}{24}\right) & c = 0, \\ i^{1/2}(-1)^{(1/2)(c+ad+1)} e\left(-\dfrac{13(a+d)}{24c} - \dfrac{a}{4} + \dfrac{3dc}{8}\right)\omega_{-d,c}^{-1} & c > 0, \end{cases} \tag{2.6}$$

in which case $\alpha = 13/24$. We point out that $\hat{\chi}(\gamma)$ is merely $\chi(\gamma)$ twisted by i when $c = 0$, and $\chi(\gamma)$ twisted by $ie(-(a+d)/2c)$ when $c > 0$. With these assumptions, we obtain the following theorem.

THEOREM 2.1. *If $mn < 0$ and $\kappa = \frac{1}{2}$, then $Z_{n,m}(s, \hat{\chi})$ is convergent at $s = 3/4$.*

REMARKS. (1) A similar proof shows that Theorem 2.1 holds in greater generality, for example, for weakly holomorphic modular forms with Nebentypus on congruence subgroups $\Gamma_0(N)$, as well as harmonic weak Maass forms with suitable multiplier Ψ. As it stands, the proof of Theorem 2.1 ultimately relies on the fact that holomorphic modular forms of weight $1/2$ have no poles, a fact which can be used to obtain convergence whenever at least one of m and n is negative. However, when m and n are both positive, a similar argument can also apply by making use of the theory of theta functions, where it is simple to characterize vanishing Fourier coefficients of weight $1/2$ modular forms thanks to results such as the Serre–Stark basis theorem.

(2) We emphasize the fact that m and n have opposite sign in Theorem 2.1. Results pertaining to the analytic behavior of $Z_{n,m}(s,\Psi)$ when $mn > 0$ are given by Selberg [**16**] and Goldfeld and Sarnak [**8**], and later generalized by Pribitkin [**13**].

2.1. *The Selberg approach après Goldfeld and Sarnak*

Here we recall facts about eigenvalues and eigenfunctions of the Laplacian, and their implications concerning the analytic behavior of $Z_{n,m}(s,\Psi)$. To make this precise, we first fix notation. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\kappa \in \mathbb{R}^+$, the functions $f : \mathcal{H} \to \mathbb{C}$ satisfying the conditions

$$f(\gamma z) = \Psi(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^{\kappa} f(z),$$
$$\iint_{\mathfrak{F}_\Gamma} |f(z)|^2 \, \frac{dx\,dy}{y^2} < \infty$$

form a Hilbert space which we denote by $L^2(\mathfrak{F}_\Gamma, \kappa)$, where $\mathfrak{F}_\Gamma$ is a fundamental domain for the quotient space $\mathcal{H}/\Gamma$. The inner product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle f, g \rangle := \iint_{\mathfrak{F}_\Gamma} f(z)\overline{g(z)} \, \frac{dx\,dy}{y^2}. \tag{2.7}$$

As defined in [8, (2.1)] (see also [14, 16]), the operator

$$\widetilde{\Delta}_\kappa := y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - i\kappa y\frac{\partial}{\partial x} \tag{2.8}$$

has a self-adjoint extension to $L^2(\mathfrak{F}_\Gamma, \kappa)$ with real spectrum [14]. Let

$$\sigma_\kappa(\widetilde{\Delta}_\kappa) := \left\{\lambda \in \mathrm{Spec}(\widetilde{\Delta}_\kappa) : 0 \leqslant \lambda < \tfrac{1}{4}\right\},$$

where $\mathrm{Spec}(\widetilde{\Delta}_\kappa)$ denotes the spectrum of $\widetilde{\Delta}_\kappa$. The spectrum $\mathrm{Spec}(\widetilde{\Delta}_\kappa)$ was first investigated seriously by Selberg in [16], who made the important observation that

$$\dim(\sigma_\kappa(\widetilde{\Delta}_\kappa)) < \infty.$$

We order the elements of the finite spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_l$. For each λ_j, $1 \leqslant j \leqslant l$, we define $s_j \in (1/2, 1]$ by $\lambda_j = s_j(1-s_j)$, and let $u_j(z)$ be the eigenfunction corresponding to λ_j, where $z = x+iy \in \mathcal{H}$. The functions $u_j(z)$, $1 \leqslant j \leqslant l$, are chosen so that they form an orthonormal basis for L^2 with respect to the inner product as defined in (2.7). Each $u_j(z)$ admits a Fourier expansion of the form

$$\begin{aligned} u_j(z) &= \rho_j(0)y^{s_j} + \rho_j'(0)y^{1-s_j} \\ &\quad + \sum_{n\in\mathbb{Z}, n\neq\alpha} \rho_j(n) W_{(\kappa/2)\mathrm{sgn}((n-\alpha)/g),\, s_j-1/2}\left(4\pi\left|\frac{n-\alpha}{g}\right|y\right) e\left(\frac{n-\alpha}{g}x\right), \end{aligned} \tag{2.9}$$

where $\rho_j(n), n \in \mathbb{Z}$, are constants, and $W_{\nu,\mu}(y)$ is the classical Whittaker function.

The next proposition of Goldfeld and Sarnak [8], which essentially realizes $Z_{n,m}(s,\Psi)$ as an inner product of Poincaré series $P_n(z,s,\Gamma,\Psi,\kappa)$, allows one to analytically continue $Z_{n,m}(s,\Psi)$ to all s with $\mathrm{Re}(s) > 1/2$. For $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, we let $\overline{\Gamma} := \Gamma/\{\pm 1\}$, and choose Ψ, κ as in the previous section. For integers n, we define the Poincaré series $P_n(z,s,\Gamma,\Psi,\kappa)$ by

$$P_n(z,s,\Gamma,\Psi,\kappa) := \sum_{\gamma\in\overline{\Gamma}/\overline{\Gamma}_\infty} \overline{\Psi}(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^{-\kappa} e((n-\alpha)\mathrm{Re}(\gamma z)/g)e^{2\pi|n-\alpha|\mathrm{Im}(\gamma z)/g}\frac{y^s}{|cz+d|^{2s}}, \tag{2.10}$$

where $\Gamma_\infty = \left\{\left(\begin{smallmatrix}1 & gn\\ 0 & 1\end{smallmatrix}\right) : n \in \mathbb{Z}\right\}$.

Lemma 2.2 (Goldfeld and Sarnak [8]). *If $m, n, \kappa > 0$, then*

$$\begin{aligned} &\iint_{\mathfrak{F}_\Gamma} P_n(z,s,\Gamma,\Psi,\kappa)\overline{P_m(z,\overline{s}+2,\Gamma,\Psi,\kappa)}\,\frac{dx\,dy}{y^2} \\ &\qquad = \frac{(-i)^\kappa 4^{-s-1}n^{-2}\Gamma(2s+1)}{\pi\Gamma(s+k/2)\Gamma(s+2-k/2)} Z_{n,m}(s,\Psi) + \widetilde{R}(s), \end{aligned}$$

where $\widetilde{R}(s)$ is holomorphic in $\mathrm{Re}(s) > 1/2$ and satisfies the condition $\widetilde{R}(s) \ll \mathrm{Re}(s)^{1/2}$.

Using Lemma 2.2 to analytically continue $Z_{n,m}(s,\Psi)$ to $\mathrm{Re}(s) > 1/2$, one finds that the poles $Z_{n,m}(s,\Psi)$ are dictated by the elements of the finite spectrum $\sigma_\kappa(\widetilde{\Delta}_\kappa)$, that they are simple, and that they can only occur at the points s_j. The following lemma bounds $\sigma_\kappa(\widetilde{\Delta}_\kappa)$ from below (see, for example, [15] or [16]).

Lemma 2.3. *The finite spectrum $\sigma_\kappa(\widetilde{\Delta}_\kappa)$ is bounded below by $\lambda_\kappa^* := (\kappa/2)(1-\kappa/2)$.*

Much is known regarding the relationship between eigenfunctions $u(z)$ of $\widetilde{\Delta}_\kappa$ with eigenvalue λ_κ^*, and modular forms. For simplicity, we describe those $u(z)$ corresponding to modular forms of half-integral weight with Nebentypus on a congruence subgroup $\Gamma_0(4N) \subseteq \mathrm{SL}_2(\mathbb{Z})$.

LEMMA 2.4. *Let $\kappa \in \frac{1}{2} + \mathbb{Z}$ and $\Gamma = \Gamma_0(4N)$, and let $\Psi = \chi^*$ be a Nebentypus character on Γ (in the sense of Shimura [17]). If $u(z)$ is an eigenfunction of $\widetilde{\Delta}_\kappa$ corresponding to λ_κ^* with Fourier expansion as in (2.9), then $\widetilde{u}(z) := y^{-\kappa/2} u(z)$ is a holomorphic modular form of weight κ.*

Proof. Suppose that $u(z)$ is an eigenfunction of $\widetilde{\Delta}_\kappa$ corresponding to λ_κ^*, and let

$$\widetilde{\Lambda}_\kappa := (z - \overline{z}) \frac{\partial}{\partial \overline{z}} + \frac{\kappa}{2}.$$

For all f, g smooth and of compact support on $\mathfrak{F}_\Gamma$, one has [14]

$$\langle g, \widetilde{\Delta}_\kappa f \rangle = \langle \widetilde{\Lambda}_\kappa g, \widetilde{\Lambda}_\kappa f \rangle + \frac{\kappa}{2}\left(1 - \frac{\kappa}{2}\right) \langle g, f \rangle. \tag{2.11}$$

If we let $\widetilde{u}(z) := y^{-\kappa/2} u(z)$, then by (2.11) one may establish that $\widetilde{u}(z)$ is a holomorphic modular form of weight κ. □

REMARK. One may strengthen Lemma 2.4 and establish a converse statement under suitable conditions. Namely, if $\widetilde{u}(z)$ is a cusp form, then define $u(z) = y^{\kappa/2} \widetilde{u}(z)$. One finds that $u(z)$ satisfies $\widetilde{\Delta}_\kappa(u) + \lambda_\kappa^* u = 0$, and since $\widetilde{u}(z)$ is a cusp form, $u(z) \in L^2(\mathfrak{F}_\Gamma, \kappa)$.

2.2. *Residue and the resolvent*

Since the arguments m and n of the Kloosterman zeta function have opposite sign in Theorem 2.1, we are unable to simply quote results from the literature. Here, we give the remaining tools required for the proof of Theorem 2.1. We first state a residue formula for $Z_{n,m}(s, \Psi)$ when $m, n > 0$ given in [8], and in the following section we discuss the case $mn < 0$.

LEMMA 2.5. *If $m, n > 0$, then the residue of $Z_{n,m}(s, \Psi)$ at a point s_i is given by*

$$\mathrm{Res}_{s=s_i}(Z_{n,m}(s, \Psi)) = \sum \frac{g^2 \overline{\rho_j(m)} \rho_j(n) \left(((m-\alpha)/g)\,((n-\alpha)/g)\right)^{1-s_j} \Gamma(s_j + \kappa/2) \Gamma(2s_j - 1)}{\Gamma(s_j - \kappa/2)(-i)^\kappa 4^{2s_j} \pi^{3s_j+1}},$$

where the sum is taken over those j for which $s_i = s_j$.

Proof. A detailed proof, which follows from standard facts from harmonic analysis and properties of the Poincaré series P_m, is not given in [8]. We offer some details to elucidate our eventual analysis of the case $mn < 0$. We first recall that each Poincaré series $P_m(z, s, \Gamma, \Psi, \kappa)$ satisfies

$$P_m(z, s, \Gamma, \Psi, \kappa) = -4\pi \left(\frac{m-\alpha}{q}\right) \left(s - \frac{\kappa}{2}\right) \mathbf{R}_{s(1-s)}(\widetilde{\Delta}_\kappa)(P_m(z, s+1, \Gamma, \Psi, \kappa)),$$

where $\mathbf{R}_\lambda(\widetilde{\Delta}_\kappa)$ is the resolvent operator with respect to $\widetilde{\Delta}_\kappa$ at λ (see, for example, [12]). This fact, combined with Lemma 2.2, allows one to interpret the residue of $Z_{n,m}$ in terms of the residue of the resolvent operator $\mathbf{R}_\lambda(\widetilde{\Delta}_\kappa)$. The singularities of the resolvent operator occur precisely at the eigenvalues λ_j, and a standard argument of harmonic analysis (see, for example, [12, I § 5]) reveals that relations satisfied by the residues of $\mathbf{R}_\lambda$ imply that they are projections. More precisely, a calculation shows that

$$\mathrm{Res}_{s=s_i}\langle P_m(z,s,\Gamma,\Psi,\kappa),P_n(z,s,\Gamma,\Psi,\kappa)\rangle=\sum\overline{\rho_j(m)}\langle u_j,P_n\rangle, \tag{2.12}$$

where an explicit formula for $\langle u_j, P_n\rangle$ may be found in [**15**]. The sum above is taken over those j for which $s_i = s_j$. Following a short computation, one obtains the lemma. □

2.3. *Proof of Theorem* 2.1

Extending results beyond the case $m, n > 0$, by a remark of Pribitkin [**13**], one may derive a formula analogous to that of Lemma 2.2 for the case $n < 0$ and $m > 0$ by considering the inner product

$$\left\langle P_m(z,s,\Gamma,\Psi,\kappa),\overline{P_{n^*}(z,w,\Gamma,\overline{\Psi},-\kappa)}\right\rangle,$$

where $n^* := -n-1$ for $\kappa \in (0,1)$. Similarly, one obtains a formula for the case $n > 0$ and $m < 0$ by using the case $n < 0$ and $m > 0$ combined with the fact that

$$\overline{Z_{n,m}(\overline{s},\Psi,\kappa)}=Z_{-n-1,-m-1}(s,\overline{\Psi},-\kappa)$$

for $\kappa \in (0,1)$. With this, one may derive a residue formula analogous to that of Lemma 2.5.

One observes that the vanishing or non-vanishing of the coefficients $\rho(m)$ and $\rho(n)$ determine a residue or non-residue of $Z_{n,m}$ at s_j. We note that the minimal eigenvalue $(\kappa/2)(1-\kappa/2)$ $= 3/16$ corresponds to $s_j = 3/4$ since $\kappa = 1/2$. Since $mn < 0$, by Lemma 2.4, we have the product $\rho(m)\rho(n) = 0$, as eigenfunctions with lowest eigenvalue correspond to cusp forms. Holomorphicity now follows, since the eigenfunctions with lowest eigenvalue correspond to holomorphic modular forms. One deduces convergence using standard methods relating the series expansions of Kloosterman–Selberg zeta functions with their integral representations, employing Perron-type formulas.

3. *Poincaré series and the proof of Theorem* 1.1

We first recall fundamental properties of Maass forms. A *weakly holomorphic modular form* is any meromorphic modular form, the poles of which (if there are any) are supported at cusps. Theorem 1.1 is a duality statement relating the Fourier coefficients of weight 3/2 weakly holomorphic modular forms and certain weight 1/2 harmonic weak Maass forms. Now we recall the notion of a harmonic weak Maass form. Suppose that $\kappa \in \frac{1}{2}+\mathbb{Z}$. If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4,\\ i & \text{if } v\equiv 3 \pmod 4.\end{cases} \tag{3.1}$$

Similar to the operator $\widetilde{\Delta}_\kappa$ defined in (2.8), we let

$$\Delta_\kappa := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+i\kappa y\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) \tag{3.2}$$

be the weight κ hyperbolic Laplacian, where $z = x + iy$ with $x, y \in \mathbb{R}$. If N is a positive integer and $\psi \pmod{4N}$ is a Dirichlet character, then a *harmonic weak Maass form* of weight κ on $\Gamma_0(4N)$ with Nebentypus ψ is any smooth function $f : \mathcal{H} \to \mathbb{C}$ satisfying the following conditions.

(1) For all $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(4N)$ and all $z \in \mathcal{H}$, we have

$$f(Az)=\psi(d)\left(\frac{c}{d}\right)^{2\kappa}\epsilon_d^{-2\kappa}\,(cz+d)^{\kappa}\,f(z).$$

(2) We have $\Delta_\kappa f = 0$.

(3) The function $f(z)$ has at most linear exponential growth at all cusps.

3.1. *Two sequences of Poincaré series*

We begin by defining the Poincaré series $P^{\Upsilon}_{\kappa,s}(r,z)$ for $r \in \mathbb{R}\setminus\{0\}$, $z \in \mathcal{H}$, and a character Υ on $\Gamma_0(2)/\pm\Gamma_\infty$ by

$$P^{\Upsilon}_{\kappa,s}(r,z) := \frac{2}{\sqrt{\pi}} \sum_{A\in\Gamma_0(2)/\pm\Gamma_\infty} \Upsilon(A)^{-1}(cz+d)^{-\kappa}\varphi_{\kappa,s}(r,Az), \tag{3.3}$$

where A is represented by $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Here, the function $\varphi_{\kappa,s}(r,z)$ is defined by the M-Whittaker function $M_{\nu,\mu}$ as follows:

$$\varphi_{\kappa,s}(r,z) := \mathcal{M}_s(4\pi|r|y)e(rx),$$

where, for $y \geqslant 0$,

$$\mathcal{M}_s(y) := y^{-\kappa/2}M_{-\kappa/2,\, s-1/2}(y).$$

The two sequences of Poincaré series that we define are

$$P^{\chi}_{\kappa,s}\left(\frac{24m-1}{24}, z\right) \quad \text{and} \quad \frac{\sqrt{\pi}}{2}P^{\chi^{-1}}_{\kappa,s}\left(\frac{24m+1}{24}, z\right),$$

where m is a non-positive integer, and the character χ is as defined in (2.5). We will show that the following theorem holds.

THEOREM 3.1. *For m a non-positive integer define $M_1 := M_1(m) = (24m-1)/24$, and for m a negative integer define $M_2 := M_2(m) = (24m+1)/24$. For $\kappa_1 \leqslant 1/2, \kappa_1 \in \frac{1}{2}+\mathbb{Z}$ and $\kappa_2 \geqslant 3/2, \kappa_2 \in \frac{3}{2}+\mathbb{Z}$, the functions*

$$P^{\chi}_{\kappa_1,1-\kappa_1/2}(M_1, 24z) \quad \text{and} \quad \frac{\sqrt{\pi}}{2}P^{\chi^{-1}}_{\kappa_2,\kappa_2/2}(M_2, 24z)$$

are harmonic weak Maass forms on $\Gamma_0(144)$ with Nebentypus $\chi_{12} := (\frac{12}{\cdot})$ of weights κ_1 and κ_2, respectively. Moreover, when $\kappa_2 = 3/2$, $(\sqrt{\pi}/2)P^{\chi^{-1}}_{3/2,3/4}(M_2, 24z)$ is a weakly holomorphic modular form.

We prove Theorem 3.1 by first computing the Fourier expansion for the series $P^{\chi}_{\kappa,s}$ and $P^{\chi^{-1}}_{\kappa,s}$, and then justifying the convergence of the expressions obtained for the Fourier coefficients. We treat the cases $P^{\chi}_{\kappa,s}$ and $P^{\chi^{-1}}_{\kappa,s}$ separately, and begin with the former. We choose a set of representatives $\left\{A = \left(\begin{smallmatrix} a & b+na \\ 2c & d+2nc\end{smallmatrix}\right)\right\}_{n\in\mathbb{Z}}$ for $\Gamma_0(2)/\pm\Gamma_\infty$ indexed by pairs $(2c,d)$, where $c \in \mathbb{N}$, $1 \leqslant d < 2c$ with $\gcd(2c,d) = 1$, and (a,b) chosen arbitrarily so that $ad - 2bc = 1$, together with the pair $(2c,d) = (0,1)$. In computing the expansion of $P^{\chi}_{\kappa,s}$, we use the facts that

$$Mz = \frac{a}{c} - \frac{1}{c(cz+d)},$$

for any $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, and also that $(-1)^{-anc}e(-3c^2n/2) = 1$. Thus, after some computation, we may write

$$\begin{aligned} P^{\chi}_{\kappa,s}\left(\frac{24m-1}{24}, z\right) &= \frac{2}{\sqrt{\pi}}\mathcal{M}_s\left(\frac{|24m-1|\pi y}{6}\right)e\left(\frac{(24m-1)x}{24}\right) \\ &\quad + \frac{2}{\sqrt{\pi}}\sum_{c>0} i^{1/2} \sum_{d \,(\mathrm{mod}\, 2c)^\times} (-1)^{-(1/2)(2c+ad+1)} \\ &\quad \times e\left(\frac{d}{48c} + \frac{a-3dc}{4}\right)e\left(\frac{ma}{2c}\right)\omega_{-d,2c}(2c)^{-\kappa}T_\kappa\left(z+\frac{d}{2c}\right), \end{aligned} \tag{3.4}$$

where

$$T_\kappa(z) := \sum_{n\in\mathbb{Z}} (z+n)^{-\kappa} \mathcal{M}_s\left(\frac{|24m-1|\pi y}{24c^2|z+n|^2}\right) e\left(-\frac{(24m-1)(x+n)}{96c^2|z+n|^2}\right) e\left(\frac{n}{24}\right).$$

REMARK. For those readers who wish to reproduce calculation (3.4), we note that the factor $e(a/48c)$ that emerges due to the presence of $\chi^{-1}(A)$ in the expansion cancels with a factor $e(-a/48c)$ that emerges due to the presence of the function $\varphi_{\kappa,s}$.

We now state a useful form of the Poisson summation formula.

LEMMA 3.2 (*Poisson summation*). *Let $h \in L^1 \cap C^0(\mathbb{R})$ satisfying $|h(x)|, |\hat{h}(x)| \leqslant K|x|^{-s}$, where $|x| \geqslant 1$, $s > 1$, and $K \in \mathbb{R}^{\geqslant 0}$. Then*

$$\sum_{n=-\infty}^{\infty} e(nA)h(B+nC) = \sum_{n=-\infty}^{\infty} e\left(-\frac{B(n+A)}{C}\right) \hat{h}\left(\frac{(n+A)}{C}\right),$$

where

$$\hat{h}(t) = \int_{-\infty}^{\infty} e(tx)h(x)\,dx.$$

Using Lemma 3.2 with $A = 1/24$, $B = x$, and $C = 1$, we find, after changing the index of summation to $-n \in \mathbb{Z}$, that the function $T_\kappa(z)$ has a Fourier expansion of the form $T_\kappa(z) = \sum_{n\in\mathbb{Z}} a_y(n)e(n-1/24)x$, where

$$a_y(n) := \int_{\mathbb{R}} z^{-\kappa} \mathcal{M}_s\left(\frac{|24m-1|\pi y}{24c^2|z|^2}\right) e\left(\frac{-(24m-1)x}{96c^2|z|^2} - \left(n-\frac{1}{24}\right)x\right) dx. \tag{3.5}$$

Similar integrals are computed in [**6**, **9**], from which we deduce that

$$a_y(n) = \frac{(2/i)^\kappa \pi \Gamma(2s)}{c^{1-\kappa}\Gamma(s-\kappa/2)} \left|\frac{24n-1}{24m-1}\right|^{(\kappa-1)/2} J_{2s-1}\left(\frac{\pi\sqrt{|(24m-1)(24n-1)|}}{12c}\right) \times \mathcal{W}_s\left(4\pi\left(n-\frac{1}{24}\right)y\right)$$

if $n < 0$,

$$a_y(n) = \frac{i^{-\kappa}4^{1-2s}\pi^{1+s}|24m-1|^s y^{1-s}\Gamma(2s)}{6^s c^{2s}(2s-1)\Gamma(s+\kappa/2)\Gamma(s-\kappa/2)}$$

if $n = 0$, and

$$a_y(n) = \frac{(2/i)^\kappa \pi \Gamma(2s)}{c^{1-\kappa}\Gamma(s+\kappa/2)} \left|\frac{24n-1}{24m-1}\right|^{(\kappa-1)/2} I_{2s-1}\left(\frac{\pi\sqrt{|(24m-1)(24n-1)|}}{12c}\right) \mathcal{W}_s\left(4\pi\left(n-\frac{1}{24}\right)y\right)$$

if $n > 0$, where $I_\alpha(Y)$ and $J_\alpha(Y)$ are the I_α-Bessel and J_α-Bessel functions, respectively.

As usual, we require Kloosterman-like sums to describe the Fourier coefficients of our Poincaré series $P^{\Upsilon}_{\kappa,s}$. We define

$$\mathcal{A}_x(n,m) := \sum_{d \,(\mathrm{mod}\, x)^\times} e\left(\frac{nd+m\bar{d}}{x}\right) \omega_{-d,x} \tag{3.6}$$

$$\widetilde{\mathcal{A}}_x(n,m) := \sum_{d \,(\mathrm{mod}\, x)^\times} e\left(\frac{nd+m\bar{d}}{x}\right) \omega_{-d,x}^{-1} \tag{3.7}$$

where $d\bar{d} \equiv 1 \bmod x$.

REMARK. The function $\mathcal{A}_x(n,m)$ is a generalization of the function $\mathcal{A}_x(n,0)=\mathcal{A}_x(n)$ that appears in Rademacher's exact formula for the partition function $p(n)$, and $\widetilde{\mathcal{A}}_x(n,m)$ is another variant that will allow us to see the duality of Theorem 1.1.

THEOREM 3.3. *For $s=1-(\kappa/2)$, $\kappa\leqslant\frac{1}{2}$, and m a non-positive integer,*

$$P^{\chi}_{\kappa,s}\left(\frac{24m-1}{24},z\right)=\frac{2}{\sqrt{\pi}}e\left(\frac{24m-1}{24}x\right)\mathcal{M}_s\left(\frac{\pi|24m-1|y}{6}\right)+\sum_{n\in\mathbb{Z}}\alpha_{\kappa,s}(n,m)q^{n-1/24},$$

where

$$\alpha_{\kappa,s}(n,m)=2\sqrt{\pi}i^{-\kappa+1/2}\frac{\Gamma(2s)}{\Gamma\left(s+\kappa/2\right)}\left|\frac{24n-1}{24m-1}\right|^{(\kappa-1)/2}$$
$$\times\sum_{c>0}\frac{(-1)^{\lfloor(c+1)/2\rfloor}\mathcal{A}_{2c}\left(n-\frac{c(1+(-1)^c)}{4},m\right)}{c}I_{2s-1}\left(\frac{\pi}{12c}\sqrt{|(24n-1)(24m-1)|}\right)$$

if $n>0$,

$$\alpha_{\kappa,s}(0,m)=\frac{2^{1-\kappa}i^{-\kappa+1/2}4^{1-2s}\pi^{1/2+s}|24m-1|^s\Gamma(2s)e(iy/24)}{y^{s-1}6^s(2s-1)\Gamma\left(s+\kappa/2\right)\Gamma\left(s-\kappa/2\right)}$$
$$\times\sum_{c>0}\frac{(-1)^{\lfloor(c+1)/2\rfloor}\mathcal{A}_{2c}\left(\frac{-c(1+(-1)^c)}{4},m\right)}{c^{2s+\kappa}},$$

and

$$\alpha_{\kappa,s}(n,m)=2\sqrt{\pi}i^{-\kappa+1/2}\frac{\Gamma\left(s-\kappa/2;4\pi\left|n-1/24\right|y\right)\Gamma(2s)}{\Gamma(s-\kappa/2)}\left|\frac{24n-1}{24m-1}\right|^{(\kappa-1)/2}$$
$$\times\sum_{c>0}\frac{(-1)^{\lfloor(c+1)/2\rfloor}\mathcal{A}_{2c}\left(n-\frac{c(1+(-1)^c)}{4},m\right)}{c}J_{2s-1}\left(\frac{\pi}{12c}\sqrt{|(24n-1)(24m-1)|}\right)$$

if $n<0$.

REMARK. We point out that for $\kappa=\frac{1}{2}$, the leading term appearing in the Fourier expansion for $P^{\chi}_{1/2,3/4}$ reduces to

$$\frac{2}{\sqrt{\pi}}e\left(\frac{24m-1}{24}x\right)\mathcal{M}_{3/4}\left(\frac{|24m-1|\pi y}{6}\right)=q^{(24m-1)/24}\left(1-\pi^{-1/2}\Gamma\left(\frac{1}{2},\frac{|24m-1|\pi y}{6}\right)\right).$$

This follows from the fact that for $\kappa=\frac{1}{2}$,

$$\mathcal{M}_{3/4}(|y|)=\tfrac{1}{2}(\sqrt{\pi}-\Gamma(\tfrac{1}{2},|y|))e^{|y|/2},$$

that $\Gamma(3/2)=\sqrt{\pi}/2$, and that m is non-positive.

We have the following useful lemma.

LEMMA 3.4. *In the notation above, we have*

$$\sum_{d(\operatorname{mod}2c)^\times}(-1)^{-(1/2)(2c+1+ad)}e\left(\frac{a-3dc}{4}+\frac{nd+ma}{2c}\right)\omega_{-d,2c}$$
$$=(-1)^{\lfloor(c+1)/2\rfloor}\mathcal{A}_{2c}\left(n-\frac{c(1+(-1)^c)}{4},m\right).$$

3.2. *Proof of Lemma* 3.4

If $c \equiv 0 \bmod 2$, then a sufficient condition to prove Lemma 3.4 is the following:

$$3c - (a + d) + ad + 1 + 3dc \equiv 0 \text{ mod } 4, \tag{3.8}$$

for each $d \in [1, 2c)$ such that $\gcd(d, 2c) = 1$. Using the fact that $ad - 2bc = 1$ we find that either $(a, d) \equiv (1, 1)$ or $(a, d) \equiv (-1, -1) \bmod 4$. An easy check shows that (3.8) holds in both cases. If $c \equiv 1 \bmod 2$ then the following condition is sufficient to prove Lemma 3.4:

$$3c - a + ad + 2 + 3dc \equiv 0 \text{ mod } 4, \tag{3.9}$$

for each $d \in [1, 2c)$ such that $\gcd(d, 2c) = 1$. Again using the fact that $ad - 2bc = 1$, we have $ad \equiv \pm 1$ mod 4. If $ad \equiv -1$ mod 4 then (3.9) holds if and only if $6c + 2 = 2(3c + 1) \equiv 0$ mod 4, which follows from the fact that $c \equiv 1$ mod 2. If $ad \equiv 1$ mod 4 then $(a, d) \equiv (1, 1)$ or $(a, d) \equiv (-1, -1)$ mod 4 and one may easily verify (3.9) in these cases as well.

3.3. *Proof of Theorem* 3.3

Strictly speaking, Theorem 3.3 follows by explicit calculation combined with the fact that our formulas for the Fourier coefficients $\alpha_{\kappa,s}$ converge. For $\kappa < \frac{1}{2}$, one obtains convergence by applying standard existing bounds on Kloosterman sums, Lemma 3.4, and the identity

$$\mathcal{W}_{1-\kappa/2}(y) = \begin{cases} e^{-y/2} & y > 0, \\ e^{-y/2}\Gamma(1-\kappa, |y|) & y < 0, \end{cases}$$

to the formulas given in the previous sections for $\alpha_{\kappa,s}$. Convergence is only questionable in the case $\kappa = \frac{1}{2}$, which we give at the end of this section.

With our second family of Poincaré series $(\sqrt{\pi}/2)P_{\kappa,s}^{\chi^{-1}}((24m+1)/24, z)$, similar to the argument given for $P_{\kappa,s}^{\chi}((24m-1)/24, z)$, we find that

$$\begin{aligned} \frac{\sqrt{\pi}}{2} P_{\kappa,s}^{\chi^{-1}}\left(\frac{24m+1}{24}, z\right) &= \mathcal{M}_s\left(\frac{|24m+1|\pi y}{6}\right) e\left(\frac{(24m+1)x}{24}\right) \\ &\quad + \sum_{c>0} i^{-1/2} \sum_{d(\text{mod } 2c)^\times} (-1)^{(1/2)(2c+ad+1)} e\left(-\frac{d}{48c} - \frac{a-3dc}{4}\right) \\ &\quad \times e\left(\frac{ma}{2c}\right) \omega_{-d,2c}^{-1} (2c)^{-\kappa} V_\kappa\left(z + \frac{d}{2c}\right), \end{aligned}$$

where

$$V_\kappa(z) := \sum_{n\in\mathbb{Z}} (z+n)^{-\kappa} \mathcal{M}_s\left(\frac{|24m+1|\pi y}{24c^2|z+n|^2}\right) e\left(-\frac{(24m+1)(x+n)}{96c^2|z+n|^2}\right) e\left(\frac{-n}{24}\right).$$

The function $V_\kappa(z)$ has a Fourier expansion of the form $V_\kappa(z) = \sum_{n\in\mathbb{Z}} b_y(n) e((n + \frac{1}{24})x)$, where

$$b_y(n) := \int_{\mathbb{R}} z^{-\kappa} \mathcal{M}_s\left(\frac{|24m+1|\pi y}{24c^2|z|^2}\right) e\left(\frac{-(24m+1)x}{96c^2|z|^2} - \left(n + \frac{1}{24}\right)x\right) dx.$$

One finds as in the case of $a_y(n)$ that

$$\begin{aligned} b_y(n) = \frac{(2/i)^\kappa \pi \Gamma(2s)}{c^{1-\kappa}\Gamma(s-\kappa/2)} \left|\frac{24n+1}{24m+1}\right|^{(\kappa-1)/2} J_{2s-1}\left(\frac{\pi\sqrt{|(24m+1)(24n+1)|}}{12c}\right) \\ \times \mathcal{W}_s\left(4\pi\left(n + \frac{1}{24}\right)y\right) \end{aligned}$$

if $n < 0$,

$$b_y(0) = \frac{i^{-\kappa} 4^{1-2s} \pi^{1+s} |24m+1|^s y^{1-s} \Gamma(2s)}{6^s c^{2s} (2s-1) \Gamma(s+\kappa/2) \Gamma(s-\kappa/2)},$$

and

$$b_y(n) = \frac{(2/i)^{\kappa} \pi \Gamma(2s)}{c^{1-\kappa} \Gamma(s+\kappa/2)} \left| \frac{24n+1}{24m+1} \right|^{(\kappa-1)/2} I_{2s-1} \left(\frac{\pi \sqrt{|(24m+1)(24n+1)|}}{12c} \right)$$
$$\times \mathcal{W}_s \left(4\pi \left(n + \frac{1}{24} \right) y \right)$$

if $n > 0$.

THEOREM 3.5. *For $s = \kappa/2$, $\kappa \geqslant \frac{3}{2}$, and m a negative integer,*

$$\frac{\sqrt{\pi}}{2} P^{\chi^{-1}}_{\kappa,\kappa/2} \left(\frac{24m+1}{24}, z \right) = \mathcal{M}_{\kappa/2} \left(\frac{|24m+1| \pi y}{6} \right) e \left(\frac{(24m+1)x}{24} \right)$$
$$+ \sum_{n \geqslant 1} \beta_{\kappa,s}(n,m) q^{n+1/24}, \tag{3.10}$$

where

$$\beta_{\kappa,s}(n,m) = \frac{\pi}{i^{\kappa+1/2}} \frac{\Gamma(2s)}{\Gamma(s+\kappa/2)} \left| \frac{24n+1}{24m+1} \right|^{(\kappa-1)/2}$$
$$\times \sum_{c>0} \frac{(-1)^{\lfloor (c+1)/2 \rfloor} \widetilde{\mathcal{A}}_{2c} \left(n + \frac{c(1+(-1)^c)}{4}, m \right)}{c}$$
$$\times I_{2s-1} \left(\frac{\pi}{12c} \sqrt{|(24n+1)(24m+1)|} \right). \tag{3.11}$$

3.4. *Proof of Theorem* 3.5

As in the proof of Theorem 3.3, for $\kappa > \frac{3}{2}$, Theorem 3.5 follows by explicit calculation combined with the fact that our formulas for the Fourier coefficients $\beta_{\kappa,s}$ converge. For $\kappa = \frac{3}{2}$, we observe the presence of the factor $\Gamma(s - (\kappa/2))$ in the denominator of the coefficients $b_y(n)$ for $n \leqslant 0$, so that the terms in the Fourier expansion corresponding to non-positive n vanish. Arguing as in Proposition 3.4 allows us to recognize the presence of the Kloosterman-like sum $\widetilde{A}_x$.

REMARK. Note that when $\kappa = \frac{3}{2}$, the leading term in (3.10) reduces to

$$\mathcal{M}_{3/4} \left(\frac{|24m+1| \pi y}{6} \right) e \left(\frac{(24m+1)x}{24} \right) = q^{(-24|m|+1)/24}.$$

This follows from the fact, for $\kappa = \frac{3}{2}$, that $\mathcal{M}_{3/4}(|y|) = e^{|y|/2}$ and that m is negative.

3.5. *The $f_m(q)$ and $g_m(q)$ series*

For $m \geqslant 1$, let

$$f_m(z) = -q^{1-m} + 1 + q^{1/24} \mathrm{Hol} \left(P^{\chi}_{1/2,3/4} \left(\frac{-24m+23}{24}, z \right) \right),$$
$$g_m(z) = -q^{-1/24} \frac{\sqrt{\pi}}{2} P^{\chi^{-1}}_{3/2,3/4} \left(\frac{-24m+1}{24}, z \right),$$

so that

$$a_m(n) = \alpha_{1/2,3/4}(n, -m+1),$$
$$b_m(n) = -\beta_{3/2,3/4}(n-1, -m),$$

where $\mathrm{Hol}(h)$ denotes the holomorphic part of h. We first argue the convergence of the expansions for the Fourier coefficients of these Poincaré series in the case where $\kappa = 1/2$. We have chosen to rewrite the Fourier coefficients of the series P^{χ} in terms of the Kloosterman-like sums $\mathcal{A}_{2c}(n,m)$ for purposes of more easily establishing the duality of Theorem 1.1. To prove convergence, we apply Lemma 3.4 and find

$$\begin{aligned}\alpha_{1/2,3/4}(n,m) &= \pi \left|\frac{24n-1}{24m-1}\right|^{-1/4} \sum_{c>0} c^{-1} I_{1/2}\left(\frac{\pi}{12c}\sqrt{|(24n-1)(24m-1)|}\right) \\ &\quad \times \sum_{d (\mathrm{mod}\, 2c)^{\times}} i^{1/2}\hat{\chi}^{-1}\left(\begin{smallmatrix} a & b \\ 2c & d\end{smallmatrix}\right) e\left(\frac{d(n-(13/24)) + \overline{d}(m-(13/24))}{2c}\right) \\ &= \pi i^{1/2} \left|\frac{24n-1}{24m-1}\right|^{-1/4} \sum_{c>0} c^{-1} I_{1/2}\left(\frac{\pi}{12c}\sqrt{|(24n-1)(24m-1)|}\right) S_{2c}(n,m,\hat{\chi}),\end{aligned} \tag{3.12}$$

where the last equality follows from the definition of $S_k(n,m,\Psi)$, and that $\alpha = 13/24$. As $c \mapsto \infty$ we have

$$I_{1/2}\left(\frac{\pi\sqrt{X}}{24c}\right) \sim \frac{|X|^{1/4}}{\sqrt{12c}}. \tag{3.13}$$

Using this and the formula for the Fourier coefficients $\alpha_{1/2,3/4}$ given in (3.12), we are left to show that the Selberg generalized Kloosterman zeta function $Z_{n,m}(s,\hat{\chi})$ converges at $s = 3/4$. This follows from Theorem 2.1.

In summary, $\alpha_{1/2}(n,m)$ converges for integers $n > 0$ and $m \leqslant 0$. With this, by definition, $a_m(n)$ converges for integers $n, m \geqslant 1$, and thus by the duality Theorem 1.1, $b_n(m)$ converges for integers $n, m \geqslant 1$.

We are left to show convergence only for the non-holomorphic part of the series $P^{\chi}_{1/2,3/4}$. In this case, $m < 0$ and $n < 0$, and we follow an argument similar to our argument above for the case $mn < 0$. We observe that the $J_{1/2}$-Bessel function behaves as the $I_{1/2}$-Bessel function does in (3.13) as $c \mapsto \infty$, so that we are left once again to prove convergence of a Selberg–Kloosterman zeta function $Z_{n,m}$ at $s = 3/4$. In this setting we are able to use the residue formula (2.12) given in [8] which holds for $nm > 0$. We again use the fact that $\lambda^{*}_{1/2}$ corresponds to $s_j = 3/4$ and apply Lemma 2.4. We observe again that $\rho(m)\rho(n) = 0$. From these things we are able to conclude convergence.

3.6. *Proof of Theorem* 3.1

Let $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2)$, and let $A' = \left(\begin{smallmatrix} a' & b' \\ c' & d'\end{smallmatrix}\right) \in \Gamma_0(2)/\pm\Gamma_{\infty}$ denote a coset representative. Using the definition of $P^{\chi}_{\kappa,M}$ and the properties of χ, it is straightforward to verify that $P^{\chi}_{\kappa,M}(1-(\kappa/2), z)$ transforms under $A \in \Gamma_0(2)$ by

$$P^{\chi}_{k,M}\left(1-\frac{\kappa}{2}, Az\right) = \chi(A)(cz+d)^{\kappa} P^{\chi}_{\kappa,M}\left(1-\frac{\kappa}{2}, z\right).$$

The transformation for $(\sqrt{\pi}/2)P^{\chi^{-1}}_{\kappa,M}(\kappa/2,z)$ follows similarly. To prove that the functions are real analytic, for the cases $\kappa < 1/2$ and $\kappa > 3/2$, respectively, we argue as in [3] for the specific case $m = 0$, where it is shown that the automorphy factor coincides with that of the Dedekind η-function and is equal to $\chi_{12} := \left(\frac{12}{\cdot}\right)$. The convergence for the cases $\kappa \in \{1/2, 3/2\}$ is as argued in Subsection 3.5, and the fact that the forms $(\sqrt{\pi}/2)P^{\chi^{-1}}_{3/2,3/4}(\cdot, 24z)$ are weakly holomorphic modular forms is as argued in Theorem 3.5.

REMARK. The Fourier expansions at cusps may be determined for these Poincaré series. For example, the mock theta function $f(q)$ is a component of a vector valued Maass Poincaré series along with Ramanujan's mock theta function $\omega(q)$. The work [7] makes this explicit for $\omega(q)$, and one may extend these arguments to the series here.

3.7. *The Kloosterman-like sums* $\mathcal{A}_x(n,m)$ *and* $\widetilde{\mathcal{A}}_x(n,m)$

In order to continue the proof of Theorem 1.1, we first make the following important observation regarding the Kloosterman-like sums $A_x(n,m)$ and $\widetilde{A}_x(n,m)$.

LEMMA 3.6. *For $m, n \in \mathbb{Z}$, we have*

$$\mathcal{A}_{2c}\left(n - \frac{c(1+(-1)^c)}{4}, m\right) = \widetilde{\mathcal{A}}_{2c}\left(-m + \frac{c(1+(-1)^c)}{4}, -n\right). \tag{3.14}$$

Proof. We first observe that

$$\begin{aligned}\mathcal{A}_{2c}\left(n - \frac{c(1+(-1)^c)}{4}, m\right) &= \sum_{d \pmod{2c}^\times} e\left(\frac{nd+ma}{2c} - \frac{d(1+(-1)^c)}{8}\right)\omega_{-d,2c} \\ &= \sum_{d \pmod{2c}^\times} e\left(-\frac{na+md}{2c} + \frac{a(1+(-1)^c)}{8}\right)\omega_{a,2c}. \end{aligned}\tag{3.15}$$

In the case that $c \equiv 1 \bmod 2$, the term $a(1+(-1)^c)/8$ vanishes, and in the case that $c \equiv 0 \bmod 2$, $(a(1+(-1)^c))/8 = a/4$. Due to the fact that $ad - 2bc = 1$, in this case $ad \equiv 1 \bmod 4$ and hence $a \equiv d \bmod 4$. Finally, we use a property of the Dedekind sums, namely that

$$s(d,x) = s(\overline{d}, x).$$

From these things we may conclude that (3.15) may be written as

$$\sum_{d \pmod{2c}^\times} e\left(\frac{(-m)d + (-n)a}{2c} + \frac{d(1+(-1)^c)}{8}\right)\omega^{-1}_{-d,2c} = \widetilde{\mathcal{A}}_{2c}\left(-m + \frac{c(1+(-1)^c)}{4}, -n\right). \qquad \square$$

3.8. *Proof of Theorem* 1.1

The duality follows by applying Theorem 3.3 in the case $\kappa = \frac{1}{2}$ and by applying Theorem 3.5 in the case $\kappa = \frac{3}{2}$. We find in the former case a term of the form $(|24n-1|/|24m-1|)^{-1/4}$, and in the latter, $(|24n+1|/|24m+1|)^{1/4}$. If we let $n \to -m$ and $m \to -n$ in either one of these terms (but not both) then they are seen to be equal. The Bessel functions appearing in the Fourier expansions for P^{χ} and $P^{\chi^{-1}}$ are both $I_{1/2}$-Bessel functions, and their arguments are seen to be equal again after replacing (n,m) by $(-m,-n)$ in one such term. In addition, by applying Lemma 3.6, and the special value $\Gamma(3/2) = \sqrt{\pi}/2$, we find that other terms in the expressions for the Fourier coefficients in question are negatives of each other. This proves the theorem.

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