Asymptotics and Ramanujan's mock theta functions: then and now

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Abstract

This article is in commemoration of Ramanujan's election as Fellow of The Royal Society 100 years ago, as celebrated at the October 2018 scientific meeting at the Royal Society in London. Ramanujan's last letter to Hardy, written shortly after his election, surrounds his mock theta functions. While these functions have been of great importance and interest in the decades following Ramanujan's death in 1920, it was unclear how exactly they fit into the theory of modular forms – Dyson called this "a challenge for the future" at another centenary conference in Illinois in 1987, honoring the 100th anniversary of Ramanujan's birth. In the early 2000s, Zwegers finally recognized that Ramanujan had discovered glimpses of special families of nonholomorphic modular forms, which we now know to be Bruinier and Funke's harmonic Maass forms from 2004, the holomorphic parts of which are called mock modular forms. As of a few years ago, a fundamental question from Ramanujan's last letter remained, on a certain asymptotic relationship between mock theta functions and ordinary modular forms. The author, with Ono and Rhoades, revisited Ramanujan's asymptotic claim, and established a connection between mock theta functions and quantum modular forms, which were not defined until 90 years later in 2010 by Zagier.

Here, we bring together past and present, and study the relationships between mock modular forms and quantum modular forms, with Ramanujan's mock theta functions as motivation. In particular, we highlight recent work of Bringmann-Rolen, Choi-Lim-Rhoades, and Griffin-Ono-Rolen in our discussion. This article is largely expository, but not exclusively: we also establish a new interpretation of Ramanujan's radial asymptotic limits in the subject of topology.

1 The mock theta functions, then.

It is well known, undoubtedly to readers of this special issue of Philosophical Transactions A in celebration of the centenary of Ramanujan's election as Fellow of The Royal Society in 1918, that Ramanujan was a prolific mathematician who has been referred to as a mathematical visionary, and whose life ended in 1920 at the early age of 32 due to illness. Shortly after Ramanujan's death, G.N. Watson, who in 1919 had also just been elected as Fellow of The Royal Society, and B.M. Wilson, began editing Ramanujan's mathematical notebooks. The pair worked on this pursuit for many years, but by the late 1930s their work had dwindled, affected by Wilson's own death in 1935 at the young age of 38 due to infection. Coinciding with his work on Ramanujan's notebooks, Watson assumed the role of President of the London Mathematical Society from 1933–1935. For his retiring presidential address in 1935, Watson chose to speak on "The Final Problem: An Account of the Mock Theta Functions," curious q-series appearing in Ramanujan's final letter to G.H. Hardy in 1920.

> *"The topic which I have selected [is] unfortunately not too well adapted for oral exposition... I make no apologies for my subject being what is now regarded as old-fashioned, because... I am an old-fashioned mathematician,"*

said Watson [31]. Ramanujan's seventeen mock theta functions referenced in the title of Watson's address include the functions

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots ,$$

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_n^2} = 1 + \frac{q^4}{(1-q)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2} + \cdots .$$

Like these two, Ramanujan's other mock theta functions are defined as q-hypergeometric series built using the q-Pochhammer symbol $(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$ $(n \in \mathbb{N}_0 \cup \infty)$.

About the mock theta function $f(q)$, Ramanujan wrote to Hardy [11, 31]

> *"I have proved that if* $f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$ *then*
>
> $$f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots) = O(1)$$
>
> *at all the points* $q=-1, q^3=-1, q^5=-1, q^7=-1, \ldots,$ *and at the same time*
>
> $$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots) = O(1)$$
>
> *at all the points* $q^2=-1, q^4=-1, q^6=-1, \ldots$ *Also obviously* $f(q)=O(1)$ *at all the points* $q=1, q^3=1, q^5=1, \ldots$ *And so* $f(q)$ *is a Mock* ϑ*-function."*

The function

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$$

which Ramanujan includes in his statement above is perhaps more easily recognized as a modular form[1] (up to multiplication by $q^{-\frac{1}{24}}$) when it is re-written in terms of the weight 1/2 modular Dedekind η-function, as

$$b(q) = q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)},$$

[1] As is standard in the subject, here and throughout we may refer to a function as automorphic of some variety, when in reality it may need to be slightly normalized before exhibiting appropriate transformation properties.

with $q = q_\tau = e^{2\pi i\tau}, \tau \in \mathbb{H} := \{\tau \in \mathbb{C} \colon \mathrm{Im}(\tau) > 0\}$. That is, the modular forms $\pm b$ asymptotically "cut out" all of the mock theta function f's exponential singularities at even ordered roots of unity. Expanding upon this, the following definition of a mock theta function captures the essence of these functions as described by Ramanujan in his letter to Hardy. For this reason, the definition below (see [15]) is attributed to Ramanujan, although he did not literally write it down exactly in this way.

Definition 1.1. A **mock theta function** is a function F, defined on $\mathbb{H}$, satisfying the following conditions:

i) There are infinitely many roots of unity ζ for which $F(\tau)$ grows exponentially as $q = e^{2\pi i\tau}$ approaches ζ radially from inside the unit disk.

ii) For every root of unity ζ, there exists a (weakly holomorphic) modular form g_ζ and a rational number α_ζ such that
$$F(\tau) - q^{\alpha_\zeta} g_\zeta(\tau)$$
is bounded as $q \to \zeta$ radially from within the unit disk.

iii) There does not exist a single (weakly holomorphic) modular form g that satisfies ii) for every root of unity ζ.

To paraphrase (omitting some technical details), mock theta functions are asymptotically close to ordinary modular forms, at their exponential singularities. In his address, Watson commented on Ramanujan's lack of rigor surrounding this definition of a mock theta function, a quality which also seeps into his claim on f and b described above:

> *"[Ramanujan's] remarks about lack of rigorous proof indicate that he was not completely convinced that the functions which he had constructed actually cannot be expressed in terms of ϑ-functions and 'trivial' functions. It would therefore seem that his work on the transformation theory of mock ϑ-functions did not lead him to the precise formulae (such as I shall describe presently) for transformations of mock ϑ-functions of the third order. The precise forms of the transformation formulae make it clear that the behaviour of mock ϑ-functions near the unit circle is of a more complex character than that of ordinary ϑ-functions."*

One such transformation result that Watson is referring to is the following, which appeared in his same 1935 address [31]:

Theorem 1.2. *Let $q = e^{-\alpha}, \beta = \pi^2/\alpha, q_1 = e^{-\beta}$, where $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0$. Then*

$$q^{-\frac{1}{24}} f(q) = 2\sqrt{\frac{2\pi}{\alpha}} q_1^{\frac{4}{3}} \omega(q_1^2) + 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty \frac{\sinh(\alpha t)}{\sinh\left(\frac{3\alpha t}{2}\right)} e^{-\frac{3\alpha t^2}{2}} dt.$$

Changing variables and setting $\alpha = -2\pi i\tau$, this may be interpreted as a modular-type transformation, under $\tau \mapsto -1/(2\tau)$, between the mock theta functions f and ω. Watson's transformation notably contains an "error" integral, which would be absent if this was a true modular transformation.

Both of these properties of the mock theta function f, its curious asymptotic closeness to modular forms at exponential singularities, as well as its modular-like transformation properties, influenced extensive further work on the mock theta functions in the decades following Ramanujan's death, by mathematicians including Andrews, Berndt, Bringmann, Dragonette, Dyson, Garvan, Gordon, Griffin, Hardy, Hickerson, McIntosh, Ono, Rademacher, Rhoades, Rolen, Watson, Zagier, Zwegers, and scores more (for example, see [15]). It turns out, as we shall describe in the next section, that a more complete picture of the modular properties of the mock theta functions emerged only within the last 15 years, in the context of *harmonic Maass forms* (defined in 2004), and *quantum modular forms* (defined in 2010).

2 The mock theta functions, now.

2.1 Harmonic Maass forms and mock modular forms

Work of Zwegers [35, 36], which emerged circa the year 2000, 80 years after Ramanujan's letter and 65 years after Watson's transformation formulae, finally resolved the question of precisely understanding the transformation theory of Ramanujan's mock theta functions: they could be completed by the addition of a suitable nonholomorphic function, in such a way that the resulting function, although no longer holomorphic, transforms like a modular form. In short, we may now view Ramanujan's functions as holomorphic parts of harmonic Maass forms, which were defined by Bruinier and Funke in 2004. For additional details on these functions and the technicalities of the definition below, we refer the reader to [15, 17].

Definition 2.1. If $k \in \frac{1}{2}\mathbb{Z}$, then a **weight k harmonic Maass form on a subgroup** $\Gamma = \Gamma_0(N)$ for some $N \in \mathbb{N}$, where $4|N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, is any smooth function $M \colon \mathbb{H} \to \mathbb{C}$ satisfying the following properties.

i) For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have

$$M\left(\frac{a\tau+b}{c\tau+d}\right) = \begin{cases} (c\tau+d)^k M(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-2k} (c\tau+d)^k M(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

ii) We have that $\Delta_k(M) = 0$, where for $\tau = x+iy$, $\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

iii) There exists a polynomial $P_M(\tau) \in \mathbb{C}[e^{-2\pi i\tau}]$ such that

$$M(\tau) - P_M(\tau) = O\left(e^{-\epsilon y}\right),$$

where $\tau = x + iy$, as $y \to \infty$ for some $\epsilon > 0$. Analogous conditions hold at all cusps.

The group $\Gamma_0(N)$ in the above definition is the set of all 2×2 matrices with integer entries and determinant 1, with lower left entry congruent to $0 \pmod N$. The operator Δ_k in condition ii) above is called the *weight k Laplacian operator*. The complex numbers ε_d appearing in condition i) are defined to be 1 or i depending on whether d is 1 or $-1 \pmod 4$.

It is a fact that harmonic Maass forms M decompose as $M = M^+ + M^-$, a sum of a holomorphic part M^+ and a nonholomorphic part M^-, where M^+ and M^- have prescribed shapes (see [15, Definition 4.4] for specific details). We refer to such holomorphic parts as *mock modular forms* [32], aptly named, since Ramanujan's mock theta functions are among the first explicit examples (up to possible multiplication by rational powers of q and/or the addition of constants, see [15, Theorem 9.4]). The theory of harmonic Maass forms has developed in many analogous ways to the theories of ordinary modular and Maass forms, though it has also led to numerous new applications (see for example [15] and the many references therein).

2.2 Quantum modular forms

Understanding Ramanujan's mock theta functions within the context of harmonic Maass forms has opened many doors in recent years. However, questions remained, as B. C. Berndt (author of numerous contemporary volumes on Ramanujan's notebooks [1]-[5] (with G. E. Andrews) and [6]-[10]) recently pointed out [12]:

> *"We emphasize that Ramanujan does not prove that $f(q)$ is actually a mock theta function according to his somewhat imprecise definition. Moreover, no one since has actually proved this statement, nor has anyone proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition."*

Armed with the start of a theory of harmonic Maass forms, as well as Zwegers' important work, Ono, Rhoades and the author revisited Ramanujan's asymptotic claim about the mock theta function f and the modular form b (described in Section 1). We sought to better understand the implied $O(1)$ constants, and wondered if there could be any connection between what Ramanujan was studying in 1920, and quantum modular forms, defined 90 years later in 2010 by Zagier. Quantum modular forms, which we more precisely define below, transform like modular forms but with some notable differences: they are defined on $\mathbb{Q}$ as opposed to $\mathbb{H}$, and they transform only up to additive error terms, which are required to extend to suitably analytic or continuous functions in $\mathbb{R}$. It may be unclear from this terse description why we imagined quantum modular forms may be at play in Ramanujan's asymptotic claim. In fact, we saw potential hints in the transformation exhibited by $f(q)$ on $\mathbb{H}$ (see Watson's theorem in Section 1), and also in Ramanujan's asymptotic claim: if we re-write $q = e^{2\pi i\tau}$ where $\tau \in \mathbb{H}$, then $q \to \zeta$ (a root of unity) radially is equivalent to $\tau \to x \in \mathbb{Q}$, a rational number, vertically.

More precisely, we have the following definition of a quantum modular form, which we attribute to Zagier [33], noting that his original definition was slightly more restrictive.

Definition 2.2. A **quantum modular form of weight** $k \in \frac{1}{2}\mathbb{Z}$ is a function $\varphi : \mathbb{Q}\backslash S \to \mathbb{C}$, for some discrete subset S, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, an appropriate subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the functions

$$h_\gamma(x) = h_{\varphi,\gamma}(x) := \varphi(x) - \varepsilon^{-1}(\gamma)(cx+d)^{-k}\varphi\left(\frac{ax+b}{cx+d}\right) \tag{2.1}$$

satisfy a suitable property of continuity or analyticity in $\mathbb{R}$.

(The multiplier systems $\varepsilon(\gamma)$ which appear satisfy $|\varepsilon(\gamma)| = 1$, in accordance with the theory of ordinary modular forms.)

To elaborate on this definition, in which the analytic property required to hold in $\mathbb{R}$ is left intentionally a bit vague by Zagier, a typical example of a quantum modular form may be (but is not necessarily required to be) such that the corresponding functions h_γ are defined on $\mathbb{Q}\setminus\{\gamma^{-1}(i\infty)\}$, and extend to functions which are C^∞, or perhaps analytic in $\mathbb{R}\setminus\{\gamma^{-1}(i\infty)\}$. Given the nature of the modular action of $\mathrm{SL}_2(\mathbb{Z})$, it does not quite make sense to define quantum modular forms exactly analogously to ordinary modular forms with only a change in domain. Zagier's definition above exhibits functions which fail to be modular in such a way that precisely offsets their failure to be analytic, and vice-versa.

We offer the following explicit example of a quantum modular form, deliberately chosen for reasons revealed in the next subsection. Let

$$U(q) := \sum_{n\geq 1}(u_e(n) - u_o(n))q^n = \sum_{n=0}^{\infty}(q;q)_n^2 q^{n+1} \tag{2.2}$$

be a certain weighted combinatorial generating function for strongly unimodal sequence ranks. (Precisely, $u_{e/o}(n) := \#\{\text{size } n \text{ strongly unimodal sequences with even/odd rank}\}$.) Bryson-Ono-Pitman-Rhoades [18] proved that $\mathcal{U}(x) := e^{-\frac{\pi i x}{12}}U(e^{2\pi i x})$ is in fact a quantum modular form of weight $3/2$. Interestingly, the authors showed that $\mathcal{U}$ simultaneously exhibits mock modular-type transformation properties when viewed as a function on $\mathbb{H}$. Generalizations and extensions of these results related to U have since been established, some of which we discuss in Section 3 (see also [15]). Moreover, many quantum modular forms have been studied in addition to U, arising from diverse areas such as number theory, combinatorics, representation theory, topology, and more [15, 33].

2.3 Ramanujan's radial limits revisited

We are now prepared to state, and explain, the following result and its generalization, established in [22]. Below and in what follows, we use the notation $\zeta_A := e^{2\pi i/A}$. Theorem 2.3 below resolves, makes explicit, and reinterprets Ramanujan's observation on the asymptotic behavior of the mock theta function f described in Section 1.

Theorem 2.3. *As q approaches a primitive even-order $2k$-th root of unity ζ_{2k}^h radially from within the unit disk, we have that*

$$f(q) - (-1)^k b(q) = -4\sum_{n=0}^{k-1}(-\zeta_{2k}^h;\zeta_{2k}^h)_n^2\zeta_{2k}^{h(n+1)}.$$

The right-hand side of the radial limit in Theorem 2.3 is closely related to the quantum modular form U discussed above. In particular, it is (up to multiplication by -4) the special value at ζ_{2k}^h of the generating function for strongly unimodal sequences, which also known to possess quantum modular properties (see [14, 21], and the narrative below). That is, Theorem 2.3 reveals, in a single asymptotic statement, a relationship between three different types of modular forms: mock theta functions, (ordinary) modular forms, and quantum modular forms, the first of which emerged in 1920, and the last of which emerged over 90 years later. In addition to offering this connection between various types of modular forms, the statement of Theorem 2.3 also asymptotically relates three different combinatorial generating functions: one arising from partition ranks, one from partition cranks, and one from strongly unimodal sequences (see [22] and the narrative below for further details).

The theory of harmonic Maass forms indeed plays a role in the proof of Theorem 2.3 and its generalization Theorem 2.4 in [22]. In particular, Zwegers' so-called μ-function, which may be viewed as a mock Jacobi form, plays a key role. Also crucial in our proof is an identity due to Ramanujan himself, listed as entry 3.4.7 in his "Lost" Notebook [2], which was unearthed by G.E. Andrews in 1976 in the Trinity College library:

Entry 3.4.7

$$\sum_{n=0}^{\infty} \frac{a^{-n-1}b^{-n}q^{n^2}}{(-a^{-1};q)_{n+1}(-qb^{-1};q)_n} + \sum_{n=1}^{\infty}(-aq;q)_{n-1}(-b;q)_n q^n$$
$$= \frac{(-aq;q)_\infty}{(q;q)_\infty(-q/b;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{b^n q^{\frac{n(n+1)}{2}}}{1+aq^n} + \frac{1}{a}\sum_{n=1}^{\infty}\frac{b^{-n}q^{\frac{n(n+1)}{2}}}{1+a^{-1}q^n}\right).$$

Theorem 2.4 below is stated in terms of the partition rank and crank generating functions R and C, respectively, and the strongly unimodal sequence rank generating function U.[2] These functions satisfy

$$R(w;q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}, \qquad C(w;q) = \frac{(q;q)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty},$$
$$U(w;q) = \sum_{n=0}^{\infty}(wq;q)_n(w^{-1}q;q)_n q^{n+1}.$$

Theorem 2.4. *Let $1 \le a < b$ and $1 \le h < k$ be integers with $\gcd(a,b) = \gcd(h,k) = 1$ and $b|k$. If $h' \in \mathbb{Z}$ satisfies $hh' \equiv -1 \pmod{k}$ then as q approaches ζ_k^h radially within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\left(R(\zeta_b^a;q) - \zeta_{b^2}^{-a^2h'k}C(\zeta_b^a;q)\right) = -(1-\zeta_b^a)(1-\zeta_b^{-a})U(\zeta_b^a;\zeta_k^h).$$

[2] We caution the reader that the function U is defined with slightly different normalizations (using the same notation for the function) in different sources. Here we have used the definition from [22].

Theorem 2.3 is deduced from Theorem 2.4 in [22] by setting $(a, b) = (1, 2)$. We also point the reader to a later proof of Theorem 2.3 not involving harmonic Maass forms by Zudilin in [34].

Similar quantum properties (to those possessed by $U(q)$ mentioned above) are possessed by the right-hand side of the radial limit displayed in Theorem 2.4. Namely, in [21], we established quantum modular properties of $U(w; q)$ when w is fixed to be a root of unity, when viewed as a function of $x \in \mathbb{Q}$, where $q = e^{2\pi i x}$. In [14], we defined the notion of a *quantum Jacobi form*, and established quantum Jacobi properties of $U(w; q)$ when viewed as a two variable function in $(x_1, x_2) \in \mathbb{Q} \times \mathbb{Q}$, with $w = e^{2\pi i x_1}$ and $q = e^{2\pi i x_2}$.

Later alternative expressions for the right-hand-side of the radial limit difference in Theorem 2.4 have also been given. We mention three such expressions in (e1)–(e3) below:

(e1) In [21], we show that the right-hand side of the radial limit in Theorem 2.4 may be expressed as

$$-(1-\zeta_b^a)(1-\zeta_b^{-a})F(\zeta_b^a; \zeta_k^{-h}),$$

where

$$F(w; q) := \sum_{n=0}^{\infty} w^{n+1}(wq; q)_n.$$

The function $F(w; q)$ is an extension of a function originally studied by Kontsevich and Zagier [33].

(e2) In [14], using quantum Jacobi properties, we show that the right-hand side of the radial limit in Theorem 2.4 may be expressed as the following simple (non-q-hypergeometric) polynomial in roots of unity:

$$\frac{1}{2}\left(1-\zeta_b^{-a}\right)\zeta_b^{-2a}\zeta_k^{-h}\sum_{j=0}^{k-1}(-1)^{j+1}\zeta_{2k}^{-5hj}\left(1-\zeta_b^{2a}\zeta_k^{h(2j+1)}\right)\zeta_b^{-3ja}\zeta_{2k}^{-3j^2h}.$$

(e3) In Section 3.4, we state and prove new expressions for the right-hand sides of the radial limits in Theorem 2.3 and Theorem 2.4 in terms of colored Jones polynomials from topology.

3 Applications and the future of the mock theta functions

Addressing Dyson's claim, the future is here, and if Watson, who called the mock theta functions (and himself) old-fashioned, only knew to what lengths Ramanujan's mock theta functions would travel over the course of the next 100 years. In the remaining sections, we discuss some modern generalizations and applications of the results described in the previous sections surrounding the mock theta functions. In particular, in Section 3.4, we discuss applications to topology, and state and prove a new interpretation of the right-hand sides of the radial limits in Theorem 2.3 and Theorem 2.4.

3.1 Radial limits of universal mock theta functions

As a companion to a function studied by Hickerson (sometimes denoted by $g_3(w;q)$), Gordon and McIntosh defined the *universal mock theta function* [23]

$$g_2(w;q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\frac{n(n+1)}{2}}}{(w;q)_{n+1}(w^{-1}q;q)_{n+1}}.$$

This function is so named due to the fact that all of Ramanujan's mock theta functions can be expressed in terms of linear combinations of specializations of g_2 and ordinary modular forms. Extending results from [22], some of which are discussed above, Rhoades [30] asked if one could similarly explicitly determine modular forms $f_{a,b,A,B,h,k}$ in a uniform way such that $g_2(\zeta_b^a q^A; q^B) - f_{a,b,A,B,h,k}(q)$ is bounded as $q \to \zeta_k^h$ radially from within the unit disk, and subsequently find finite formulas for these radial limits. These questions were answered by Bringmann and Rolen [16] in 2015, who determined modular forms $f_{a,b,A,B,h,k}$ such that

$$Q_{a,b,A,B,h,k} := \lim_{q \to \zeta_k^h} \left(g_2(\zeta_b^a q^A; q^B) - f_{a,b,A,B,h,k}(q)\right)$$

is bounded as $q \to \zeta_k^h$ radially from within the unit disk. Moreover, they obtained explicit finite formulas for the constants $Q_{a,b,A,B,h,k}$, and in parallel to Theorem 2.3 and Theorem 2.4 from [22], they showed that the functions $Q(h/k) := Q_{a,b,A,B,h,k}$ are quantum modular forms. Similar results related to the universal mock theta function g_3 are established in [29]. To prove their results, the authors of [16] use results from [19], which we discuss in the next section.

3.2 Radial limits of mock modular forms

Given the radial limit results described above related to mock theta functions, it is a natural question to ask whether similar results hold for more arbitrary mock modular forms (holomorphic parts of harmonic Maass forms). Indeed this is the case, as established in the following results of Choi, Lim, and Rhoades from 2016 [19].

Theorem 3.1. *Let M^+ be a mock modular form, so that $M = M^+ + M^-$ (with $M^- \neq 0$) is a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ and level N with $t > 1$ inequivalent cusps $\{c_1, c_2, \ldots, c_t\} \subset \mathbb{Q} \cup \{\infty\}$. Then there is a collection $\{g_j\}_{j=1}^t$ of weakly holomorphic modular forms such that $M^+ - g_j$ is bounded towards all rationals equivalent to the cusp c_j.*

In [19], the authors generalize a result of Borcherds from [13] in order to prove Theorem 3.1. They also establish the analogous result that these radial limit differences may be realized as special values of quantum modular forms.

3.3 Ramanujan's definition of a mock theta function

As stated in Section 2.2, Berndt observed that no one had actually proved that Ramanujan's mock theta functions actually satisfied his definition of a mock theta function. By making use of the realization that Ramanujan's examples are essentially mock modular forms of weight 1/2, Griffin, Ono and Rolen proved this in [24]. A key result to this end (established in [24]) is Theorem 3.2 below. The group $\Gamma_1(N)$ appearing is the set of all 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with integer entries a, b, c, d, and determinant 1, satisfying $a \equiv d \equiv 1 \pmod{N}$ and $c \equiv 0 \pmod{N}$.

Theorem 3.2. *Suppose that $M = M^+ + M^-$ is a weight k harmonic Maass form on $\Gamma_1(N)$, where $k \in \frac{1}{2}\mathbb{Z}$. If M^- is non-trivial*[3] *and g is a weight k weakly holomorphic modular form on $\Gamma_1(N)$, then there are infinitely many roots of unity ζ for which $M^+ - g$ has exponential growth as q radially approaches ζ.*

This theorem confirmed Ramanujan's original claim about his examples. Namely, if F is one of Ramanujan's mock theta functions and $a \in \mathbb{N}, b \in \mathbb{Z}, c \in \mathbb{C}$ such that $q^b F(a\tau) + c$ is the holomorphic part of a weight 1/2 harmonic Maass form, then it is proved in [24] using the above theorem, that there does not exist a weakly holomorphic modular form g of any half-integral weight on any group $\Gamma_1(N)$ such that at every root of unity ζ we have $\lim_{q \to \zeta}(q^b F(a\tau) + c - g(\tau)) = O(1)$.

3.4 Colored Jones polynomials for torus knots

We conclude with an application to the area of topology, which has been shown to exhibit interesting intersections with the subject of quantum modular forms in recent years. This connection has been seen through certain Jones polynomials, Laurent polynomials which are a type of knot invariant, unchanged under isotopy (see [26]). Zagier made this observation in his original paper on quantum modular forms [33], and this subject is an active and current area of research [27, 28]. To provide a specific example, which turns out to be related to the subject of this paper, we set some notation. Let $J_N(T^*_{2,3}; q)$ be the N-colored Jones polynomial for the left-handed trefoil knot $T^*_{(2,3)}$. Hikami and Lovejoy [25] observed that when evaluated at $q = \zeta_N$, where ζ_N is an Nth root of unity, the colored Jones polynomial $J_N(T^*_{(2,3)}, \zeta_N)$ equals $\zeta_N^{-1} U(\zeta_N)$, where $U(q)$ is the quantum combinatorial generating function from (2.2) discussed in Section 2.2. We extend this idea, and establish the following new result, which shows that the radial limit sums or differences discussed previously, including those for the mock theta function $f(q)$, may be realized as special values of colored Jones polynomials.

Theorem 3.3. *Let $1 \leq a < b$ and $1 \leq h < k$ be integers with $\gcd(a,b) = \gcd(h,k) = 1$ and $b|k$, so that $bb' = k$ for some integer b'. Let $h' \in \mathbb{Z}$ satisfy $hh' \equiv -1 \pmod{k}$. Then for any positive integer $N = N_{a,b,h,k} \equiv -ab'h' \pmod{k}$, as q approaches ζ_k^h radially within the unit disk, we have that*

$$\lim_{q \to \zeta_k^h} (R(\zeta_b^a; q) - \zeta_{b^2}^{-a^2 h' k} C(\zeta_b^a; q)) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a})\zeta_k^h J_N(T^*_{(2,3)}; \zeta_k^h),$$

where $J_N(T^*_{(2,3)}; q)$ is the N-colored Jones polynomial for the left-handed trefoil knot $T^*_{(2,3)}$.

[3] If $N = 1$ then M^- is trivial.

In particular, with $k = 2\kappa$, for any positive integer $N = N_{h,\kappa} \equiv -\kappa h' \pmod{2\kappa}$, we have that

$$\lim_{q \to \zeta_{2\kappa}^h} \left(f(q) - (-1)^\kappa b(q)\right) = -4\zeta_{2k}^h J_N(T_{(2,3)}^*; \zeta_{2\kappa}^h).$$

Proof. We begin with Theorem 2.4, so that establishing Theorem 3.3 is equivalent to establishing that $U(\zeta_b^a; \zeta_k^h)$ equals $\zeta_k^h J_N(T_{(2,3)}^*; \zeta_k^h)$ for any $N = N_{a,b,h,k} \equiv -ab'h' \pmod{k}$. This was recently established by the author in the course of the proof of Theorem 3 in [20], drawing from work of Hikami and Lovejoy. For completeness, we give the argument here. From [25, (1.7)], for any positive integer N, we deduce that $\zeta_k^h J_N(T_{(2,3)}^*; \zeta_k^h) = U(\zeta_k^{hN}; \zeta_k^h)$. Now suppose $N = N_{a,b,h,k} \equiv -ab'h' \pmod{k}$. Then

$$\zeta_k^{hN} = \zeta_k^{-ab'h'h} = \zeta_k^{ab'} = \zeta_b^a.$$

Thus we have proved the first assertion for $N = N_{a,b,h,k}$ under the prescribed hypotheses. The second assertion follows by setting $(a, b) = (1, 2)$ and letting $k = 2\kappa$. $\square$

The interested reader may also wish to consult the websites of the recent related workshops at Banff International Research Station (BIRS) in Banff, AB [27], and at The Institute for Computational and Experimental Research in Mathematics (ICERM) in Providence, RI [28].

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