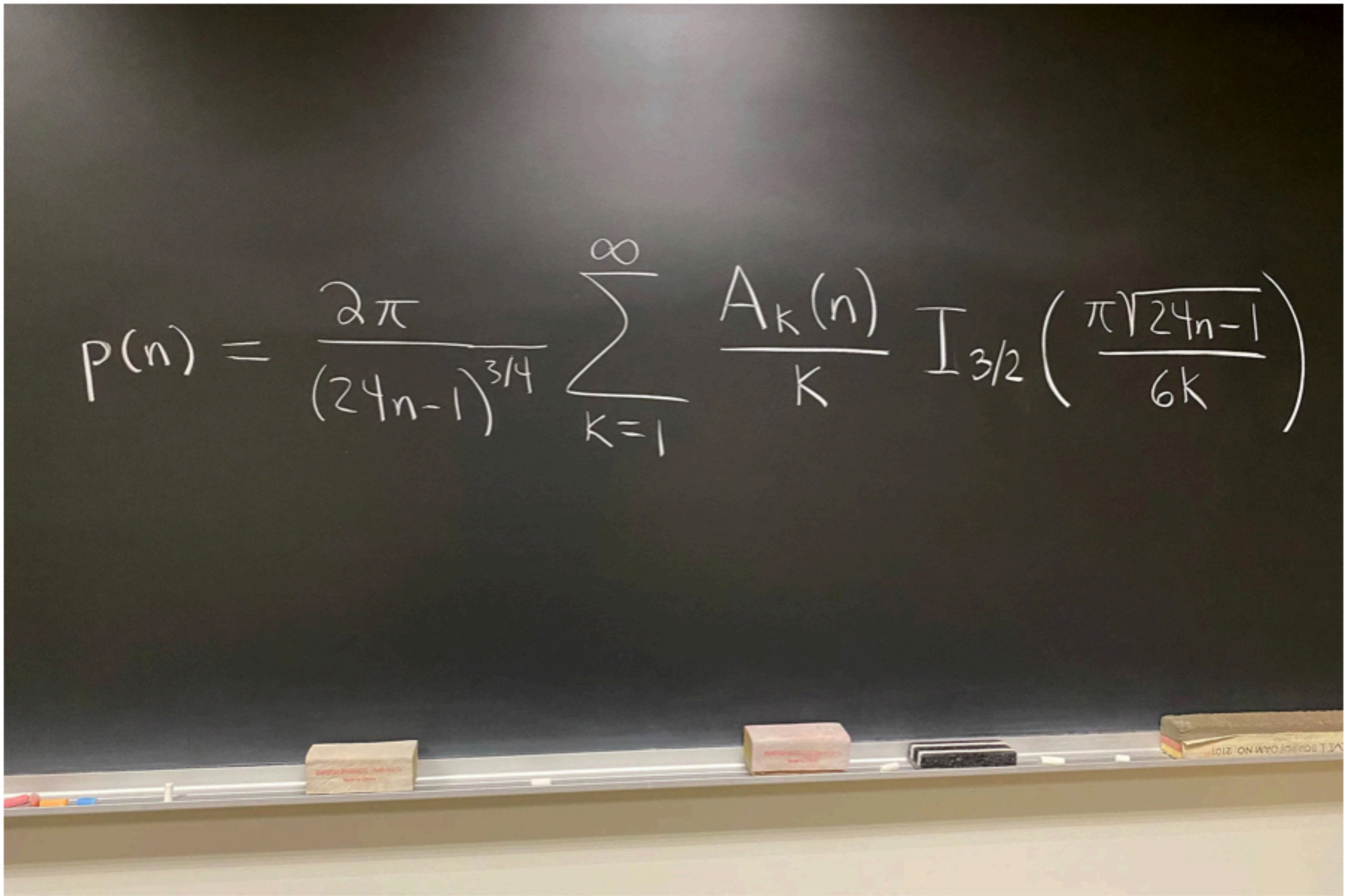
AN INFINITE SUM



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I don’t think there’s a single most beautiful mathematical equation, but this one stands out to me. I showed a photograph of it to a young child while working on this piece. We first talked about what an equation is (a mathematical statement relating two things with an equals sign, such as $4 = 2 + 1 + 1$), and they asked why this equation is so big. We talked about how the right-hand side is visually the big side but the left-hand side isn’t—how the right-hand side is a sum (like $2 + 1 + 1$) but of an infinite number of terms and how the left-hand side has to do with integer partitions in number theory. For example, there are five partitions of $n = 4$ (4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1), so the partition function $p(n)$ evaluated at $n = 4$ is 5 ($p(4) = 5$).

This important and seemingly basic function having to do with adding and counting is beautifully and perhaps unexpectedly complex. For example, $p(100)$ is more than 190 million, but surely the way to figure that out isn't by writing out the millions of partitions of $n = 100$ and counting how many there are. The right-hand side of this equation is an exact formula for $p(n)$ thanks to the 1937 work of Hans Rademacher, who extended related earlier work of G. H. Hardy and Srinivasa Ramanujan. Some may view the (big) right-hand side, an infinite sum involving sums of complex (imaginary) numbers ($A_k(n)$), fractional powers ($^3/_4$ and the square root), the transcendental number pi, and more, as the opposite of beautiful or as intimidating—especially given that it replaces the (visually small) left-hand side that even a child can understand. It's the beauty of analytic number theory and Rademacher and Hardy and Ramanujan's work that shows that this formula for $p(n)$ exists—a feat in and of itself—and that it turns out to be practical for computing $p(n)$ by truncating the infinite sum and essentially rounding. It's beautiful that the infinite sum that appears here converges, meaning it sums to something finite, a real number that counts something of importance, and doesn't keep on accumulating—not to mention the mathematical legacy, further research and connections to other areas that persist today, now close to a century later. —*Amanda Folsom, Amherst College*