

The *spt*-function of Andrews

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Recently, Andrews introduced the function $s(n) = spt(n)$ which counts the number of smallest parts among the integer partitions of n . We show that its generating function satisfies an identity analogous to Ramanujan's mock theta identities. As a consequence, we are able to completely determine the parity of $s(n)$. Using another type of identity, one based on Hecke operators, we obtain a complete multiplicative theory for $s(n)$ modulo 3. These congruences confirm unpublished conjectures of Garvan and Sellers. Our methods generalize to all integral moduli.

generating | Maass form | partition

1. Introduction and Statement of Results

Recently, Andrews (1) introduced the function $s(n) = spt(n)$ which counts the number of smallest parts among the integer partitions of n . As an illustration, one sees that $s(4) = 10$ by examining the partitions of 4 below:

$$4, 3 + \underline{1}, \underline{2} + \underline{2}, 2 + \underline{1} + \underline{1}, \underline{1} + \underline{1} + \underline{1} + \underline{1}.$$

The generating function for $s(n)$ is

$$S(z) := \sum_{n=0}^{\infty} s(n)q^n = \frac{1}{(q)_{\infty}} \cdot \sum_{n=1}^{\infty} \frac{q^n \cdot \prod_{m=1}^{n-1} (1 - q^m)}{1 - q^n} = q + 3q^2 + 5q^3 + 10q^4 + \dots, \quad [1.1]$$

where $q := e^{2\pi iz}$, and $(q)_{\infty} := \prod_{n=1}^{\infty} (1 - q^n)$. Andrews obtained an elegant description of $s(n)$ in terms of the partition function $p(n)$, and the 2nd rank moment function*

$$N_2(n) := \sum_{m \in \mathbb{Z}} m^2 N(m, n).$$

The rank of a partition is its largest part minus its number of parts, and $N(m, n)$ denotes the number of partitions of n with rank m . Andrews proved that

$$s(n) = np(n) - \frac{1}{2}N_2(n). \quad [1.2]$$

In view of the recent works on ranks (for example, see refs. 3 and 4 among other subsequent works), one suspects that 1.2 can be placed within the framework of the theory of Maass forms. In a recent preprint, Bringmann (5) indeed produces a harmonic weak Maass form related to $q^{-1}S(24z)$. As in refs. 3 and 4, she makes great use of this result to obtain asymptotics, and congruences modulo primes $\ell > 3$. Recent works by Bringmann, Garvan, and Mahlburg (6, 7) give more results concerning such congruences.

Here, we investigate $s(n)$ modulo 2 and 3. To place this in proper context, we note that very little is known about $p(n)$ modulo 2 and 3. This difficulty stems from the fact that there are no known methods of mapping the generating function for $p(n)$ to spaces of holomorphic modular forms mod 2 or 3, where the powerful theory of Galois representations gives results for primes $\ell \geq 5$ (see refs. 8–10). Bringmann's work shows that these technical difficulties persist for $s(n)$ modulo 2 and 3.

We circumvent these problems by invoking the "principle" of Ramanujan's mock theta function identities. Loosely speaking, this is the fact that the difference of 2 harmonic weak Maass forms,

whose nonholomorphic parts agree, is a weakly holomorphic modular form (for example, see refs. 11–13). Our task is to find such pairs of harmonic weak Maass forms whose difference contains the generating function for $s(n)$ as a component. These pairs provide further relations that are presently unavailable in the case of $p(n)$, thereby allowing us to leverage information such as congruences.

Remark. The methods here apply to all integral moduli ℓ , not just $\ell = 2$ and 3. For brevity, we restrict attention to these troublesome primes.

We require 2 further mock theta functions. The first arises from the nearly modular Eisenstein series $E_2(z)$. Define $D(z)$ by

$$D(z) := \frac{q^{-\frac{1}{24}}}{(q)_{\infty}} \left(1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \right) = \frac{q^{-\frac{1}{24}}}{(q)_{\infty}} E_2(z) = q^{-\frac{1}{24}} - 23q^{\frac{23}{24}} - \dots. \quad [1.3]$$

The second mock theta function $L(z)$ is defined by

$$L(z) := \frac{(q^6)_{\infty}^2 (q^{24})_{\infty}^2}{(q^{12})_{\infty}^5} \cdot \left(\sum_{n \in \mathbb{Z}} \frac{(12n - 1)q^{6n^2 - \frac{1}{24}}}{1 - q^{12n - 1}} - \sum_{n \in \mathbb{Z}} \frac{(12n - 5)q^{6n^2 - \frac{25}{24}}}{1 - q^{12n - 5}} \right) = q^{\frac{23}{24}} + q^{\frac{47}{24}} + q^{\frac{71}{24}} - 4q^{\frac{95}{24}} - 6q^{\frac{119}{24}} + 12q^{\frac{143}{24}} - \dots. \quad [1.4]$$

Our first result provides a modular linear combination of these mock theta functions.

Theorem 1.1. In the notation above, the function

$$D(24z) - 12L(24z) - 12q^{-1}S(24z) = q^{-1} - 47q^{23} - 142q^{47} - 285q^{71} - 547q^{95} - \dots$$

is a weight 3/2 weakly holomorphic modular form on $\Gamma_0(576)$ with Nebentypus $(\frac{12}{\cdot})$.

Remark. Using the Dedekind eta-function $\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, in Section 3 we shall construct a modular function $\tilde{F}(z)$ with the property that

$$D(24z) - 12L(24z) - 12q^{-1}S(24z) = \tilde{F}(24z) \cdot q^{23} (-q^{24})_{\infty} (q^{96})_{\infty} (q^{144})_{\infty} (q^{288})_{\infty}. \quad [1.5]$$

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* Moments were introduced by Atkin and Garvan (2) in their work on partition ranks and cranks.

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Now we turn to questions of congruence. In ref. 1, Andrews proved that

$$\begin{aligned} s(5n + 4) &\equiv 0 \pmod{5}, \\ s(7n + 5) &\equiv 0 \pmod{7}, \\ s(13n + 6) &\equiv 0 \pmod{13}. \end{aligned}$$

Modulo primes $\ell > 3$, there are infinitely many congruences of the form

$$s(an + b) \equiv 0 \pmod{\ell}.$$

Bringmann essentially proves this in ref. 5. To obtain this result, one simply combines her work for $N_2(n)$ with results for $p(n)$ (8–10) using lemmas 2.63 and 3.30 of ref. 14. Garvan (7) has obtained explicit examples modulo 11, and a forthcoming article by Bringmann, Garvan, and Mahlburg (6) promises further results.

For the moduli $\ell = 2$ and 3, different techniques are required. *Theorem 1.1* allows us to determine the parity of $s(n)$, a result that proves phenomena first observed numerically by both Garvan and Sellers.

Theorem 1.2. *We have that $s(n)$ is odd if and only if $24n - 1 = pm^2$, where m is an integer and $p \equiv 23 \pmod{24}$ is prime.*

Remark. *Note that $s(n)$ is even for almost all n (in terms of arithmetic density).*

Ramanujan-type congruences follow immediately from *Theorem 1.2*. For example, if $3 < \ell \not\equiv 23 \pmod{24}$ is prime, and if $0 < k < \ell$, then for all n , $s(\ell^2 n + \ell k - (\ell^2 - 1)/24) \equiv 0 \pmod{2\ell}$. For example, when $\ell = 7$, one has

$$\begin{aligned} s(49n + 5) &\equiv s(49n + 12) \equiv s(49n + 19) \equiv s(49n + 26) \\ &\equiv s(49n + 33) \equiv s(49n + 40) \equiv 0 \pmod{2}. \end{aligned} \quad [1.6]$$

Remark. *By 1.2, Theorem 1.2 gives congruences for $N_2(n)$ for even integers n .*

Garvan conjectured a surprising multiplicative congruence modulo 3. Here, we prove this phenomenon by leveraging a Maass form by using the Hecke algebra, a method that plays a central role in recent work of K.O. and Bruinier on Heegner divisors and derivatives of modular L -functions (see the proof of theorem 7.6 of ref. 15). For notational convenience, if $p \geq 5$ is prime, let

$$\delta(p) := (p^2 - 1)/24. \quad [1.7]$$

In what follows, we let $(\frac{\cdot}{\cdot})$ denote the Legendre symbol.

Theorem 1.3. *If $p \geq 5$ is prime, then for every nonnegative integer n we have*

$$\begin{aligned} s(p^2 n - \delta(p)) + \left(\frac{3 - 72n}{p}\right) s(n) + ps \left(\frac{n + \delta(p)}{p^2}\right) \\ \equiv \left(\frac{3}{p}\right) (1 + p) s(n) \pmod{3}. \end{aligned}$$

Remark. *Theorem 1.3 asserts that $q^{-1}S(24z) \pmod{3}$ is a weight $3/2$ Hecke eigenform with Nebentypus. We will see that it is related to $\eta(6z)^4 = \sum_{n=1}^{\infty} a(n)q^n$, and the elliptic curve*

$$E: y^2 = x^3 + 1.$$

Remark. *A straightforward calculation involving Theorem 1.3, combined with Dirichlet's theorem on primes in arithmetic progressions, easily implies*

$$\#\{0 \leq n \leq X : s(n) \equiv r \pmod{3}\} \gg \begin{cases} X & \text{if } r \equiv 0 \pmod{3}, \\ \frac{\sqrt{X}}{\log X} & \text{if } r \equiv 1, 2 \pmod{3}. \end{cases}$$

Remark. *Theorem 1.3 also yields Ramanujan congruences thanks to the fact that the Legendre symbol satisfies $(\frac{a}{p}) = 0$ when $p \mid n$. Specifically, let $p > 3$ be a prime such that $p \equiv 2 \pmod{3}$. If $0 < k < p - 1$, then for all n , $s(p^4 n + p^3 k - (p^4 - 1)/24) \equiv 0 \pmod{3}$. For example, when $p = 5$ we have*

$$\begin{aligned} s(625n + 99) &\equiv s(625n + 224) \\ &\equiv s(625n + 349) \equiv s(625n + 474) \equiv 0 \pmod{3}. \end{aligned}$$

When $3 \mid n$, Theorem 1.3 and 1.2 implies congruences for $N_2(n) \pmod{3}$.

In Section 2 we prove *Theorem 1.1*, and in Section 3 we produce 1.5. In Section 4 we provide a proof of *Theorem 1.2*, and in Section 5 we conclude with a proof of *Theorem 1.3*.

2. The New Mock Theta Identity

Theorem 1.1 follows easily from the lemma in this section that expresses “completed” combinations of the series $L(24z)$, $D(24z)$, and $q^{-1}S(24z)$ as harmonic weak Maass forms. We then use these forms to produce a weakly holomorphic modular form, a meromorphic form whose poles (if any) are supported at cusps.

2.1. Preliminaries. We first recall the notion of a harmonic weak Maass form (see also refs. 3, 4, 6, 11, and 16). We suppose $k \in \frac{1}{2} + \mathbb{Z}$, and define ϵ_ν , for ν odd, by

$$\epsilon_\nu := \begin{cases} 1 & \text{if } \nu \equiv 1 \pmod{4}, \\ i & \text{if } \nu \equiv 3 \pmod{4}. \end{cases} \quad [2.1]$$

As usual, we let

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) +iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad [2.2]$$

be the weight k hyperbolic Laplacian, where $z = x + iy$ with $x, y \in \mathbb{R}$. If N is a positive integer and $\psi \pmod{4N}$ is a Dirichlet character, then a harmonic weak Maass form of weight k on $\Gamma_0(4N)$ with Nebentypus ψ is any smooth function $f: \mathbb{H} \rightarrow \mathbb{C}$ satisfying:

1. For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ and all $z \in \mathbb{H}$, we have

$$f(Az) = \psi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz + d)^k f(z).$$

2. We have that $\Delta_k f = 0$.

3. The function $f(z)$ has at most linear exponential growth at all cusps.

Ramanujan’s “Lost” Notebook contains a number of identities involving Ramanujan’s original mock theta functions. By the “mock theta conjectures,” we refer to the surprising identities involving the 5th-order mock theta functions. These identities remained unproven until the important work of Hickerson (13) in 1988.

Recently, a complete understanding of the modular properties of Ramanujan’s mock theta functions (see refs. 3, 4, 17, and 18) has been obtained. These functions are holomorphic projections of harmonic weak Maass forms. Thanks to this theory, we have a conceptual understanding of the phenomenon of the mock theta conjectures. Such identities arise when two harmonic weak Maass forms share a nonholomorphic part. Folsom (12) has obtained a short proof of the original identities arguing in this way, and Bringmann, Ono, and Rhoades (11) have developed this further to obtain many infinite families of modular forms as Eulerian series. *Theorem 1.1* is a unique mock theta identity.

2.2. Proof of Theorem 1.1. *Theorem 1.1* follows immediately from the next lemma.

Lemma 2.1. *The two real analytic functions*

$$\mathcal{M}(z) := q^{-1}S(24z) - \frac{D(24z)}{12} - \frac{i}{4\pi\sqrt{2}} \int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau,$$

$$\mathcal{N}(z) := L(24z) + \frac{i}{4\pi\sqrt{2}} \int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau$$

are weight $3/2$ harmonic weak Maass forms on $\Gamma_0(576)$ with Nebentypus $(\frac{12}{\cdot})$.

Remark. *We refer to the period integrals*

$$\pm \frac{i}{4\pi\sqrt{2}} \int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau$$

as the nonholomorphic parts of these Maass forms.

Remark. *We note that Lemma 2.1 corrects a sign error in front of the nonholomorphic part of $\mathcal{M}(z)$ in ref. 5.*

Proof: The claim about $\mathcal{M}(z)$ follows from theorem 1.1 of ref. 5, 1.2, and the identity

$$\sum_{n=0}^{\infty} np(n)q^n = \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \frac{1-E_2(z)}{24(q)_{\infty}}.$$

The claim about $\mathcal{N}(z)$ is easily derived from ref. 16 (theorem 2.1, proposition 3.1), the fact that $\psi_{a,b}(\tau) = \psi_{b-a,b}(\tau)$ for $0 < a < b$ as defined in ref. 16 (equation 1.20), and the two identities

$$\eta(24z) = \sum_{k \in \mathbb{Z}} (-1)^k q^{(6k+1)^2} \text{ and } \Theta_0(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = \frac{\eta(2z)^5}{\eta(z)^2 \eta(4z)^2}.$$

3. Explicit Description of the Modular Form

By *Theorem 1.1*, we have that

$$\eta(24z) \cdot (D(24z) - 12L(24z) - 12q^{-1}S(24z)) = 1 - 48q^{24} - 96q^{48} - 96q^{72} - \dots$$

is a weight 2 weakly holomorphic modular form on $\Gamma_0(576)$ whose nonzero Fourier coefficients are supported on multiples of 24. By standard facts (for example, see ref. 14), it equals $\Omega(24z)$ for some modular form $\Omega(z)$ on $\Gamma_0(24)$. We obtain an explicit description of $\Omega(z)$. To this end we define modular functions $F_1(z)$ and $F_2(z)$. First, we let

$$f_1(z) := \frac{\eta(2z)^2 \eta(4z)^2}{\eta(z) \eta(3z) \eta(8z) \eta(24z)} = q^{-1} + 1 + 2q^2 - q^3 - 2q^4 + 4q^5 - \dots,$$

$$f_2(z) := \frac{\eta(4z)^4 \eta(6z)^4}{\eta(z) \eta(2z)^2 \eta(3z) \eta(8z) \eta(12z)^2 \eta(24z)} = q^{-1} + 1 + 4q + 6q^2 + 11q^3 + \dots,$$

$$f_3(z) := \frac{\eta(2z)^3 \eta(3z)^9}{\eta(z)^3 \eta(6z)^9} = q^{-1} + 3 + 6q + 4q^2 - 3q^3 - 12q^4 - 8q^5 + \dots,$$

$$f_4(z) := \frac{\eta(z)^2 \eta(3z)^2}{\eta(4z)^2 \eta(12z)^2} = q^{-1} - 2 - q + 7q^3 - 9q^5 + \dots,$$

$$f_5(z) := \frac{\eta(2z)^2 \eta(4z)^2 \eta(6z)^2 \eta(12z)^2}{\eta(z)^2 \eta(3z)^2 \eta(8z)^2 \eta(24z)^2} = q^{-1} + 2 + 3q + 8q^2 + 11q^3 + 16q^4 + 31q^5 + \dots.$$

As usual, if m is a positive integer, let $U(m)$ denote the Atkin- U -operator defined by

$$\left(\sum b(n)q^n \right) \Big| U(m) := \sum b(mn)q^n. \quad [3.1]$$

The modular function $F_1(z)$ is given by

$$F_1(z) = \sum_{n=-1}^{\infty} A_1(n)q^n := -6f_1 - 28f_1|U(2) + 24f_1|U(3) - 18f_2 + 14f_3 + 15f_4 - 4f_5 = q^{-1} - 24 - 95q - 88q^2 - 213q^3 - 856q^4 - 1003q^5 - 904q^6 - 1694q^7 - \dots. \quad [3.2]$$

To define $F_2(z)$, we require

$$f_6(z) := \frac{\eta(4z)^4 \eta(6z)^2}{\eta(2z)^2 \eta(12z)^4} = q^{-1} + 2q + q^3 - 2q^7 - \dots,$$

$$f_7(z) := \frac{\eta(z)^4 \eta(4z)^4 \eta(6z)^4}{\eta(2z)^4 \eta(3z)^4 \eta(12z)^4} = q^{-1} - 4 + 6q - 4q^2 - 3q^3 + 12q^4 - 8q^5 - \dots,$$

$$f_8(z) := \frac{\eta(3z)^4 \eta(4z)^4}{\eta(z)^4 \eta(12z)^4} = q^{-1} + 4 + 14q + 36q^2 + 85q^3 + 180q^4 + 360q^5 + \dots,$$

$$f_9(z) := \frac{\eta(z)^4 \eta(2z)^4}{\eta(3z)^4 \eta(6z)^4} = q^{-1} - 4 - 2q + 28q^2 - 27q^3 - 52q^4 + 136q^5 - \dots.$$

Using these functions and $F_1(z)$, we define $F_2(z)$ by

$$F_2(z) = \sum_{n=-1}^{\infty} A_2(n)q^n := 76 + F_1|U(2) - 2f_6 + 2f_7 + 6f_8 - 20f_3 + 14f_9 = -48 - 144q - 336q^2 - 720q^3 - 1584q^4 - 3168q^5 - 6192q^6 - 11520q^7 - \dots. \quad [3.3]$$

Using the coefficients of $F_1(z)$ and $F_2(z)$, define $\tilde{F}(z)$ by

$$\tilde{F}(z) := \sum_{n=0}^{\infty} A_1(2n-1)q^{2n-1} + F_2(2z) = q^{-1} - 48 - 95q - 144q^2 - 213q^3 - 336q^4 - 1003q^5 - 720q^6 - 1694q^7 - \dots. \quad [3.4]$$

Although these harmonic weak Maass forms are proven to be of level 576, all of their images under $SL_2(\mathbb{Z})$ are known. For brevity, they are not given here. Nevertheless, the holomorphic

parts of these images are always reciprocals of explicit weight $1/2$ theta functions (which are eta-products) multiplied with explicit Lambert-type series (see refs. 5 and 16). By considering poles at cusps for these series (using the valence formula), it turns out that

$$\Omega(z) = \tilde{F}(z) \cdot \eta(2z)\eta(4z)\eta(6z)\eta(12z),$$

which in turns implies that

$$D(24z) - 12L(24z) - 12q^{-1}S(24z) = \frac{\tilde{F}(24z) \cdot \eta(48z)\eta(96z)\eta(144z)\eta(288z)}{\eta(24z)}. \quad [3.5]$$

Remark. It would be very interesting to know whether there is a simpler description of $\tilde{F}(z)$ involving just one or two “natural” modular functions.

4. Parity of $s(n)$

We first reduce the proof of *Theorem 1.2* to an elementary congruence.

Lemma 4.1. *Theorem 1.2 is equivalent to the congruence*

$$q^{-1}S(24z) \equiv L(24z) \pmod{2}.$$

Proof: It suffices to prove that

$$L(24z) \equiv \sum_{p=23} \sum_{(\text{mod } 24)} \text{prime} \sum_{\substack{m \geq 1 \\ \gcd(m,6)=1}} q^{pm^2} \pmod{2}.$$

Since

$$\frac{(q^6)_\infty^2 (q^{24})_\infty^2}{(q^{12})_\infty^5} \equiv 1 \pmod{2},$$

it follows that

$$\begin{aligned} L(24z) &\equiv \sum_{n \in \mathbb{Z}} \frac{q^{144n^2-1}}{1 - q^{288n-24}} + \sum_{n \in \mathbb{Z}} \frac{q^{144n^2-25}}{1 - q^{288n-120}} \pmod{2} \\ &\equiv \sum_{\substack{m \geq 1 \\ -12m < d < 0 \\ \gcd(d,6)=1}} q^{144m^2-d^2} \equiv \sum_{\substack{m \geq 1 \\ -12m < d < 0 \\ \gcd(d,6)=1}} q^{(12m+d)(12m-d)} \pmod{2}. \end{aligned}$$

The proof now follows easily by rewriting the expression $(12m + d)(12m - d)$ as a sum over divisors of integers.

Proof of Theorem 1.2: By *Lemma 4.1*, it suffices to prove that

$$q^{-1}S(24z) \equiv L(24z) \pmod{2}.$$

By *Theorem 1.1* and **1.5** (or **3.5**), it suffices to show that

$$D(24z) \equiv \frac{\tilde{F}(24z) \cdot \eta(48z)\eta(96z)\eta(144z)\eta(288z)}{\eta(24z)} \pmod{24}.$$

Since $\eta(24z) = \sum_{n=1}^\infty \binom{12}{n} q^{n^2} = q + \dots$, and since $\eta(24z)D(24z) \equiv 1 \pmod{24}$, we must show that

$$\begin{aligned} \tilde{F}(24z) \cdot \eta(48z)\eta(96z)\eta(144z)\eta(288z) \\ = 1 - 48q^{24} - 96q^{48} - \dots \equiv 1 \pmod{24}. \end{aligned}$$

Since the nonzero coefficients of this form are supported on exponents which are multiples of 24, it follows that $\Psi(z) := \tilde{F}(z) \cdot \eta(2z)\eta(4z)\eta(6z)\eta(12z)$ is a weakly holomorphic modular form of weight 2 on $\Gamma_0(24)$. From the definitions in the introduction, or a simple analysis of the orders at cusps of the form $\tilde{F}(z)$, it follows that $\Psi(z)\eta(z)^{24}E_4(z)E_6(z)$ is a weight 24 cusp form on $\Gamma_0(24)$, where $E_4(z)$ and $E_6(z)$ are the classical Eisenstein series

$$E_4(z) = 1 + 240 \sum_{n=1}^\infty \sum_{d|n} d^3 q^n \quad \text{and} \quad E_6(z) = 1 - 504 \sum_{n=1}^\infty \sum_{d|n} d^5 q^n.$$

Since $E_4(z) \equiv E_6(z) \equiv 1 \pmod{24}$, the claim is implied by the following congruence between two weight 24 modular forms:

$$\Psi(z)\eta(z)^{24}E_4(z)E_6(z) \equiv \eta(z)^{24}E_6(z)^2 \pmod{24}.$$

This is easily verified by a result of Sturm (for example, see theorem 2.58 of ref. 14).

5. Sketch of the Proof of Theorem 1.3

Because many of the methods of the proof already appear previously in this article, and because this theorem is merely about congruences rather than explicit formulas, for brevity, we provide just a brief sketch of the proof.

The classical weight $3/2$ Hecke operators $T(m^2)$ act (for example, see section 7 of ref. 15) on the Maass form $\mathcal{M}(z)$ from *Lemma 2.1*. Since $\eta(24z)$ is a Hecke eigenform, it follows that the nonholomorphic part of this Maass form is also an eigenform with explicit eigenvalues. Combined with elementary facts about p -adic modular forms [e.g., that E_2 is a weight two 3-adic modular form, and the algebra structure of p -adic modular forms (19)], we may, for every prime $p \geq 5$, combine $\mathcal{M}(z)$ with $\mathcal{M}(z)|T(p^2)$ to obtain a weakly holomorphic modular form. Numerical calculations, combined with Hecke theory and standard facts about p -adic modular forms reveal that

$$\begin{aligned} q^{-1}S(24z)|T(25) &\equiv 0 \pmod{3}, \\ q^{-1}S(24z)|T(49) &\equiv q^{-1}S(24z) \pmod{3}. \end{aligned}$$

Such formulas, combined with the corresponding Hecke action on the other summands of $\mathcal{M}(z)$, allow us to conclude that $q^{-1}S(24z)$ is the reduction modulo 3 of a weakly holomorphic modular form. It turns out to be the reduction of a cusp form whose image modulo 3 under the Shimura correspondence is the 12-quadratic twist of the newform $\eta(6z)^4 = \sum_{n=1}^\infty a(n)q^n \in S_2(\Gamma_0(36))$. The Shimura correspondence is Hecke equivariant, and so it suffices to show, for primes $p \geq 5$, that $a(p) \equiv 1 + p \pmod{3}$.

The newform $\eta(6z)^4$ corresponds to the $\mathbb{Q}(\sqrt{-3})$ CM elliptic curve

$$E : y^2 = x^3 + 1.$$

The desired congruence follows by the theory of Hecke Größencharakteren, which in this case is equivalent to multiplying out q -series identities of Euler and Jacobi. One uses the fact that the ring of integers of $\mathbb{Q}(\sqrt{-3})$ is a UFD. Alternatively, one may employ the fact that the reduction map modulo p for elliptic curves is injective on torsion. In this case, the proof follows since the torsion subgroup of E/\mathbb{Q} has nontrivial 3-rank. In particular, the torsion subgroup is $\mathbb{Z}/6\mathbb{Z}$, which implies that $a(p) \equiv 1 + p \pmod{6}$.

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