# Symmetry, Almost

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Some definitions of the word symmetry include "correct or pleasing proportion of the parts of a thing," "balanced proportions," and "the property of remaining invariant under certain changes, as of orientation in space." One might think of snowflakes, butterflies, and our own faces as naturally symmetric objects—or at least close to it.

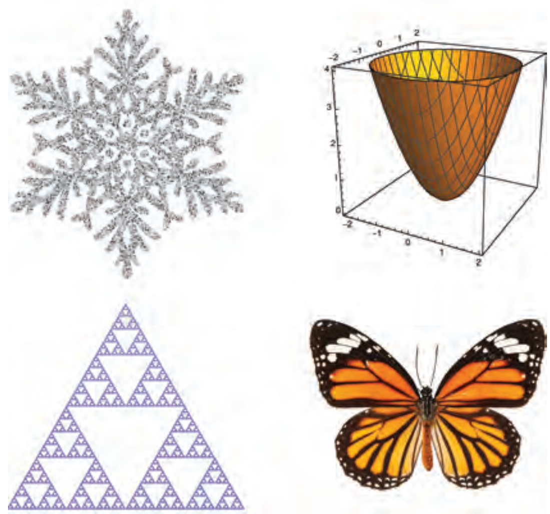

Figure 1. Natural and mathematical symmetries

Mathematically one can also conjure up many symmetric objects: even and odd functions, fractals, certain matrices, and *modular forms*, a type of symmetric complex function.

In more detail, modular forms, defined on the upper half of the complex plane $\mathbb{H}$, are (among other things) essentially symmetric with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$ (the set of $2 \times 2$ integer matrices with determinant 1) on $\mathbb{H}$. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\tau$ in $\mathbb{H}$ by fractional linear (a.k.a. Möbius) transformation, $\gamma \cdot \tau = (a\tau + b)/(c\tau + d)$, and this action yields beautiful symmetries in $\mathbb{H}$.

Namely, in Figure 2, we see the upper half-plane divided into *fundamental domains*—loosely speaking, each region (including the gray shaded region, a standard fundamental domain) is a set of representatives for the orbits of the group action described above (where we must more precisely consider which boundary points to include). Due

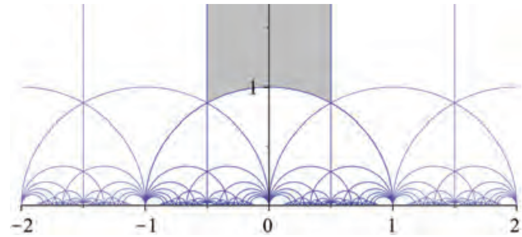


Figure 2. Fundamental domains

to the symmetries satisfied by a modular form, we need only to understand it on a fundamental domain in order to understand it everywhere in $\mathbb{H}$. For example, if $f$ is a *modular function* (a *weight* 0 modular form), it satisfies $f(\gamma \cdot \tau) = f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. "Modular forms are everywhere," as the title of Don Zagier's 2017 birthday conference boasted, and their properties surpass aesthetics; e.g., modular forms are known to be connected to a number of well-known problems, including Fermat's Last Theorem, Monstrous Moonshine, the Birch and Swinnerton-Dyer Conjecture, and more.

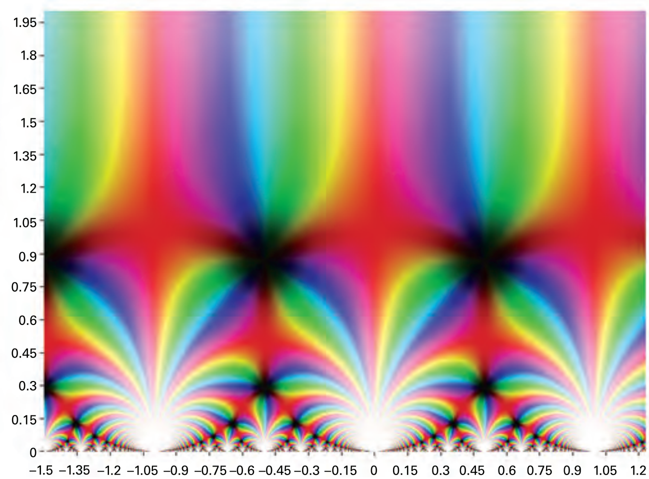


Figure 3. The modular $j$-function

In Figure 3, we see one interpretation of a graph of the modular function denoted by $j$, a *hauptmodul* which generates the field of modular functions. There, we see the upper half-plane, with the argument of a complex value of $j$ encoded by the color shown; red corresponds to the positive real axis, and movement in the counterclockwise direction runs through the colors of the rainbow. The magnitude of a complex value is represented by the darkness of the color. (For example, the figure suggests $j(0)$ is infinite, $j(i)$ (at center) is fairly large and real, and $j(e^{2\pi i/3})$ is zero, all of which are true.) The symmetries of $j$ with respect to the tiling of $\mathbb{H}$ by fundamental domains suggested in Figure 3 are apparent.

All of the things discussed above, whether naturally symmetric or mathematically symmetric, exhibit a kind of beauty, so would they lose some of their innate beauty if their symmetries were altered? Alternatively, what could possibly be

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DOI: https://doi.org/10.1090/noti/1775

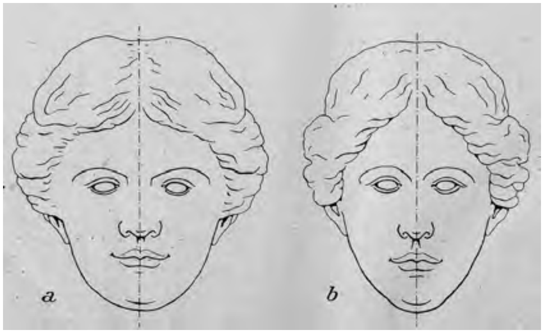


Figure 4. Facial symmetries

gained with slight symmetric imperfections?
Consider Figure 4, in which the images shown are rendered by reflecting one half of a drawing of a face over a central axis: what to make of any "biological preference" towards symmetric human faces? We now also have various notions of perturbed modular forms: as their names may suggest, *quasimodular forms*, *false* or *partial theta functions*, and *mock modular forms* are all not-quite-but-close to being modular in the sense of the symmetric characterization given above. For example, (with notation as above) a mock modular form $m$ satisfies $m(\gamma\cdot\tau) = \rho_{\gamma,\tau} m(\tau) + h_m(\tau)$, for some (nontrivially produced) "error function" $h_m$ and explicit factor $\rho_{\gamma,\tau}$. Even more, we can get rid of the error to modularity $h_m$ attached to a mock modular form $m$. That is, by (nontrivially) adding to $m$ a suitable function $m^-$, the sum $\widehat{m} := m + m^-$ becomes more or less as symmetric as $j$: we have $\widehat{m}(\gamma\cdot\tau) = \rho_{\gamma,\tau}\widehat{m}(\tau)$. But at what expense does $m$ gain symmetry by this addition of $m^-$?

If modular forms are everywhere, then perhaps mock modular forms are almost everywhere. Over the course of the last 10–15 years, a more general theory of harmonic Maass forms has developed, however, earlier footprints appear in Maass' work from the 1950s and, as we have more recently discovered, in Ramanujan's mock theta functions from 1920. The theory of mock modular forms has also seen applications in a number of subjects including combinatorics, mathematical physics, elliptic curves, *quantum modular forms*, and more.

This last subject, quantum modular forms, initially developed by Zagier in 2010, has been of particular interest lately. Like mock modular forms, quantum modular forms exhibit symmetry properties up to an error function as explained in the preceding paragraph, however, the domains of a quantum modular form and a mock modular form are notably different: quantum modular forms are defined in $\mathbb{Q}$, the set of rational numbers, as opposed to $\mathbb{H}$, the upper half of the complex plane. Under the mapping $\tau \mapsto e^{2\pi i\tau}$, mock modular forms are defined inside the unit disk, and rational numbers correspond to roots of unity on the boundary. Despite their differing domains,

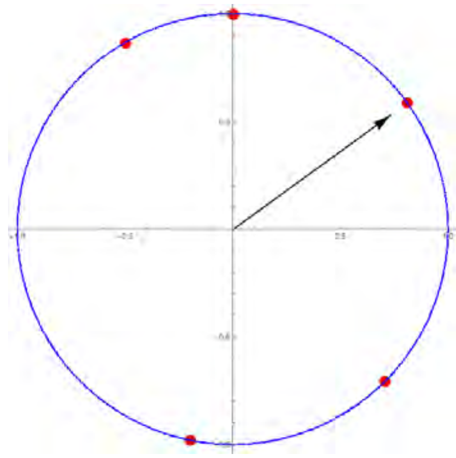

Figure 5. A radial limit

it has been a question of interest to understand the relationships between these almost symmetric functions and to farm the fruits of such relationships. See Figure 5, where a radial limit from inside the unit disk to a root of unity on the boundary is suggestively drawn.

Please join me at the 2019 AMS–MAA Joint Meetings for an accessible discussion of these and other questions surrounding symmetry, almost, guided by the topic of modular forms. The origins of these questions are rooted in the past, while some fascinating and surprising answers come from just the last 10–15 years.

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