A. Folsom *Res Math Sci(2019)6:25*
https://doi.org/10.1007/s40687-019-0188-y

Research in the Mathematical Sciences

RESEARCH

Quantum Jacobi forms in number theory, topology, and mathematical physics

Amanda Folsom

*Correspondence:
afolsom@amherst.edu
Department of Mathematics and Statistics, Amherst College, Amherst, MA 01002, USA

Abstract

We establish three infinite families of quantum Jacobi forms, arising in the diverse areas of number theory, topology, and mathematical physics, and unified by partial Jacobi theta functions.

1 Introduction and statement of results

There has been a great deal of interest in the subject of quantum modular forms since the time of their definition in 2010 by Zagier [41]. Loosely speaking, they are functions f which exhibit modular transformation properties on $\mathbb{Q}$, as opposed to $\mathbb{H}$ the upper half-plane, up to the addition of nontrivial error functions. Such error functions must exhibit appropriate analytic properties in $\mathbb{R}$. (See [5,41] for a more precise definition.) In their relatively short lifetime, extending from the foundation laid in [41], quantum modular forms have seen applications to a variety of subjects including the celebrated Riemann hypothesis [33], combinatorics [6,10,19,20,28], mock theta functions in number theory [9,12,18,21,27,29], Hecke operators in number theory [31,32], topology [25,26,35], and mathematical physics [7,14,36]. (A number of other references on the applications of quantum modular forms also exist, some of which may be found in [5, Chapter 21].)

Jacobi forms, defined on $\mathbb{C}\times\mathbb{H}$, are two-variable analogues to modular forms on $\mathbb{H}$, and their theory was largely developed by Eichler and Zagier in the 1980s [5,16]. Naturally marrying the definition of a Jacobi form with that of a quantum modular form, Bringmann and the author defined the notion of a quantum Jacobi form in 2016 in [3] and provided the first example of such a function, arising from combinatorics. Precisely, we have the following definition.

Definition 1 A weight $k \in \frac{1}{2}\mathbb{Z}$ and index $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form** is a complex-valued function ϕ on $\mathbb{Q}\times\mathbb{Q}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}\times\mathbb{Z}$, the functions $h_\gamma : \mathbb{Q}\times(\mathbb{Q}\setminus\gamma^{-1}(i\infty)) \to \mathbb{C}$ and $g_{(\lambda,\mu)} : \mathbb{Q}\times\mathbb{Q}\to\mathbb{C}$ defined by

$$h_\gamma(z;\tau) := \phi(z;\tau) - \varepsilon_1^{-1}(\gamma)(c\tau+d)^{-k}\mathrm{e}^{\frac{-2\pi i m c z^2}{c\tau+d}}\phi\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right),$$
$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \varepsilon_2^{-1}((\lambda,\mu))\mathrm{e}^{2\pi i m(\lambda^2\tau+2\lambda z)}\phi(z+\lambda\tau+\mu;\tau),$$

satisfy a “suitable” property of continuity or analyticity in a subset of $\mathbb{R}\times\mathbb{R}$.

Remarks (1) The complex numbers $\varepsilon_1(\gamma)$ and $\varepsilon_2((\lambda,\mu))$ satisfy $|\varepsilon_1(\gamma)| = |\varepsilon_2((\lambda,\mu))| = 1$; in particular, the $\varepsilon_1(\gamma)$ are such as those appearing in the theory of half-integral weight modular forms.

(2) We may modify the definition to allow modular transformations on appropriate subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We may also restrict the domain to be a suitable subset of $\mathbb{Q}\times\mathbb{Q}$.

(2) The "suitable" property of continuity or analyticity required is intentionally left somewhat vague in order to mimic Zagier's definition of a quantum modular form [41].

Since this definition, initial example, and application emerged in [3], only a small handful of other quantum Jacobi forms have been found (see [1,2]), all of which are combinatorial. The first infinite family of quantum Jacobi forms was established in [11], with applications to mock theta functions.

Here, our results and motivations are multifaceted. We establish three new infinite families of quantum Jacobi forms, arising in a different manner from all quantum Jacobi forms which are known thus far (in [1–3,11]). These infinite families of quantum Jacobi forms arise in the diverse areas of number theory, topology, and mathematical physics, especially motivating our results. Our main results are explicitly stated in Theorem 1 (infinite family and applications to number theory), Theorem 4 (infinite family and applications to topology), and Theorem 6 (applications to mathematical physics). Sections 1.1, 1.2, and 1.3 below are devoted to developing and stating these and other results in the context of number theory, topology, and mathematical physics, respectively. On the one hand, these results lie in diverse areas; on the other hand, as the remaining narrative reveals, they are unified by the partial Jacobi theta functions $C_{\alpha,\beta}(z;\tau)$ defined in (1.1).

1.1 An infinite family of quantum Jacobi forms

Throughout, unless otherwise stated, we let $q = e(\tau)$, $w = e(z)$, with $e(u) := \mathrm{e}^{2\pi i u}$. We also let $\alpha,\beta\in\mathbb{N}$ be such that $0<\alpha<\beta$, $4|\beta$ and $\gcd(\alpha,\beta)=1$. With these conventions, we define the function

$$C_{\alpha,\beta}(z;\tau) := q^{\frac{\alpha^2}{2\beta^2}} w^{\frac{\alpha}{2\beta}} \sum_{n\geq 0} q^{\frac{n^2}{2}} \left(w^{\frac{1}{2}} q^{\frac{\alpha}{\beta}}\right)^n. \tag{1.1}$$

For fixed α,β, the function $C_{\alpha,\beta}(z;\tau)$ may be regarded as a Jacobi partial theta function; historically, such functions have been important in number theory, in the theory of q-hypergeometric series, in connection with mock theta functions, and most recently in relation to quantum modular forms (when viewed as one-variable functions of τ with z fixed). This last subject is of particular historic interest: In Ramanujan's final letter to Hardy in 1920, Ramanujan states that partial (or "false") theta functions *"do not enter into mathematics as beautifully as the ordinary theta functions,"* which are modular forms. We now know how partial/false theta functions are intimately connected to the subject of quantum modular forms and mock modular forms. (See [5,21] and references therein.)

In what follows, $\chi_{C,D}$ and $\psi_{B,C,D}(\alpha,\beta)$ are characters defined in (5.17) and (5.7), respectively, with respect to matrices $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ in appropriate subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Similarly, $\left(\frac{C}{D}\right)$ denotes the Kronecker symbol on such matrices. The set $Q_{\alpha,\beta}\subseteq\mathbb{Q}\times\mathbb{Q}$ and groups G_β, H_β are defined in Sect. 3, and the theta functions $g_{A,B}$ are defined in (2.14).

Our first result in Theorem 1 shows that $\{C_{\alpha,\beta}\}$, indexed by pairs $(\alpha, \beta) \in \mathbb{N}^2$ satisfying the hypotheses above, forms an infinite family of quantum Jacobi forms.

Theorem 1 *Assume the hypotheses above. The following are true.*
(1) *The function $C_{\alpha,\beta}(z;\tau)$ is a quantum Jacobi form on $Q_{\alpha,\beta}$ of weight $1/2$, index $-1/8$, group G_β, and character $\chi_{C,D}$.*

In particular, for any $\epsilon_{\alpha,\beta} > 0$ satisfying $\frac{\beta-\alpha}{\beta^2} < \epsilon_{\alpha,\beta} < \frac{1}{\beta}$, if $z \in \left(-\frac{\alpha}{\beta^2}, \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon_{\alpha,\beta}\right)$, we have that

$$C_{\alpha,\beta}(z;\tau) - (-2\beta\tau+1)^{-\frac{1}{2}}\chi_{2\beta,1}^{-1} e\left(\frac{2\beta z^2}{8(-2\beta\tau+1)}\right) C_{\alpha,\beta}\left(\frac{z}{-2\beta\tau+1}; \frac{\tau}{-2\beta\tau+1}\right)$$
$$= \frac{-1}{2}\int_0^\infty \frac{\sum_{\pm} g_{-\frac{\alpha}{2\beta}+\frac{3\mp 1}{4}, -z}\left(\frac{2}{\beta}+it\right)}{\sqrt{-i(\frac{2}{\beta}+it-4\tau)}} \mathrm{d}t, \tag{1.2}$$

and the difference in (1.2) extends to a C^∞ function on

$$\left(\mathbb{R} \setminus \left(\tfrac{2}{\beta}\mathbb{Z} - \tfrac{\alpha}{\beta^2} + \left\{0, \tfrac{1}{\beta}, \tfrac{\alpha}{\beta^2}, \tfrac{1}{\beta} \pm \epsilon_{\alpha,\beta}\right\}\right)\right) \times \left(\mathbb{R} \setminus \left\{\tfrac{1}{2\beta}\right\}\right).$$

(2) *The function $C_{\alpha,\beta}(z;-\tau)$ is a mock Jacobi form of weight $1/2$, index $-1/8$, group H_β, and character $\psi_{B,C,D}(\alpha,\beta)$.*

Remarks (1) Part (1) of Theorem 1 shows that $\{C_{\alpha,\beta}(z;\tau)\}$, when its members are viewed as functions on $Q_{\alpha,\beta} \subseteq \mathbb{Q} \times \mathbb{Q}$, forms an infinite family of quantum Jacobi forms. It also holds that $\{C_{\alpha,\beta}(z;-\tau)\}$ forms an infinite family of quantum Jacobi forms on $Q'_{\alpha,\beta} := \left\{\left(\frac{a}{b}, \frac{h}{k}\right) \in \mathbb{Q} \times \mathbb{Q} \colon \left(\frac{a}{b}, \frac{-h}{k}\right) \in Q_{\alpha,\beta}\right\}$.
(2) Part (2) of Theorem 1 above shows that $\{C_{\alpha,\beta}\}$, when its members are viewed as functions in $\mathbb{C} \times \mathbb{H}^-$, where $\mathbb{H}^\pm := \{\tau \in \mathbb{C} \mid \pm \mathrm{Im}(\tau) > 0\}$, also forms an infinite family of mock Jacobi forms. By mock Jacobi form, we mean a function which is the holomorphic part of a nonholomorphic Jacobi form. (See [5,42].)
(3) Fix α, β, a, b. The quantum modular properties of $C_{\alpha,\beta}(\frac{a}{b};\tau)$ as a single-variable function of τ follow from work in [8,21], and the mock modular properties of $C_{\alpha,\beta}(\frac{a}{b};-\tau)$ as a single-variable function of τ follow from work in [21]. Thus, we have restricted our attention to establishing the two-variable quantum Jacobi and mock Jacobi properties of the $C_{\alpha,\beta}$ in Theorem 1.

Before we state our remaining theorems, we discuss the so-called Eichler integral appearing in (1.2). Historically, Eichler integrals have played important roles in number theory, such as in the work of Eichler [15] and Shimura [39]. More recently, they have been rather prominent in the growing world of quantum modular forms, as shown in [8,41] for example. Example 4 from [41], taken from [30] and which incorporates Eichler integrals, is particularly interesting, as it comes from topology and the theory of quantum invariants of 3-manifolds. Developing this topic remains an active and current area of research, bridging different areas of mathematics [36].

In Theorem 2 below, we give an asymptotic expansion for $C_{\alpha,\beta}\left(\frac{a}{b}, \frac{h}{k} + \frac{it}{\pi}\right)$, with $(\frac{a}{b}, \frac{h}{k}) \in Q_{\alpha,\beta}$, as $t \to 0^+$, and we also give two different ways to explicitly evaluate $C_{\alpha,\beta}(z;\tau)$ on the quantum Jacobi set $Q_{\alpha,\beta}$.

Theorem 2 *Assume the hypotheses above, and let* $\left(\frac{a}{b}, \frac{h}{k}\right) \in Q_{\alpha,\beta}$. *We have the asymptotic expansion*

$$C_{\alpha,\beta}\left(\frac{a}{b}, \frac{h}{k} + \frac{it}{\pi}\right) \sim \zeta_{2\beta b}^{\alpha a} \sum_{r=0}^{\infty} \frac{L(-2r, c)}{r!} \left(\frac{-t}{\beta^2}\right)^r$$

as $t \to 0^+$, *where for* $r \in \mathbb{N}_0$, $L(-r, c) := -\frac{(k\beta^2)^r}{r+1} \sum_{n=1}^{k\beta^2} c(n) B_{r+1}\left(\frac{n}{k\beta^2}\right)$. *The coefficients* $c(n) = c_{\alpha,\beta,a,b,h,k}(n)$ *are defined explicitly in* (6.1), *and* $B_r(x)$ *denotes the rth Bernoulli polynomial.*

Moreover, the function $C_{\alpha,\beta}$ *may be evaluated explicitly on* $Q_{\alpha,\beta}$ *in the following two (different) ways:*

$$\begin{aligned} C_{\alpha,\beta}\left(\frac{a}{b}, \frac{h}{k}\right) &= -\zeta_{2\beta b}^{\alpha a} \sum_{n=1}^{k\beta^2} c(n) B_1\left(\frac{n}{k\beta^2}\right) \\ &= \zeta_{2\beta b}^{\alpha a} \zeta_{2\beta^2 k}^{\alpha^2 h} \sum_{n=0}^{M} \zeta_{2k}^{nh} \zeta_{k\beta}^{nh\alpha} \zeta_{2b}^{na} \frac{(\zeta_{2b}^{a} \zeta_{k\beta}^{\alpha h} \zeta_{2k}^{h}; \zeta_k^{2h})_n}{(\zeta_{2b}^{a} \zeta_{k\beta}^{\alpha h} \zeta_{2k}^{3h}; \zeta_k^{2h})_n}, \end{aligned}$$

where $M = M_{a,b,h,k}(\alpha, \beta)$ *is the smallest nonnegative integer such that* $\frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2M\right) \in \mathbb{Z}$.

In Theorem 2 and throughout, for $n \in \mathbb{N}_0$, $(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$ denotes the q-Pochhammer symbol.

Example 1 Let $(\alpha, \beta) = (1, 4)$. Then $\left(\frac{a}{b}, \frac{h}{k}\right) = \left(\frac{-1}{8}, \frac{3}{4}\right) \in Q_{1,4}$ (with $M = 1$). By Theorem 2, we have that

$$C_{1,4}\left(\tfrac{-1}{8}, \tfrac{3}{4}\right) = -\zeta_{64}^{-1} \sum_{n=1}^{64} c(n) B_1\left(\frac{n}{64}\right) = -\zeta_{128}^{33} \approx .0490677 - .998795i.$$

To calculate this value, we have used the fact that mod 64, the 16 nonzero coefficients $c(n)$ are:

ζ_{128}^{3}	$c(1), c(21), c(33), c(53)$
ζ_{128}^{-61}	$c(5), c(17), c(37), c(49)$

ζ_{128}^{-29}	$c(9), c(13), c(41), c(45)$
ζ_{128}^{35}	$c(25), c(29), c(57), c(61)$

On the other hand, by Theorem 2 we also have that

$$C_{1,4}\left(\tfrac{-1}{8}, \tfrac{3}{4}\right) = \zeta_{128} \sum_{n=0}^{1} (-1)^n \frac{(\zeta_2; \zeta_2^3)_n}{(\zeta_4^5; \zeta_2^3)_n} = \zeta_{128}\left(1 - \frac{2}{1-i}\right) \approx .0490677 - .998795i.$$

Example 2 Let $(\alpha, \beta) = (5, 24)$. Then $\left(\frac{a}{b}, \frac{h}{k}\right) = \left(\frac{1}{40}, \frac{-3}{10}\right) \in Q_{5,24}$ (with $M = 3$). By Theorem 2, we have that

$$\begin{aligned} C_{5,24}\left(\tfrac{1}{40}, \tfrac{-3}{10}\right) &= -\zeta_{384} \sum_{n=1}^{5760} c(n) B_1\left(\frac{n}{5760}\right) \\ &= \zeta_{1280}^{379}(1 + \zeta_{10} - \zeta_{10}^2) \\ &\approx -.0802333716 + 1.5412749973i, \end{aligned} \tag{1.3}$$

where we have calculated the 240 nonzero coefficients $c(n)$ mod 5760 explicitly for this evaluation, using (6.1).

On the other hand, by Theorem 2 we also have that

$$\begin{aligned} C_{5,24}\left(\tfrac{1}{40}, \tfrac{-3}{10}\right) &= \zeta_{256}^{-1} \sum_{n=0}^{3} \zeta_5^{-n} \frac{(\zeta_5^{-1};\zeta_5^{-3})_n}{(\zeta_2^{-1};\zeta_5^{-3})_n} \\ &= \zeta_{256}^{-1}\Bigg(1 + \frac{\zeta_5^{-1}}{2}(1-\zeta_5^{-1}) + \frac{\zeta_5^{-2}}{2}\frac{(1-\zeta_5^{-1})(1-\zeta_5^{-4})}{(1+\zeta_5^{-3})} \\ &\quad + \frac{\zeta_5^{-3}}{2}\frac{(1-\zeta_5^{-1})(1-\zeta_5^{-2})(1-\zeta_5^{-4})}{(1+\zeta_5^{-1})(1+\zeta_5^{-3})}\Bigg) \\ &\approx -.0802333716 + 1.5412749973i. \end{aligned} \tag{1.4}$$

Without Theorem 2, it is unlikely that one would immediately expect that the two different exponential-type sums in (1.3) and (1.4) are equal.

Remark By combining the simple closed form expressions from Theorem 2 with Theorem 1 (1), we immediately obtain simple closed form expressions in roots of unity for rational evaluations of Eichler integrals of weight 3/2 modular forms. (Similar corollaries have been explicitly written down in [18,20].)

1.2 Jones polynomials for torus knots as quantum Jacobi forms

One of the first fundamental examples of a quantum modular form is (a slightly normalized version of) the function

$$F(q) = \sum_{n=0}^{\infty} (q;q)_n$$

when viewed as a function of $x \in \mathbb{Q}$, with $q = e^{2\pi i x}$. The function F was originally studied by Zagier [41], and it was later shown to be dual to the combinatorial generating function for strongly unimodal sequences $U(-1;q)$ by Bryson et al. [10], in the sense that $F(\zeta) = U(-1;\zeta^{-1})$, where ζ is a root of unity (equivalently, $F(q) = U(-1;q^{-1})$ when $q = e^{2\pi i x}$, with $x \in \mathbb{Q}$). The function $U(-1;q)$ is not only a quantum modular form by virtue of the duality and quantum modularity of $F(q)$ just mentioned, but it is also essentially a mock modular form by results established in [10].

This function $U(-1;q)$ arises as the special value at $w = -1$ of the two-variable strongly unimodal sequence rank generating function

$$U(w;q) := \sum_{n=0}^{\infty} (-wq;q)_n(-w^{-1}q;q)_n q^{n+1}.$$

Work in [20] generalized the duality and quantum modular properties associated with $F(q)$ and $U(-1;q)$ just mentioned; namely, in [20] we defined a two-variable function $F(w;q)$ with the properties that (1) $F(1;q) = F(q)$, the Kontsevich–Zagier function, (2) $F(\zeta_b^a;\zeta) = U(-\zeta_b^a;\zeta^{-1})$ for any primitive kth root of unity ζ and any bth root of unity ζ_b^a where $b \mid k$, and (3) for fixed ζ_b^a, $F(\zeta_b^a;q)$ (and hence $U(-\zeta_b^a;q^{-1})$ gives rise to a quantum modular form when viewed as a one-variable function of $x \in \mathbb{Q}$, with $q = e^{2\pi i x}$.[1]

[1] We note that the same or similar notation for F and U appears in different sources (such as [6,10,20,26,41]) but may in reality define slightly different normalizations of these functions; the reader should proceed with caution when consulting the literature.

In addition to [10,20], to date, there are three more works which establish mock and quantum properties associated with the functions F and U. First, work of Bringmann et al. [6] established an infinite family of quantum (and mock) modular forms from rank generating functions for unimodal sequences, the first member of which is the Kontsevich–Zagier function $F(q)$. Second, Bringmann and colleague [3] provided the first example of a quantum Jacobi form by proving that the function $U(w;q)$, when viewed as a two-variable function in (z,τ), with $w = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$, is (up to a slight normalization) such a form.

The last work along these lines exists in another direction and has topological applications. Namely, Hikami in [22] defined an infinite family of q-hypergeometric series $\{F_t(q)\}_{t\in\mathbb{N}}$, the members of which give rise to Jones polynomials for torus knots $T_{2,2t+1}$. (See also Hikami's [23].) When $t = 1$, Hikami's $F_1(q) = qF(q)$, where $F(q)$ is the Kontsevich–Zagier function. Hikami and Lovejoy established the duality associated with $F_t(q)$ and $U_t(-1;q)$, where (as defined by the authors in [26])

$$U_t(w;q) := q^{-t} \sum_{k_t\geq\cdots k_1\geq 1}^{\infty} (-wq;q)_{k_t-1}(-w^{-1}q;q)_{k_t-1}q^{k_t} \times \prod_{j=1}^{t-1} q^{k_j^2} \begin{bmatrix} k_{j+1}+k_j-j+2\sum_{\ell=1}^{j-1}k_\ell \\ k_{j+1}-k_j \end{bmatrix}_q .$$

Here,

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}}$$

denotes the q-binomial coefficient. Analogous to the fact that $F_1(q) = qF(q)$, the function $U_t(w;q)$ satisfies $U_1(w;q) = q^{-1}U(w;q)$. One of the main results in [26] shows that $F_t(\zeta) = U_t(-1;\zeta^{-1})$, where ζ is a root of unity, generalizing the duality from [10] in another direction. Playing a key role in [26] is the interesting observation that $U_t(-1;q)$ and $F_t(q)$ may be interpreted as colored Jones polynomials, when specialized appropriately at roots of unity.

Here, we generalize the previous duality and quantum properties associated with $U_t(w;q)$ and $F_t(q)$ by first defining, for $t\in\mathbb{N}$, a two-variable analogue to $F_t(q)$, namely

$$F_t(w;q) := q^t(-w)^t \sum_{k_t\geq\cdots\geq k_1\geq 0}^{\infty} (-w)^{k_t}(-wq;q)_{k_t} \prod_{j=1}^{t-1} q^{k_j(k_j+1)}(-w)^{2k_j} \begin{bmatrix} k_{j+1} \\ k_j \end{bmatrix}_q . \tag{1.5}$$

This function first appeared in work of Hikami [22,24]. We point out that $F_t(-1;q) = F_t(q)$ (and hence $F_1(-1;q) = qF(q)$). In Theorem 3 below, we establish the duality between $F_t(w;q^{-1})$ and $U_t(w;q)$ both as a polynomial in q for fixed w, and at suitable pairs of roots of unity $(w;q) = (\zeta_b^a;\zeta_k^h)$, generalizing the duality established in [10,20,26]. Specifically, we obtain [26, Theorem 1.2] as a special case of Theorem 3 part (2) below when $a = b = 1$, $h = 1$, $k = N$, and we obtain [20, Corollary 1.3] when $t = 1$.

Theorem 3 *Let $t\in\mathbb{N}$. The following are true.*
(1) *For any $N\in\mathbb{N}$, we have the polynomial identity*

$$F_t(-q^N;q^{-1}) = U_t(-q^N;q) \in \mathbb{Z}[q].$$

(2) *Let $h, a \in \mathbb{Z}$, and $k, b \in \mathbb{N}$, be such that* $\gcd(h, k) = \gcd(a, b) = 1$, *and such that* $b|k$. *Then we have that*

$$F_t(-\zeta_b^a; \zeta_k^{-h}) = U_t(-\zeta_b^a; \zeta_k^h).$$

The proof of Theorem 3 in Sect. 7, which extends original work and observations of Hikami and Lovejoy [26], reveals that the polynomials in Theorem 3 (1) may be interpreted as N-colored Jones polynomials for torus knots $T_{2,2t+1}$ and their mirrors. The example below provides an illustration.

Example We have from Theorem 3 (1) with $t = 3$ and $N = 4$, after simplifying, that

$$\begin{aligned} q^{-9}F_3(-q^4, q^{-1}) &= q^{-9}U_3(-q^4, q) \\ &= 1+q^4-q^7+q^8-q^{11}+q^{12}-q^{15}+q^{16}-q^{19}+q^{20}-q^{22}-q^{23}+q^{24} \\ &\quad -q^{26}+q^{28}-q^{30}+q^{32}-q^{34}+q^{36}+q^{37}-q^{38}+q^{41}-q^{42}. \end{aligned}$$

This polynomial may be viewed as q^{-9} times the 4-colored Jones polynomial for the torus knot $T_{(2,7)}$ (in the variable q^{-1}). That is, it equals $q^{-9}J_4(T_{(2,7)}; q^{-1})$. (See Sect. 7.)

Now let $(h, k) = (5, 12)$ and $(a, b) = (2, 3)$. By Theorem 3 (2), or equivalently, by its proof, combined with the above when $N = 4$, we also have (after simplifying) that

$$\begin{aligned} iF_3(-\zeta_3^2; \zeta_{12}^{-5}) &= iU_3(-\zeta_3^2; \zeta_{12}^5) \\ &= 6 - i - 2\zeta_{12}^{-1} + \zeta_{12} - 2\zeta_6^{-1} - 2\zeta_6 + 3\zeta_3^{-1} + 3\zeta_3 - 2\zeta_{12}^{-5} + \zeta_{12}^5 \\ &= 2i + 1. \end{aligned}$$

Next, we turn to establishing the quantum Jacobi and mock Jacobi properties of the functions $F_t(w; q)$ and $U_t(w; q)$. We define slight normalizations of the functions $F_t(w; q)$ and $U_t(w; q)$ as follows:

$$\begin{aligned} \mathcal{F}_t(z; \tau) &:= (1 - w)q^{\frac{(2t-1)^2}{16t+8} - t} w^{-\frac{1}{2}} F_t(-w; q), \\ \mathcal{U}_t(z; \tau) &:= (1 - w)q^{\frac{(2t-1)^2}{16t+8} - t} w^{-\frac{1}{2}} U_t(-w; q^{-1}). \end{aligned}$$

Here and throughout, we let

$$\begin{aligned} &\beta_t := 4(2t + 1),\ \alpha_t^{(1)} = \alpha_t := 2t - 1,\ \alpha_t^{(2)} := 2t + 3, \\ &\alpha_t^{(3)} := 6t + 1,\ \alpha_t^{(4)} = A_t := 6t + 5. \end{aligned}$$

The periodic function χ_{8t+4} appearing in Theorem 4 is as defined in Sect. 8. The subset $Q_2 \subseteq \mathbb{Q} \times \mathbb{Q}$ and groups K_t, L_t are defined in Sect. 3.

Theorem 4 *Assume the notation and hypotheses above. The following are true.*

(1) *The function* $\mathcal{F}_t(z; \tau) = \mathcal{U}_t(z; \tau)$ *is a quantum Jacobi form on* Q_2 *of weight* $1/2$, *index* $-t - \frac{1}{2}$, *group* K_t, *and character* $\chi_{C/\beta_t, D}$.

In particular, for any $\epsilon_t > 0$ *satisfying* $\frac{\beta_t-\alpha_t}{\beta_t^2} < \epsilon_t < \frac{1}{\beta_t}$*, if* $z \in \left(-\frac{\alpha_t}{\beta_t^3}, \frac{1}{\beta_t^2} - \frac{A_t}{\beta_t^3} - \frac{\epsilon_t}{\beta_t}\right)$*, we have that*

$$\begin{aligned}\mathcal{F}_t(z;\tau) &- (-2\beta_t^2\tau+1)^{-\frac{1}{2}}\chi_{2\beta_t,1}^{-1} \\ &\times e\left(\frac{2\beta_t^3 z^2}{8(-2\beta_t^2\tau+1)}\right)\mathcal{F}_t\left(\frac{z}{-2\beta_t^2\tau+1};\frac{\tau}{-2\beta_t^2\tau+1}\right) \\ &= -\frac{1}{2}\int_0^\infty \frac{\sum_{j=1}^4 \chi_{8t+4}(\alpha_t^{(j)})\sum_\pm g_{-\frac{\alpha_t^{(j)}}{2\beta_t}+\frac{3\mp 1}{4},-\beta_t z}\left(\frac{2}{\beta_t}+is\right)}{\sqrt{-i(\frac{2}{\beta_t}+is-4\beta_t\tau)}}\mathrm{d}s, \quad (1.6)\end{aligned}$$

and the difference in (1.6) *extends to a* C^∞ *function on*

$$\left(\mathbb{R}\setminus\bigcup_{j=1}^4\left(\frac{2}{\beta_t^2}\mathbb{Z}-\frac{\alpha_t^{(j)}}{\beta_t^3}+\{0,\frac{1}{\beta_t^2},\frac{\alpha_t^{(j)}}{\beta_t^3},\frac{1}{\beta_t^2}\pm\frac{\epsilon_t}{\beta_t}\}\right)\right)\times\left(\mathbb{R}\setminus\{\frac{1}{2\beta_t^2}\}\right).$$

(2) *The function* $\mathcal{F}_t(z;-\tau)$ *is a mock Jacobi form of weight* $1/2$*, index* $-t-\frac{1}{2}$*, group* L_t*, and character* $\left(\frac{C}{D}\right)$*.*

Remarks (1) Theorem 4 combined with the discussion prior (see also Sect. 7) brings the N-colored Jones polynomials for the torus knots $T_{(2,2t+1)}$ and their mirrors into the theory of quantum Jacobi forms.

(2) Asymptotics and various explicit evaluations of the functions $\mathcal{F}_t$ and $\mathcal{U}_t$ follow readily from Theorem 2, the proof of Theorem 3, the proof of Theorem 4, and the definitions of $\mathcal{F}_t$ and $\mathcal{U}_t$.

Theorem 5 below establishes the quantum modular properties associated with $\mathcal{F}_t(\frac{a}{b},\tau) = \mathcal{U}_t(\frac{a}{b};\tau)$ when viewed as a single-variable function of τ. We state this result explicitly for completeness, extending results from [20] (which pertain to the case $t=1$). Below, ρ_D is 1 or i, depending on whether D is 1 or 3 $\pmod 4$, and $\ell_{b,\beta} := \mathrm{lcm}(b,\beta)$. The group X_{b,β_t} is defined in Sect. 3.

Theorem 5 *We have that* $\mathcal{F}_t(\frac{a}{b};\tau) - \mathcal{F}_t(-\frac{a}{b};\tau) = \mathcal{U}_t(\frac{a}{b};\tau) - \mathcal{U}_t(-\frac{a}{b};\tau)$ *is a quantum modular form on* $\mathbb{Q}$ *of weight* $1/2$*, group* X_{b,β_t}*, and character* $\left(\frac{\ell_{b,\beta_t}}{D}\right)\left(\frac{2C\beta_t/\ell_{b,\beta_t}}{D}\right)\rho_D^{-1}$*. The errors to modularity on* $\mathbb{Q}$ *extend to real analytic functions in* $\mathbb{R}\setminus\{-\frac{D}{C}\}$*.*

1.3 Applications to mathematical physics

The $(1,p)$-singlet vertex algebra admits atypical (regularized) irreducible characters

$$\mathrm{ch}[M_{1,s}^{iz}](\tau) = \frac{1}{\eta(\tau)}\sum_{n\geq 0}\left(w^{\frac{1}{\sqrt{2p}}(2pn-s+p)}q^{\frac{1}{4p}(2pn-s+p)^2} - w^{\frac{1}{\sqrt{2p}}(2pn+s+p)}q^{\frac{1}{4p}(2pn+s+p)^2}\right),$$

where $1\leq s\leq p-1$. As originally studied, these characters did not include the complex parameter z. However, motivated by the Verlinde formula in conformal field theory, these original characters have since been regularized to include the new complex parameter z, which can be viewed in terms of the $U(1)$-charge in physics. Within mathematical physics and number theory, these functions have recently been studied in [4,13,14]. Understanding the modular properties of such functions in general has been of interest, most basically referencing Monstrous Moonshine, mock modular Moonshine, and similar results attached to Lie superalgebras. (See [5, Chapter 20] and references therein.) Here we establish the quantum Jacobi and mock Jacobi properties of these functions, when viewed as two-variable functions in (z, τ). The subset $\mathcal{Q}_{s,p} \subseteq \mathbb{Q} \times \mathbb{Q}$ and groups M_p, W_p are defined in Sect. 3.

Theorem 6 *Let $p = 2m^2$ for some $m \in \mathbb{N}$. The following are true.*
(1) *The atypical characters $\eta(\tau)\mathrm{ch}[M_{1,s}^{iz}](\tau)$ are quantum Jacobi forms on $\mathcal{Q}_{s,p}$ of weight $1/2$, index -1, group M_p, and character $\left(\frac{C}{D}\right)$.*
(2) *The functions $\eta(\tau)\mathrm{ch}[M_{1,s}^{iz}](-\tau)$ are mock Jacobi forms of weight $1/2$, index -1, group W_p, and character $\zeta_{8p}^{-Bs^2}(-1)^{\frac{Bs}{2}}\left(\frac{C}{D}\right)$.*

Remarks (1) Asymptotics and explicit evaluations of these characters at rationals follow readily from Theorem 2 and the proof of Theorem 6. Such expressions are of interest in representation theory and mathematical physics as related to the quantum dimension of the characters. (See [4, 13, 14].)
(2) As remarked after Theorem 1, the quantum modular properties of these functions for fixed z as a one-variable function of τ follow from work in [8].

1.3.1 The remainder of the paper is organized as follows. In Sect. 2, we define certain functions required in the proofs of our main results and give some of their known properties. In Sect. 3, we define and study groups and sets appearing in the statements of our main results above. In Sect. 4, we explicitly establish the mock Jacobi properties of a certain function, as well as some of its related nonholomorphic transformations. Each of the six Sects. 5–10 are devoted to proving one of the six Theorems 1–6. For convenience, a partial index of the main notations and definitions has been included, and this appears toward the end of the paper before the references.

2 Preliminaries

In this section, we collect certain functions which are used in the proofs of our main results, and state some of their known identities and transformation properties.

2.1 *q*-series identities

First, we define the partial Jacobi theta function H by

$$H(w;q) := \sum_{n=0}^{\infty} q^{\frac{(n+\frac{1}{2})^2}{2}} w^{-n}.$$

Next, we define a q-hypergeometric function K, which may be viewed as a universal mock theta function, defined in [34] by

$$K(w;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(wq^2;q^2)_n(w^{-1}q^2;q^2)_n}.$$

In Lemma 1 below, we relate[2] H and K to Fine's q-hypergeometric series [17]

$$F(A,B,x;q) := \sum_{n=0}^{\infty} \frac{(Aq;q)_n x^n}{(Bq;q)_n}.$$

[2]Note that the function H defined here is not the same as the function with the same name in [34].

Lemma 1 *We have that*

$$H(w;q) = q^{\frac{1}{8}} F(w^{-1}q^{-1}, w^{-1}, w^{-1}q; q^2) \tag{2.1}$$

$$K(wq;q)(1-w^{-1}q^{-1})^{-1} = F(w, wq, w^{-1}q^{-1}; q^2). \tag{2.2}$$

Lemma 1 follows from known identities: (2.1) is a slight generalization of [21, Eq. (1.11)], see [38], and (2.2) is [21, Eq. (2.6)].

We require the following identity, which follows from results in [34], combined with the definition of the Appell function A_2 given in Sect. 2.3.

Lemma 2 *We have that*

$$K(w;q) = (w^{-1}-1)\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2} A_2(z, -\tau; 2\tau). \tag{2.3}$$

2.2 Modular and Jacobi forms

We make use of the weight $1/2$ modular form η, and the weight $1/2$ Jacobi form ϑ, defined for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \quad \vartheta(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} e^{\pi i n^2\tau + 2\pi i n(z+\frac{1}{2})}. \tag{2.4}$$

These functions satisfy the transformation properties given in Lemmas 3 and 4 below [37].

Lemma 3 *For* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *and* $\tau \in \mathbb{H}$,

$$\eta(\gamma\tau) = \varepsilon(\gamma)(C\tau + D)^{\frac{1}{2}}\eta(\tau),$$

where for $C > 0$,

$$\varepsilon(\gamma) = \begin{cases} \frac{1}{\sqrt{i}}\left(\frac{D}{C}\right) i^{(1-C)/2} e^{\pi i (BD(1-C^2)+C(A+D))/12} & \text{if } C \text{ is odd}, \\ \frac{1}{\sqrt{i}}\left(\frac{C}{D}\right) e^{\pi i D/4} e^{\pi i (AC(1-D^2)+D(B-C))/12} & \text{if } D \text{ is odd}. \end{cases} \tag{2.5}$$

Lemma 4 *For* $\lambda, \mu \in \mathbb{Z}$, $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, *and* $(z,\tau) \in \mathbb{C} \times \mathbb{H}$,

(i) $\vartheta(z + \lambda\tau + \mu; \tau) = (-1)^{\lambda+\mu} q^{-\frac{\lambda^2}{2}} e^{-2\pi i \lambda z}\vartheta(z;\tau)$,

(ii) $\vartheta\left(\frac{z}{C\tau + D}; \gamma\tau\right) = \varepsilon^3(\gamma)(C\tau+D)^{\frac{1}{2}} e^{\frac{\pi i C z^2}{C\tau + D}}\vartheta(z;\tau)$,

(iii) $\vartheta(z;\tau) = -iq^{\frac{1}{8}} w^{-\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n)(1-wq^{n-1})(1-w^{-1}q^n)$.

Using η and ϑ we define the functions

$$N(\tau) := \frac{\eta(\tau)}{\eta^2(2\tau)}, \quad \text{and } T(\tau) := \vartheta(-\tau + \tfrac{1}{2}; 4\tau).$$

A short calculation using the definition of η in (2.4) as well as Lemma 4 (iii) reveals that

$$N(\tau)T(\tau) = -q^{-\frac{1}{8}}, \tag{2.6}$$

a fact which we will use later. We also require the following lemma which gives some of the modular transformation properties of N and T.

Lemma 5 *Let* $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *with C even, and define* $\widetilde{\gamma} := \left(\begin{smallmatrix} A & 2B \\ C/2 & D \end{smallmatrix}\right)$*. We have that*

$$N(\gamma\tau) = (c\tau + d)^{-\frac{1}{2}}\varepsilon(\gamma)\varepsilon^{-2}(\widetilde{\gamma})N(\tau). \tag{2.7}$$

Moreover, for $\beta \in \mathbb{N}$ *with* $4 \mid \beta$*, we have that*

$$T\left(\frac{\tau}{2\beta\tau + 1}\right) = \varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0 \\ \frac{\beta}{2} & 1 \end{smallmatrix}\right)\right)(2\beta\tau + 1)^{\frac{1}{2}}\mathrm{e}^{\frac{\pi i\beta}{8}}\mathrm{e}^{\frac{\pi i\beta\tau^2}{2(2\beta\tau+1)}}T(\tau). \tag{2.8}$$

Proof The proof of (2.7) follows immediately from Lemma 3. To prove (2.8), we employ Lemma 4 (ii) and find that

$$T\left(\frac{\tau}{2\beta\tau + 1}\right) = \varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0 \\ \frac{\beta}{2} & 1 \end{smallmatrix}\right)\right)(2\beta\tau + 1)^{\frac{1}{2}}\mathrm{e}^{\frac{\pi i\beta\left(-\tau+\frac{1}{2}(2\beta\tau+1)\right)^2}{2(2\beta\tau+1)}}\vartheta\left(-\tau + \frac{1}{2} + \beta\tau; 4\tau\right).$$

Using Lemma 4 (i), this is

$$\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0 \\ \frac{\beta}{2} & 1 \end{smallmatrix}\right)\right)(2\beta\tau + 1)^{\frac{1}{2}}\mathrm{e}^{\frac{\pi i\beta\left(-\tau+\frac{1}{2}(2\beta\tau+1)\right)^2}{2(2\beta\tau+1)}}q^{-\frac{\beta^2}{8}+\frac{\beta}{4}}T(\tau).$$

The result in (2.8) now follows after a simplification. □

2.3 The level 2 Appell function

After Zwegers (see [5]), we define the level 2 Appell function for $z_1, z_2 \in \mathbb{C}$, $\tau \in \mathbb{H}$ by

$$A_2(z_1, z_2; \tau) := \xi_1\sum_{n\in\mathbb{Z}}\frac{\xi_2^n q^{n(n+1)}}{1 - \xi_1 q^n}, \tag{2.9}$$

where $\xi_j = e(z_j)$, $j \in \{1, 2\}$. This function may be decomposed as

$$\begin{aligned} A_2(z_1, z_2; \tau) &= \vartheta(z_2 + \tfrac{1}{2}; 2\tau)\mu(2z_1, z_2 + \tfrac{1}{2}; 2\tau) \\ &\quad + \xi_1\vartheta(z_2 + \tau + \tfrac{1}{2}; 2\tau)\mu(2z_1, z_2 + \tau + \tfrac{1}{2}; 2\tau), \end{aligned} \tag{2.10}$$

where

$$\mu(z_1, z_2; \tau) := \frac{\xi_1^{\frac{1}{2}}}{\vartheta(z_2; \tau)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\xi_2^n q^{\frac{n(n+1)}{2}}}{1 - \xi_1 q^n}.$$

The completed level 2 Appell functions $\widehat{A}_2$ are defined by

$$\begin{aligned} &\widehat{A}_2(z_1, z_2; \tau) \\ &:= A_2(z_1, z_2; \tau) + \frac{i}{2}\sum_{j=0}^{1}\mathrm{e}^{2\pi ijz_1}\vartheta\left(z_2 + j\tau + \frac{1}{2}; 2\tau\right)R\left(2z_1 - z_2 - j\tau - \frac{1}{2}; 2\tau\right), \end{aligned} \tag{2.11}$$

where the nonholomorphic function R is defined by

$$R(z; \tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}}\left\{\mathrm{sgn}(\nu) - E\left((\nu + \lambda)\sqrt{2y}\right)\right\}(-1)^{\nu-\frac{1}{2}}\mathrm{e}^{-\pi i\nu^2\tau - 2\pi i\nu z}, \tag{2.12}$$

with $y := \mathrm{Im}(\tau)$, $\lambda := \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}$ and

$$E(z) := 2\int_0^z \mathrm{e}^{-\pi u^2}\mathrm{d}u.$$

We have the following transformation properties of $\widehat{A}_2$ [5]:

Lemma 6 *With hypotheses as above, for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the functions $\widehat{A}_2$ satisfy the following transformation properties:*

(i) $\widehat{A}_2(-z_1, -z_2; \tau) = -\widehat{A}_2(z_1, z_2; \tau)$,
(ii) $\widehat{A}_2(z_1 + n_1\tau + m_1, z_2 + n_2\tau + m_2; \tau) = \xi_1^{2n_1-n_2}\xi_2^{-n_1}q^{n_1^2-n_1n_2}\widehat{A}_2(z_1, z_2; \tau)$,
(iii) $\widehat{A}_2\left(\frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}; \gamma\tau\right) = (c\tau + d)e^{\frac{\pi ic}{c\tau+d}(-2z_1^2+2z_1z_2)}\widehat{A}_2(z_1, z_2; \tau)$.

From [42, Propositions 1.9, 1.10], we have the following transformation properties of R.

Lemma 7 *With hypotheses as above, R satisfies the following transformation properties:*

(i) $R(z; \tau + 1) = e^{-\frac{\pi i}{4}}R(z; \tau)$,
(ii) $\frac{1}{\sqrt{-i\tau}}e^{\frac{\pi iz^2}{\tau}}R\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) + R(z; \tau) = h(z; \tau)$,
(iii) $R(z; \tau) = R(-z; \tau)$,
(iv) $R(z; \tau) + e^{-2\pi iz-\pi i\tau}R(z + \tau; \tau) = 2e^{-\pi iz-\pi i\tau/4}$,
(v) $R(z + 1; \tau) = -R(z; \tau)$.

As in Lemma 7 (ii), for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, the Mordell integral h is given by

$$h(z; \tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau u^2-2\pi zu}}{\cosh(\pi u)}du. \tag{2.13}$$

Under certain conditions, h can be rewritten using the weight $3/2$ theta functions $g_{A,B}$ (see Lemma 8), defined for $A, B \in \mathbb{R}$ and $\tau \in \mathbb{H}$ by

$$g_{A,B}(\tau) := \sum_{\nu\in A+\mathbb{Z}} \nu e^{\pi i\nu^2\tau+2\pi i\nu B}. \tag{2.14}$$

The functions $g_{A,B}$ transform as follows [40,42].

Lemma 8 *With hypotheses as above, the functions $g_{A,B}$ satisfy:*

(i) $g_{A+1,B}(\tau) = g_{A,B}(\tau)$,
(ii) $g_{A,B+1}(\tau) = e^{2\pi iA}g_{A,B}(\tau)$,
(iii) $g_{A,B}(\tau + 1) = e^{-\pi iA(A+1)}g_{A,A+B+\frac{1}{2}}(\tau)$,
(iv) $g_{A,B}\left(-\frac{1}{\tau}\right) = ie^{2\pi iAB}(-i\tau)^{\frac{3}{2}}g_{B,-A}(\tau)$,
(v) $g_{-A,-B}(\tau) = -g_{A,B}(\tau)$.

The following result relates the functions h and $g_{A,B}$ [42].

Lemma 9 *For $A, B \in (-\frac{1}{2}, \frac{1}{2})$,*

$$\int_0^{i\infty} \frac{g_{A+\frac{1}{2},B+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}}dz = -e^{-\pi iA^2\tau+2\pi iA\left(B+\frac{1}{2}\right)}h(A\tau - B; \tau).$$

3 Groups and sets

Here we define a number of subgroups of $SL_2(\mathbb{Z})$ and study their Jacobi action on various subsets of $\mathbb{Q} \times \mathbb{Q}$. We use the notation $\langle S\rangle$ to denote the group generated by the set S. The parameters β, t, β_t, p (and b, as occurring in $a/b \in \mathbb{Q}$), are as in the previous section, and $\ell_{b,\beta} := \mathrm{lcm}(b, \beta)$. We specifically recall that $p = 2r^2$ for some $r \in \mathbb{N}$. We define

$$\begin{aligned}
H_\beta &:= \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon A, D \equiv 1 \pmod{2\beta}, B, C \equiv 0 \pmod{\beta}\right\}, \\
L_t &:= \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon A, D \equiv 1 \pmod{2\beta_t}, B \equiv 0 \pmod{2\beta_t}, C \equiv 0 \pmod{\beta_t^2}\right\}, \\
W_p &:= \left\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon A, D \equiv 1 \pmod{4p}, B \equiv 0 \pmod{2}, C \equiv 0 \pmod{2p^2}\right\}, \\
X_{b,\beta} &:= \Bigg\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon A, D \equiv 1 \pmod{2\ell_{b,\beta}}, B \equiv 0 \pmod{2\beta}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad C \equiv 0 \pmod{2\ell_{b,\beta}^2/\beta}\Bigg\},
\end{aligned}$$

$$G_\beta := \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 2\beta & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2\beta^2 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle, \ K_t := \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 2\beta_t^2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2\beta_t \\ 0 & 1 \end{smallmatrix}\right) \right\rangle, \ M_p := \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 4p^2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 8p \\ 0 & 1 \end{smallmatrix}\right) \right\rangle.$$

Next, we define certain subgroups of $\mathbb{Q} \times \mathbb{Q}$ on which the above groups act. Throughout, we call a fraction $r/s \in \mathbb{Q}$ *reduced* if $\gcd(r, s) = 1$, with the additional assumption that if $r/s < 0$ then $r < 0$. (That is, we always take $s \in \mathbb{N}$.) With this, we define the following subsets of $\mathbb{Q} \times \mathbb{Q}$ (as usual, $0 < \alpha < \beta$, $4|\beta$, and $\gcd(\alpha, \beta) = 1$):

$$Q_2 := \left\{ \left(\frac{a}{b}, \frac{h}{k}\right) \in \mathbb{Q} \times \mathbb{Q}\colon \ \frac{a}{b} \text{ and } \frac{h}{k} \text{ are reduced, and } b \mid k \right\},$$

$$Q_{\alpha,\beta} := \left\{ \left(\frac{a}{b}, \frac{h}{k}\right) \in \mathbb{Q} \times \mathbb{Q}\colon \begin{array}{l} \frac{a}{b} \text{ and } \frac{h}{k} \text{ are reduced, } k \text{ is even, } \exists\, m \in \mathbb{Z} \text{ s.t.} \\ \frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right) \in \mathbb{Z}, \text{ and} \\ \text{if } k \equiv 0 \pmod{2\beta}, \text{ then } h \not\equiv \pm 1 \pmod{4\beta^3} \end{array} \right\},$$

$$\mathcal{Q}_{s,p} := \left\{ \left(\frac{a}{b}, \frac{h}{k}\right) \in \mathbb{Q} \times \mathbb{Q}\colon \begin{array}{l} \frac{a}{b} \text{ and } \frac{h}{k} \text{ are reduced, } \nexists \text{ odd integer } n \\ \text{s.t. } phn \equiv 0 \pmod{k}, \text{ and } \exists\, m_\mp \in \mathbb{Z} \text{ s.t.} \\ \frac{a\sqrt{2p}}{b} + \frac{ph}{k}\left(\frac{p \mp s}{2p} + \frac{1}{2} + 2m_\mp\right) \in \mathbb{Z}, \text{ and} \\ \text{if } k \equiv 0 \pmod{4p^2}, \text{ then } h \not\equiv \pm 1 \pmod{32p^3} \end{array} \right\}.$$

We refer the reader to Lemma 15 for more on $Q_{\alpha,\beta}$, and note that $\mathcal{Q}_{s,p}$ is defined similarly. See also Example 1 and Example 2 in Sect. 1.1.

Lemma 10 *The set Q_2 is closed under the Jacobi action of $K_t \ltimes (\mathbb{Z} \times \mathbb{Z})$.*

Proof We divide the proof into two parts.
1. Jacobi modular action It suffices to establish closure under the generators of K_t. Let $(\frac{a}{b}; \frac{h}{k}) \in K_t$. We have the Jacobi action

$$\left(\begin{smallmatrix} 1 & 0 \\ 2\beta_t^2 & 1 \end{smallmatrix}\right) \cdot \left(\frac{a}{b}, \frac{h}{k}\right) = \left(\frac{a'}{b'}, \frac{h'}{k'}\right),$$

where $b\widetilde{b} = k$, $k' := 2\beta_2^2 h + k$, $h' := h$, and we have written $a\widetilde{b} = ga'$ and $2\beta_t^2 h + k = gb'$ where $g := \gcd(a\widetilde{b}, 2\beta_t^2 h + k)$, and $a', b' \in \mathbb{Z}$ with $\gcd(a', b') = 1$. Since $\gcd(h, k) = 1$, we have that $\gcd(h', k') = 1$. We also have that $b' \mid k'$ by definition. If $b' > 0$ and $k' > 0$, the proof is complete. Otherwise, we first note that neither can equal zero. (If $b' = 0$ then $k' = 0$, which would imply $k = -2\beta_t^2 h$, e.g., $\gcd(h, k) > 1$, a contradiction.) If either b' or k' is negative, we rewrite $a'/b' = -a'/|b'|$ or $h'/k' = -h'/|k'|$, and by the above discussion, all other conditions required by the definition of $Q_{\alpha,\beta}$ are satisfied.

We also have that

$$\left(\begin{smallmatrix} 1 & 2\beta_t \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\tfrac{a}{b}, \tfrac{h}{k}\right) = \left(\tfrac{a}{b}, \tfrac{h+2\beta_t k}{k}\right),$$

and it is easily verified that this pair is in Q_2.

2. Jacobi elliptic action Let $\left(\frac{a}{b}, \frac{h}{k}\right) \in Q_2$, and let $(\lambda, \mu) \in (\mathbb{Z} \times \mathbb{Z})$. We seek to show that $\frac{a}{b} + \lambda \frac{h}{k} + \mu$ can be expressed as a fraction a'/b' such that $\left(\frac{a'}{b'}, \frac{h}{k}\right) \in Q_2$. To show this, we let $g := \gcd(ak/b + \lambda h + \mu k, k)$, and let a', b' be such that $a'g = ak/b + \lambda h + \mu k$ and $b'g = k$. Then $\frac{a}{b} + \lambda \frac{h}{k} + \mu = a'/b'$, $\gcd(a', b') = 1$ and $b' \mid k$. Finally, since $k > 0$, we have that $b' > 0$. □

Lemma 11 *$Q_{\alpha,\beta}$ is closed under the Jacobi action of $G_\beta \ltimes (4\mathbb{Z} \times 2\mathbb{Z})$.*

Proof We divide the proof into two parts.

1. Jacobi modular action It suffices to establish closure under the generators of G_β. Let $(\frac{a}{b}; \frac{h}{k}) \in Q_{\alpha,\beta}$. We have the Jacobi action

$$\left(\begin{smallmatrix} 1 & 0 \\ 2\beta & 1 \end{smallmatrix}\right) \cdot \left(\tfrac{a}{b}, \tfrac{h}{k}\right) = \left(\tfrac{a'}{b'}, \tfrac{h'}{k'}\right)$$

where $a'g = ak$, $b(2\beta h + k) = b'g$ with $g := \gcd(ak, b(2\beta h + k)$, $h' := h$, and $k' := 2\beta h + k$. Then it follows that $\gcd(a', b') = \gcd(h', k') = 1$, and k' is even. Now if $k' \equiv 0 \pmod{2\beta}$, then we have that $k \equiv 0 \pmod{2\beta}$, and so $h \not\equiv \pm 1 \pmod{4\beta^3}$ Since $h' = h$, we also have that $h' \not\equiv \pm 1 \pmod{4\beta^3}$.

We first assume $b' > 0$ and $k' > 0$, in which case, a'/b' and h'/k' are reduced. Note that $b' \neq 0$ and $k' \neq 0$, for otherwise, either would imply that $h/k = -1/(2\beta)$, a contradiction. Because $(\frac{a}{b}, \frac{h}{k}) \in Q_{\alpha,\beta}$, there exists some integer m such that $\frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right) = x \in \mathbb{Z}$. Let $\ell = m + \beta x$. Then

$$\begin{aligned} \frac{a'}{2b'} + \frac{h'}{k'}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2\ell\right) &= \frac{k}{k'}\left(\frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m + 2\beta x\right)\right) \\ &= \frac{kx}{k'} + \frac{2\beta x h}{k'} \\ &= x, \end{aligned}$$

which is an integer. In this case, the proof is complete.

If either of b' or k' is negative, then both are negative. In this case, we rewrite $a'/b' = -a'/|b'|$ and $h'/k' = -h'/|k'|$; all required hypotheses in the definition of $Q_{\alpha,\beta}$ are met using the above arguments, and the proof is complete in this case as well.

As for the second generator, we have the Jacobi action

$$\left(\begin{smallmatrix} 1 & 2\beta^2 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\tfrac{a}{b}, \tfrac{h}{k}\right) = \left(\tfrac{a'}{b'}, \tfrac{h'}{k'}\right),$$

where $a' := a$, $b' := b$, $h' := 2\beta^2 k + h$ and $k' := k$. Then it is clear that $\gcd(a', b') = \gcd(h', k') = 1$, k' is even, and $b' > 0$ and $k' > 0$. We take m and x as above, and find after a short calculation that

$$\frac{a'}{2b'} + \frac{h'}{k'}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right) = x + 2\beta^2\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right),$$

which is an integer. Finally, if $k' \equiv 0 \pmod{2\beta}$, then $k \equiv 0 \pmod{2\beta}$, which means $h \not\equiv \pm 1 \pmod{4\beta^3}$. Using the definition of h' (and the fact that $k \equiv 0 \pmod{2\beta}$), we have that $h' \equiv h \pmod{4\beta^3}$, hence $h' \not\equiv \pm 1 \pmod{4\beta^3}$. This completes the proof.

2. Jacobi elliptic action Let $\left(\frac{a}{b}, \frac{h}{k}\right) \in Q_{\alpha,\beta}$, and let $(\lambda, \mu) \in (4\mathbb{Z} \times 2\mathbb{Z})$. We seek to show that $\frac{a}{b} + \lambda\frac{h}{k} + \mu$ can be expressed as a fraction a'/b' such that $\left(\frac{a'}{b'}, \frac{h}{k}\right) \in Q_{\alpha,\beta}$. First, we write $a/b + \lambda h/k + \mu = a'/b'$ where $\gcd(a', b') = 1$, and $b' > 0$. (It is not difficult to check that $b' \neq 0$.) By hypothesis, there is some $m \in \mathbb{Z}$ such that $\frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right) \in \mathbb{Z}$. Let $\ell = m - \lambda/4$ (which is an integer, since $4 \mid \lambda$). Then

$$\frac{a'}{2b'} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2\ell\right) = \frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right) + \frac{\mu}{2}$$

is an integer (also using that $2 \mid \mu$). □

Lemma 12 *$\mathcal{Q}_{s,p}$ is closed under the Jacobi action of $M_p \ltimes (\sqrt{2p}\,\mathbb{Z} \times \mathbb{Z})$.*

Proof We divide the proof into two parts.
1. Jacobi modular action It suffices to establish closure under the generators of M_p. Let $(\frac{a}{b}; \frac{h}{k}) \in \mathcal{Q}_{s,p}$. We have the Jacobi action

$$\begin{pmatrix} 1 & 0 \\ 4p^2 & 1 \end{pmatrix} \cdot \left(\frac{a}{b}, \frac{h}{k}\right) = \left(\frac{a'}{b'}, \frac{h'}{k'}\right),$$

where $a'g = ak, b(4p^2h + k) = b'g$ with $g := \gcd(ak, b(4p^2h + k)$, $h' := h$, and $k' := 4p^2h + k$. Then it follows that $\gcd(a', b') = \gcd(h', k') = 1$. Suppose for contradiction's sake that there is some odd number n such that $ph'n \equiv 0 \pmod{k'}$. Then there is some integer c such that $phn = c(4p^2h + k)$, equivalently, $ph(n - 4pc) = ck$. But since n is odd, $n - 4pc$ is odd. Hence, we contradict the original assumption that there is no odd integer n' such that $phn' \equiv 0 \pmod{k}$.

We first assume $b' > 0$ and $k' > 0$, in which case, a'/b' and h'/k' are reduced. Observe that if either of b' or k' is equal to zero, then $-4p^2h = k$, so that $\gcd(h, k) > 1$, a contradiction. Because $(\frac{a}{b}, \frac{h}{k}) \in \mathcal{Q}_{s,p}$, there exist some integers $m_\mp$ such that $\frac{a\sqrt{2p}}{b} + \frac{ph}{k}\left(\frac{p\mp s}{2p} + \frac{1}{2} + 2m_\mp\right) = x_\mp \in \mathbb{Z}$. Let $\ell_\mp = m_\mp + 2px_\mp$. Then

$$\begin{aligned}\frac{a'\sqrt{2p}}{b'} + \frac{ph'}{k'}\left(\frac{p \mp s}{sp} + \frac{1}{2} + 2\ell_\mp\right) &= \frac{k}{k'}\left(\frac{a\sqrt{2p}}{b} + \frac{ph}{k}\left(\frac{p \mp s}{sp} + \frac{1}{2} + 2m_\mp + 4px_\mp\right)\right) \\ &= \frac{k}{k'}x_\mp + \frac{ph}{k'}4px_\mp \\ &= x_\mp,\end{aligned}$$

which are integers.

Now if $k' \equiv 0 \pmod{4p^2}$, then $k \equiv 0 \pmod{4p^2}$, which means h, and hence h', is not equivalent to $\pm 1 \pmod{32p^3}$.

If instead either of b' or k' is negative, then both are, and we rewrite $a'/b' = -a'/|b'|$ and $h'/k' = -h'/|k'|$; all other conditions required by the definition of $\mathcal{Q}_{s,p}$ are satisfied as argued above.

As for the second generator, we have the Jacobi action

$$\begin{pmatrix} 1 & 8p \\ 0 & 1 \end{pmatrix} \cdot \left(\frac{a}{b}, \frac{h}{k}\right) = \left(\frac{a'}{b'}, \frac{h'}{k'}\right),$$

where $a' := a, b' := b, h' := 8pk + h$ and $k' := k$. Then it is clear that $\gcd(a', b') = \gcd(h', k') = 1$ and k' satisfies the required divisibility condition. We also have that $b' > 0$

and $k' > 0$. We now take $m_\mp$ and $x_\mp$ as above. Then after a short calculation we find that

$$\frac{a'\sqrt{2p}}{b'} + \frac{ph'}{k'}\left(\frac{p\mp s}{2p} + \frac{1}{2} + 2m_\mp\right) = x_\mp + 8p^2\left(\frac{p\mp s}{2p} + \frac{1}{2} + 2m_\mp\right),$$

which are integers.

Finally, if $k' \equiv 0 \pmod{4p^2}$, then $k \equiv 0 \pmod{4p^2}$, hence $h \not\equiv \pm 1 \pmod{32p^3}$. Using the definition of h' and the fact that $k \equiv 0 \pmod{4p^2}$, we have that $h' \equiv h \pmod{32p^3}$, and hence $h' \not\equiv \pm 1 \pmod{32p^3}$.

2. Jacobi elliptic action Let $\left(\frac{a}{b}, \frac{h}{k}\right) \in \mathcal{Q}_{s,p}$, and let $(\lambda, \mu) \in (\sqrt{2p}\ \mathbb{Z} \times \mathbb{Z})$. We seek to show that $\frac{a}{b} + \lambda\frac{h}{k} + \mu$ can be expressed as a fraction a'/b' such that $\left(\frac{a'}{b'}, \frac{h}{k}\right) \in \mathcal{Q}_{s,p}$. First, we write $a/b + \lambda h/k + \mu = a'/b'$ where $\gcd(a', b') = 1$, and $b' > 0$. (It is not difficult to check that $b' \neq 0$.) By hypothesis, there are some $m_\mp \in \mathbb{Z}$ such that $\frac{a\sqrt{2p}}{b} + \frac{ph}{k}\left(\frac{p\mp s}{2p} + \frac{1}{2} + 2m_\mp\right) \in \mathbb{Z}$. Let $\ell_\mp = m_\mp - \lambda/\sqrt{2p}$ (which are integers, since $\sqrt{2p} \mid \lambda$). Then

$$\frac{a'\sqrt{2p}}{b'} + \frac{ph}{k}\left(\frac{p\mp s}{2p} + \frac{1}{2} + 2\ell_\mp\right) = \frac{a\sqrt{2p}}{b} + \frac{ph}{k}\left(\frac{p\mp s}{2p} + \frac{1}{2} + 2m_\mp\right) + \mu\sqrt{2p},$$

which are integers (also using that $\sqrt{2p} \in \mathbb{Z}$). □

4 A mock Jacobi form

In this section, building from functions and results from the previous section, we explicitly establish the mock Jacobi properties of a certain function in Sect. 4.1, as well as a related nonholomorphic transformation in Sect. 4.2. We later use the results established in this section toward the proofs of our main theorems from Sects. 1.1–1.3.

4.1 Modularity and an Appell sum

For $0 < \alpha < \beta$ with $\gcd(\alpha, \beta) = 1$ and $4 \mid \beta$, we define

$$B_{\alpha,\beta}(z;\tau) := e\left(\frac{\alpha z}{2\beta}\right) q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}} A_2\left(\frac{-z}{2} + \frac{\alpha}{\beta}\tau - \frac{\tau}{2}, -\tau; 2\tau\right),$$

and

$$\widehat{B}_{\alpha,\beta}(z;\tau) := e\left(\frac{\alpha z}{2\beta}\right) q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}} \widehat{A}_2\left(\frac{-z}{2} + \frac{\alpha}{\beta}\tau - \frac{\tau}{2}, -\tau; 2\tau\right).$$

Proposition 1 *With notation and hypotheses as above, the function $\widehat{B}_{\alpha,\beta}(z;\tau)$ is a nonholomorphic Jacobi form of weight 1, index $-1/8$, group H_β, and character $\zeta_8^{AB}\zeta_{2\beta^2}^{-AB\alpha^2}$, defined for matrices $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in H_\beta$.*

Proof of Proposition 1 We let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in H_\beta$. For ease of notation, let $u = u_{\alpha,\beta,z,\tau} = \frac{-z}{2} + \frac{\alpha}{\beta}\tau - \frac{\tau}{2}$ and $v = v_\tau = -\tau$. Then

$$\begin{aligned}
&\widehat{B}_{\alpha,\beta}\left(\frac{z}{C\tau+D}; \gamma\tau\right) \\
&= e\left(\frac{\alpha z}{2\beta(C\tau+D)}\right) e\left(\frac{(-4\alpha^2+\beta^2)(A\tau+B)}{8\beta^2(C\tau+D)}\right) \\
&\quad \times \widehat{A}_2\left(\frac{\widetilde{u}}{C\tau+D}, \frac{\widetilde{v}}{C\tau+D}; \frac{2A\tau+2B}{C\tau+D}\right),
\end{aligned}$$

where $\widetilde{u} = \widetilde{u}_{\alpha,\beta,z,\tau,\gamma} := -\frac{z}{2} + \left(\frac{\alpha}{\beta} - \frac{1}{2}\right)(A\tau + B)$, and $\widetilde{v} = \widetilde{v}_{\tau,\gamma} := -(A\tau + B)$. Using Lemma 6, this equals

$$e\left(\frac{\alpha z}{2\beta(C\tau + D)}\right) e\left(\frac{(-4\alpha^2 + \beta^2)(A\tau + B)}{8\beta^2(C\tau + D)}\right) \widetilde{f}_\gamma(\widetilde{u}, \widetilde{v}) \widehat{A}_2(\widetilde{u}, \widetilde{v}; 2\tau), \tag{4.1}$$

where $\widetilde{f}_\gamma(\widetilde{u}, \widetilde{v}) := (C\tau + D) e\left(\frac{C(-\widetilde{u}^2 + \widetilde{u}\widetilde{v})}{2(C\tau + D)}\right)$. The prescribed congruence conditions on γ imply that

$$n_1 := \frac{\alpha(A-1)}{2\beta} - \frac{A-1}{4}, \quad m_1 := B\left(\frac{\alpha}{\beta} - \frac{1}{2}\right), \quad n_2 := \frac{1-A}{2}, \quad m_2 := -B$$

are all integers. Thus, we may apply Lemma 6 and rewrite (4.1) as

$$\begin{aligned} &e\left(\frac{\alpha z}{2\beta(C\tau + D)}\right) e\left(\frac{(-4\alpha^2 + \beta^2)(A\tau + B)}{8\beta^2(C\tau + D)}\right) \widetilde{f}_\gamma(\widetilde{u}, \widetilde{v}) \\ &\times e\left(\left(\frac{-z}{2} + \frac{\alpha}{\beta}\tau - \frac{\tau}{2}\right)(2n_1 - n_2)\right) q^{2n_1^2 - 2n_1 n_2 + n_1} \widehat{A}_2(u, v; 2\tau). \end{aligned} \tag{4.2}$$

A rather lengthy, yet straightforward, simplification shows that (4.2) equals

$$\begin{aligned} &(C\tau + D) e\left(\frac{\alpha z}{2\beta}\right) q^{\frac{-4\alpha^2 + \beta^2}{8\beta^2}} \zeta_8^{AB} \zeta_{2\beta^2}^{-AB\alpha^2} e\left(\frac{-Cz^2}{8(C\tau + D)}\right) \widehat{A}_2(u, v; 2\tau) \\ &= (C\tau + D) \zeta_8^{AB} \zeta_{2\beta^2}^{-AB\alpha^2} e\left(\frac{-Cz^2}{8(C\tau + D)}\right) \widehat{B}(z; \tau). \end{aligned}$$

The Jacobi elliptic properties of $\widehat{B}_{\alpha,\beta}$ are similarly deduced from those of $\widehat{A}_2$, which are given in Lemma 6. □

4.2 A nonholomorphic transformation

Let α, β be as above. We first define

$$r_\pm(z; \tau) = r_{\pm,\alpha,\beta}(z; \tau) := R\left(-z + 2\tau\frac{\alpha}{\beta} - (1 \mp 1)\tau - \frac{1}{2}; 4\tau\right).$$

For convenience, we let

$$\begin{aligned} z_1^\pm &= z_1^\pm(\alpha, \beta, z, \tau) := -\frac{z}{2\beta\tau + 1} + \frac{2\tau\alpha}{\beta(2\beta\tau + 1)} - (1 \mp 1)\frac{\tau}{2\beta\tau + 1} - \frac{1}{2}, \\ z_2^\pm &= z_2^\pm(\alpha, \beta, z, \tau) := \frac{1}{2} - \frac{2\alpha\tau}{\beta} + (1 \mp 1)\tau + z, \\ \tau_1 &= \tau_1(\beta, \tau) := \frac{-1}{4\tau} - \frac{\beta}{2}. \end{aligned}$$

Lemma 13 *We have that*

$$r_\pm\left(\frac{z}{2\beta\tau + 1}; \frac{\tau}{2\beta\tau + 1}\right) = a_\pm(z; \tau) h(z_1^\pm \tau_1; \tau_1) - b_\pm(z; \tau) h(z_2^\pm; 4\tau) + b_\pm(z; \tau) r_\pm(z; \tau),$$

where

$$a_\pm(z;\tau) := \sqrt{-i\tau_1} e\left(\frac{-(z_1^\pm)^2\tau_1}{2}\right),$$
$$b_\pm(z;\tau) := a_\pm(z;\tau)(-1)^{\beta/4}\zeta_8^{\beta/2}\sqrt{-4i\tau}\, e\left(\frac{-(z_2^\pm)^2}{8\tau}\right).$$

Proof of Lemma 13 We have by definition of $r_\pm$, after a short simplification and application of Lemma 7, that

$$r_\pm\left(\frac{z}{2\beta\tau+1};\gamma\tau\right) = R\left(z_1^\pm;\frac{-1}{\tau_1}\right) = \sqrt{-i\tau_1}\mathrm{e}^{-\pi i (z_1^\pm)^2\tau_1}\left(-R(z_1^\pm\tau_1;\tau_1)+h(z_1^\pm\tau_1;\tau_1)\right). \tag{4.3}$$

A short calculation reveals that $z_1^\pm\tau_1 = z_2^\pm/4\tau + \beta/4$. Using this, as well as Lemma 7 again, we have that

$$\begin{aligned} R(z_1^\pm\tau_1;\tau_1) &= \zeta_8^{\frac{\beta}{2}} R\left(z_1^\pm\tau_1;\frac{-1}{4\tau}\right) = \zeta_8^{\frac{\beta}{2}} R\left(\frac{z_2^\pm}{4\tau}+\frac{\beta}{4};\frac{-1}{4\tau}\right) \\ &= (-1)^{\frac{\beta}{4}}\zeta_8^{\frac{\beta}{2}}\sqrt{-i4\tau}\mathrm{e}^{-\pi i \frac{(z_2^\pm)^2}{4\tau}}\left(-r_\pm(z;\tau)+h(z_2^\pm;4\tau)\right). \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) proves the lemma. □

5 Proof of Theorem 1

5.1 Proof of Theorem 1 (2)

In this section, we prove part (2) of Theorem 1. We begin by establishing the following proposition.

Proposition 2 *The function*

$$F_{\alpha,\beta}(z;\tau) := \sum_{n=0}^{\infty}\frac{(w^{\frac12}q^{\frac{\alpha}{\beta}+\frac12};q^2)_n}{(w^{\frac12}q^{\frac{\alpha}{\beta}+\frac32};q^2)_n}\left(w^{\frac12}q^{\frac{\alpha}{\beta}+\frac12}\right)^n$$

is defined for $\tau\in\mathbb{H}\cup\mathbb{H}^-$. In particular, for $\tau\in\mathbb{H}$ (and hence for $-\tau\in\mathbb{H}^-$),

$$F_{\alpha,\beta}(z;\tau) = q^{-\frac18}H(w^{-\frac12}q^{-\frac{\alpha}{\beta}+\frac12};q), \tag{5.1}$$
$$F_{\alpha,\beta}(z;-\tau) = -q^{\frac{\alpha^2}{2\beta^2}}w^{-\frac{\alpha}{2\beta}}N(\tau)B_{\alpha,\beta}(z;\tau). \tag{5.2}$$

Moreover, for $(\frac{a}{b},\frac{h}{k})\in Q_{\alpha,\beta}$, we have that

$$F_{\alpha,\beta}(\tfrac{a}{b};\tfrac{h}{k}) = \sum_{n=0}^{M}\frac{(\zeta_{2b}^a\zeta_{\beta k}^{\alpha h}\zeta_{2k}^h;\zeta_k^{2h})_n}{(\zeta_{2b}^a\zeta_{\beta k}^{\alpha h}\zeta_{2k}^{3h};\zeta_k^{2h})_n}\left(\zeta_{2b}^a\zeta_{\beta k}^{\alpha h}\zeta_{2k}^h\right)^n, \tag{5.3}$$

where $M = M_{a,b,h,k}(\alpha,\beta)\in\mathbb{N}_0$.

Remark The number M is explicitly determined in the proof of Proposition 2.

Proof The identity in (5.1) follows from (2.1). Moreover,

$$\begin{aligned} F_{\alpha,\beta}(z;-\tau) &= \sum_{n=0}^{\infty} \frac{\left(w^{\frac{1}{2}}q^{-\frac{\alpha}{\beta}-\frac{1}{2}};q^{-2}\right)_n}{\left(w^{\frac{1}{2}}q^{-\frac{\alpha}{\beta}-\frac{3}{2}};q^{-2}\right)_n} \left(w^{\frac{1}{2}}q^{-\frac{\alpha}{\beta}-\frac{1}{2}}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{\left(w^{-\frac{1}{2}}q^{\frac{\alpha}{\beta}+\frac{1}{2}};q^{2}\right)_n}{\left(w^{-\frac{1}{2}}q^{\frac{\alpha}{\beta}+\frac{3}{2}};q^{2}\right)_n} \left(w^{\frac{1}{2}}q^{-\frac{\alpha}{\beta}+\frac{1}{2}}\right)^n \\ &= F\left(w^{-\frac{1}{2}}q^{\frac{\alpha}{\beta}-\frac{3}{2}}, w^{-\frac{1}{2}}q^{\frac{\alpha}{\beta}-\frac{1}{2}}, w^{\frac{1}{2}}q^{-\frac{\alpha}{\beta}+\frac{1}{2}};q^2\right), \end{aligned}$$

where we have used that $(A;q^{-1})_n = (-A)^n q^{-\frac{n(n-1)}{2}}(A^{-1};q)_n$. By (2.2), this equals $(1-w^{\frac{1}{2}}q^{-\frac{\alpha}{\beta}+\frac{1}{2}})^{-1}K\big(w^{-\frac{1}{2}}q^{\frac{\alpha}{\beta}-\frac{1}{2}};q\big)$. Combining this with (2.3) yields (5.2) after a short calculation.

Finally, we consider $F_{\alpha,\beta}(z;\tau)$ on $Q_{\alpha,\beta}$. Let

$$j(m) = j_{a,b,h,k,\alpha,\beta}(m) := \frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right). \tag{5.4}$$

By hypothesis, there is some $m \in \mathbb{Z}$ such that $j(m) \in \mathbb{Z}$. Thus, for any $t \in \mathbb{N}$, $j(m+tk) \in \mathbb{Z}$, and for $t \geq t_0$, for some $t_0 \in \mathbb{N}$, we have that $m + tk \in \mathbb{N}_0$. Hence, there is some nonnegative integer m' such that $j(m') \in \mathbb{Z}$, and we let M denote the smallest nonnegative integer such that $j(M) \in \mathbb{Z}$. Then we have for any integer $n > M$ that $(w^{\frac{1}{2}}q^{\frac{\alpha}{\beta}+\frac{1}{2}};q^2)_n$ contains $1 - w^{\frac{1}{2}}q^{\frac{\alpha}{\beta}+\frac{1}{2}}q^{2M}$ as a factor. This can be written as $1 - e(j(M))$, which equals 0, since $j(M) \in \mathbb{Z}$. Thus, the numerators defining $F_{\alpha,\beta}(z;\tau)$ are equal to zero for any $n > M$. Further, the denominators can never be zero. Suppose for contradiction's sake that there is some $n \geq 0$ such that

$$1 - w^{\frac{1}{2}}q^{\frac{\alpha}{\beta}+\frac{3}{2}+2n} = 0,$$

equivalently, $j(n+\frac{1}{2}) \in \mathbb{Z}$. Then (with m as above) this implies $\frac{h}{k}(2(n-m)+1) \in \mathbb{Z}$. But k is even, hence h is odd, so this is impossible. □

Resuming the proof of Theorem 1 part (2), we have by definition of $C_{\alpha,\beta}$ and H that

$$q^{-\frac{\alpha^2}{2\beta^2}}w^{-\frac{\alpha}{2\beta}}C_{\alpha,\beta}(z;\tau) = \sum_{n\geq 0} q^{\frac{n^2}{2}}\left(w^{\frac{1}{2}}q^{\frac{\alpha}{\beta}}\right)^n = q^{-\frac{1}{8}}H\left(w^{-\frac{1}{2}}q^{-\frac{\alpha}{\beta}+\frac{1}{2}};q\right), \tag{5.5}$$

and so by Proposition 2, (2.7), and Proposition 1, we have that $C_{\alpha,\beta}(z;-\tau)$ is the holomorphic part of a nonholomorphic Jacobi form of weight $1/2$, index $-1/8$, group H_β, and character $\zeta_8^{AB}\zeta_{2\beta^2}^{-AB\alpha^2}\varepsilon(\gamma)\varepsilon^{-2}(\widetilde{\gamma})$ (defined for matrices $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in H_\beta$). The character simplifies as follows. Since D is odd,

$$\varepsilon\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\varepsilon^{-2}\left(\begin{smallmatrix} A & 2B \\ C/2 & D\end{smallmatrix}\right) = \zeta_8^{1-D}\left(\frac{C}{D}\right)\zeta_8^{-DB} = \zeta_8^{1-D}(-1)^{B/4}\left(\frac{C}{D}\right), \tag{5.6}$$

where we have used that $4|B$. Moreover, we have that $\zeta_8^{AB}\zeta_{2\beta^2}^{-AB\alpha^2} = (-1)^{B/4}\zeta_{2\beta}^{-\alpha^2 B/\beta}$, where we have used that $A \equiv 1 \pmod{2\beta}$ and $\beta \mid B$. Thus, the character in Theorem 1 (2) is

$$\psi_{B,C,D}(\alpha,\beta) := \zeta_8^{1-D}\zeta_{2\beta}^{-\alpha^2 B/\beta}\left(\frac{C}{D}\right). \tag{5.7}$$

This proves Theorem 1 (2).

5.2 Proof of Theorem 1 (1)

In this section, we establish the quantum Jacobi properties in Theorem 1 (1). That $C_{\alpha,\beta}$ is defined on $Q_{\alpha,\beta}$ follows from Theorem 2. (In fact, there we see how to evaluate the function explicitly.) That $Q_{\alpha,\beta}$ is closed under $G_\beta \ltimes (4\mathbb{Z} \times 2\mathbb{Z})$ follows from Lemma 11. The quantum Jacobi modular transformations in Theorem 1 (1) are established in Sect. 5.2.1 below. In particular, the transformation given in Theorem 1 (1) under $\left(\begin{smallmatrix} 1 & 0 \\ -2\beta & 1\end{smallmatrix}\right)$, which may be viewed as a generator of G_β along with $\left(\begin{smallmatrix} 1 & 2\beta^2 \\ 0 & 1\end{smallmatrix}\right)$, is deduced from Proposition 3 below, with $\tau \mapsto -\tau$. The claimed C^∞ properties in Theorem 1 (1) are established in Sect. 5.2.2, and the quantum Jacobi elliptic transformations are treated in Sect. 5.2.3.

5.2.1 Quantum Jacobi modular transformations

First, we observe that the function $C_{\alpha,\beta}(z;\tau)$ is easily seen to be invariant under the action of matrix $\left(\begin{smallmatrix} 1 & 2\beta^2 \\ 0 & 1\end{smallmatrix}\right)$ using its definition. Turning to $\left(\begin{smallmatrix} 1 & 0 \\ 2\beta & 1\end{smallmatrix}\right)$, we define

$$f_\pm(z;\tau) = f_{\pm,\alpha,\beta}(z;\tau) := \frac{i}{2}e\left(\left(\frac{1\mp 1}{2}\right)\left(-\frac{z}{2}+\frac{\alpha\tau}{\beta}-\frac{\tau}{2}\right)\right)$$

and state the following proposition.

Proposition 3 *Assume the notation and hypotheses as above. We have that*

$$\begin{aligned} &C_{\alpha,\beta}(z;-\tau) - (2\beta\tau+1)^{-\frac{1}{2}}\chi_{2\beta,1}^{-1}e\left(\frac{2\beta z^2}{8(2\beta\tau+1)}\right)C_{\alpha,\beta}\left(\frac{z}{2\beta\tau+1};\frac{-\tau}{2\beta\tau+1}\right)\\ &\quad = q^{-\frac{1}{8}-\frac{\beta^2}{8}+\frac{\beta}{4}}(2\beta\tau+1)^{-1/2}\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0 \\ \beta/2 & 1\end{smallmatrix}\right)\right)e\left(\frac{\beta z^2}{4(2\beta\tau+1)}+\frac{\alpha z}{2\beta(2\beta\tau+1)}\right)\\ &\quad\times e\left(\frac{\tau(-4\alpha^2+\beta^2)}{8\beta^2(2\beta\tau+1)}+\frac{\beta(-\tau+\frac{1}{2}(2\beta\tau+1))^2}{4(2\beta\tau+1)}\right)\\ &\quad\times\sum_\pm f_\pm\left(\frac{z}{2\beta\tau+1};\frac{\tau}{2\beta\tau+1}\right)\left(a_\pm(z;\tau)h(z_1^\pm\tau_1;\tau_1)-b_\pm(z;\tau)h(z_2^\pm;4\tau)\right), \end{aligned} \tag{5.8}$$

and for $z \in (-\frac{\alpha}{\beta^2}, -\frac{\alpha}{\beta^2}+\frac{1}{\beta}-\epsilon)$, the right-hand side of Eq. (5.8) equals

$$\frac{-1}{2}\int_0^\infty \frac{\sum_\pm g_{-\frac{\alpha}{2\beta}+\frac{3\mp 1}{4},-z}\left(\frac{2}{\beta}+it\right)}{\sqrt{-i(\frac{2}{\beta}+it+4\tau)}}\mathrm{d}t. \tag{5.9}$$

Proof of Proposition 3 We divide the lengthy proof of Proposition 3 into two parts.

Part 1. We first establish (5.8). From Theorem 1 (2), using Proposition 2 and Proposition 1, for suitable $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, we have that

$$\begin{aligned} &C_{\alpha,\beta}(z;-\tau) - (C\tau+D)^{-\frac{1}{2}}\psi_{B,C,D}^{-1}(\alpha,\beta)e\left(\frac{Cz^2}{8(C\tau+D)}\right)C_{\alpha,\beta}\left(\frac{z}{C\tau+D};-\gamma\tau\right) \\ &\quad = -\widetilde{C}^{-}(z;\tau) + (C\tau+D)^{-\frac{1}{2}}\psi_{B,C,D}^{-1}(\alpha,\beta)e\left(\frac{Cz^2}{8(C\tau+D)}\right)\widetilde{C}^{-}\left(\frac{z}{C\tau+D};\gamma\tau\right), \end{aligned} \tag{5.10}$$

where $\widetilde{C}^{-}(z;\tau) := -N(\tau)B^{-}(z;\tau)$, with

$$\begin{aligned} B^{-}(z;\tau) &= \frac{i}{2}e\left(\frac{\alpha z}{2\beta}\right)q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}}T(\tau) \\ &\quad \times \sum_{k=0}^{1} e\left(k\left(-\frac{z}{2}+\frac{\alpha\tau}{\beta}-\frac{\tau}{2}\right)\right)R\left(-z+2\tau\left(\frac{\alpha}{\beta}-k\right)-\frac{1}{2};4\tau\right). \end{aligned}$$

To simplify B^{-} as above, we have used that $\vartheta(z;\tau)$ is an odd function in z, and that $\vartheta(z+1;\tau) = -\vartheta(z;\tau)$. A short calculation shows that this expression for $B^{-}(z;\tau)$ may be further rewritten as

$$e\left(\frac{\alpha z}{2\beta}\right)q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}}T(\tau)\sigma(z;\tau),$$

where

$$\sigma(z;\tau) := \sum_{\pm} f_{\pm}(z;\tau)r_{\pm}(z;\tau).$$

Combining this with (2.7) we have that (5.10) is equal to

$$N(\tau)\left(B^{-}(z;\tau) - (C\tau+D)^{-1}\zeta_8^{-AB}\zeta_{2\beta^2}^{AB\alpha^2}e\left(\frac{Cz^2}{8(C\tau+D)}\right)B^{-}\left(\frac{z}{C\tau+D};\gamma\tau\right)\right). \tag{5.11}$$

We now apply Lemma 5, and find for $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 2\beta & 1 \end{smallmatrix}\right)$, that

$$\begin{aligned} &B^{-}(z;\tau) - (2\beta\tau+1)^{-1}e\left(\frac{2\beta z^2}{8(2\beta\tau+1)}\right)B^{-}\left(\frac{z}{2\beta\tau+1};\gamma\tau\right) \\ &\quad = T(\tau)\Bigg[e\left(\frac{\alpha z}{2\beta}\right)q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}}\sigma(z;\tau) - (2\beta\tau+1)^{-1}e\left(\frac{\beta z^2}{4(2\beta\tau+1)}\right)e\left(\frac{\alpha z}{2\beta(2\beta\tau+1)}\right) \\ &\qquad \times e\left(\frac{\tau}{2\beta\tau+1}\cdot\frac{-4\alpha^2+\beta^2}{8\beta^2}\right)\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0 \\ \frac{\beta}{2} & 1 \end{smallmatrix}\right)\right)(2\beta\tau+1)^{\frac{1}{2}} \\ &\qquad \times \mathrm{e}^{\frac{\pi i\beta\left(-\tau+\frac{1}{2}(2\beta\tau+1)\right)^2}{2(2\beta\tau+1)}}q^{-\frac{\beta^2}{8}+\frac{\beta}{4}}\sigma\left(\frac{z}{2\beta\tau+1};\gamma\tau\right)\Bigg] \end{aligned} \tag{5.12}$$

The term in brackets [·] in (5.12) is equal to

$$\begin{aligned}
&e\left(\frac{\alpha z}{2\beta}\right) q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}} \sum_{\pm} f_{\pm}(z;\tau) r_{\pm}(z;\tau)\\
&- (2\beta\tau+1)^{-1/2}\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0\\ \frac{\beta}{2} & 1\end{smallmatrix}\right)\right) e\left(\frac{\beta z^2}{4(2\beta\tau+1)}\right)\\
&\times e\left(\frac{\alpha z}{2\beta(2\beta\tau+1)}\right) e\left(\frac{\tau}{2\beta\tau+1}\cdot\frac{-4\alpha^2+\beta^2}{8\beta^2}\right) e\left(\frac{\beta(-\tau+\frac{1}{2}(2\beta\tau+1))^2}{4(2\beta\tau+1)}\right) q^{-\frac{\beta^2}{8}+\frac{\beta}{4}}\\
&\times \sum_{\pm} f_{\pm}\left(\frac{z}{2\beta\tau+1};\frac{\tau}{2\beta\tau+1}\right) r_{\pm}\left(\frac{z}{2\beta\tau+1};\frac{\tau}{2\beta\tau+1}\right),
\end{aligned}$$

which, after applying Lemma 13, we find to be equal to

$$e\left(\frac{\alpha z}{2\beta}\right) q^{\frac{-4\alpha^2+\beta^2}{8\beta^2}} \sum_{\pm} f_{\pm}(z;\tau) r_{\pm}(z;\tau) \tag{5.13}$$

$$\begin{aligned}
&- (2\beta\tau+1)^{-1/2}\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0\\ \frac{\beta}{2} & 1\end{smallmatrix}\right)\right) e\left(\frac{\beta z^2}{4(2\beta\tau+1)}\right)\\
&\times e\left(\frac{\alpha z}{2\beta(2\beta\tau+1)}\right) e\left(\frac{\tau}{2\beta\tau+1}\cdot\frac{-4\alpha^2+\beta^2}{8\beta^2}\right)\\
&\times e\left(\frac{\beta(-\tau+\frac{1}{2}(2\beta\tau+1))^2}{4(2\beta\tau+1)}\right) q^{-\frac{\beta^2}{8}+\frac{\beta}{4}} \sum_{\pm} f_{\pm}\left(\frac{z}{2\beta\tau+1};\frac{\tau}{2\beta\tau+1}\right) b_{\pm}(z;\tau)\, r_{\pm}(z;\tau)
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&- (2\beta\tau+1)^{-1/2}\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0\\ \frac{\beta}{2} & 1\end{smallmatrix}\right)\right) e\left(\frac{\beta z^2}{4(2\beta\tau+1)}\right)\\
&\times e\left(\frac{\alpha z}{2\beta(2\beta\tau+1)}\right) e\left(\frac{\tau}{2\beta\tau+1}\cdot\frac{-4\alpha^2+\beta^2}{8\beta^2}\right)\\
&\times e\left(\frac{\beta(-\tau+\frac{1}{2}(2\beta\tau+1))^2}{4(2\beta\tau+1)}\right) q^{-\frac{\beta^2}{8}+\frac{\beta}{4}} \sum_{\pm} f_{\pm}\left(\frac{z}{2\beta\tau+1};\frac{\tau}{2\beta\tau+1}\right) G_{\pm}(z;\tau),
\end{aligned} \tag{5.15}$$

where

$$G_{\pm}(z;\tau) = G_{\pm,\alpha,\beta}(z;\tau) := a_{\pm}(z;\tau)h(z_1^{\pm}\tau_1;\tau_1) - b_{\pm}(z;\tau)h(z_2^{\pm};4\tau). \tag{5.16}$$

Notice that the functions in lines (5.13) and (5.14) both involve $r_{\pm}(z;\tau)$. A very long, yet explicit, calculation, which also uses that

$$\varepsilon^3\left(\left(\begin{smallmatrix} 1 & 0\\ \frac{\beta}{2} & 1\end{smallmatrix}\right)\right) = \zeta_8^{-\beta/2},$$

shows that these two lines of the (large) expression in (5.13), (5.14), and (5.15) above entirely cancel. Thus, we have that the term in brackets [·] in (5.12) is equal to (5.15). Applying this to (5.11) and using (2.6) yields (5.8) in Proposition 3, together with the fact that $\psi_{B,C,D}(\alpha,\beta) = \chi_{C,D}$ when $B \equiv 0 \pmod{2\beta^2}$, where

$$\chi_{C,D} := \zeta_8^{1-D}\left(\frac{C}{D}\right). \tag{5.17}$$

Part 2. Next we establish (5.9) in Proposition 3. To do so, we study the function $G_{\pm}(z;\tau)$ from (5.16) and begin by rewriting

$$z_1^{\pm}\tau_1 = a_2\tau_1 - b_1^{\pm}, \quad z_2^{\pm} = a_1^{\pm}4\tau - a_2$$

where $a_2 = a_2(z) := -\frac{1}{2} - z$, $b_1^{\pm} = b_1^{\pm}(\alpha,\beta;z) := \frac{\beta z}{2} + \frac{\alpha}{2\beta} - \frac{1}{4}(1\mp 1)$, and $a_1^{\pm} = a_1^{\pm}(\alpha,\beta) := \frac{-\alpha}{2\beta} + \frac{(1\mp 1)}{4}$. □

Lemma 14 *For α, β, and $a_1^{\pm}$ as above, we have that*

(i) $a_1^+ \in (-\frac{1}{2}, 0)$ *and* $a_1^- \in (0, \frac{1}{2})$.

Further, let $\epsilon = \epsilon_{\alpha,\beta} > 0$ *satisfy*

$$\frac{\beta - \alpha}{\beta^2} < \epsilon < \frac{1}{\beta},$$

and suppose

$$z \in \left(-\frac{\alpha}{\beta^2}, \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon\right).$$

Then under these additional hypotheses, we have that

(ii) $b_1^+ \in (0, \frac{1}{2} - \frac{\epsilon\beta}{2}) \subset (0, \frac{1}{2})$, *and* $b_1^- \in (-\frac{1}{2}, -\frac{\epsilon\beta}{2}) \subset (-\frac{1}{2}, 0)$,

(iii) $-a_2 \in \left(\frac{1}{2} - \frac{\alpha}{\beta^2}, \frac{1}{2} + \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon\right) \subset \left(\frac{1}{4}, \frac{1}{2}\right)$.

Proof of Lemma 14 First, we note that (i) follows from the fact that $0 < \alpha < \beta$.

Before proving (ii) and (iii), we first show that the hypotheses given on ϵ and z are well defined. Since $0 < \alpha < \beta$, we have that $0 < \frac{\beta-\alpha}{\beta^2} < \frac{1}{\beta}$, so we may indeed choose some $\epsilon > 0$ satisfying $\frac{\beta-\alpha}{\beta^2} < \epsilon < \frac{1}{\beta}$. Further, we have that $-\frac{\alpha}{\beta^2} < \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon$ since $\epsilon < \frac{1}{\beta}$, so we may indeed let $z \in \left(-\frac{\alpha}{\beta^2}, \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon\right)$.

Using the hypothesis given on z together with the definition of b_1^+, it follows that $b_1^+ \in (0, \frac{1}{2} - \frac{\epsilon\beta}{2})$. Moreover, since $\frac{1}{2} > \frac{\epsilon\beta}{2} > 0$, we have that $(0, \frac{1}{2} - \frac{\epsilon\beta}{2}) \subset (0, \frac{1}{2})$. The assertions pertaining to b_1^- now follow from the fact that $b_1^- = b_1^+ - \frac{1}{2}$. This establishes (ii).

Similarly, using the hypothesis given on z together with the definition of a, it follows that $-a_2 \in \left(\frac{1}{2} - \frac{\alpha}{\beta^2}, \frac{1}{2} + \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon\right)$. Moreover, since $0 < \alpha < \beta$ and $\beta \geq 4$, we have that $\frac{1}{4} < \frac{1}{2} - \frac{\alpha}{\beta^2}$. Additionally, since $\frac{\beta-\alpha}{\beta^2} < \epsilon$, we have that $\frac{1}{2} + \frac{1}{\beta} - \frac{\alpha}{\beta^2} - \epsilon < \frac{1}{2}$. This establishes (iii). □

Resuming the proof of (5.9) from Proposition 3, by Lemma 9, we find for ϵ and z satisfying the hypotheses of Lemma 14 that

$$h(z_1^{\pm}\tau_1; \tau_1) = h(a_2\tau_1 - b_1^{\pm}; \tau_1)$$
$$= -e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^{\pm} + \tfrac{1}{2})\right) \int_0^{i\infty} \frac{g_{a_2+\frac{1}{2}, b_1^{\pm}+\frac{1}{2}}(u)}{\sqrt{-i(u+\tau_1)}} \mathrm{d}u, \tag{5.18}$$

$$h(z_2^{\pm}; 4\tau) = h(a_1^{\pm}4\tau - a_2; 4\tau)$$
$$= -e\left(\frac{(a_1^{\pm})^2 4\tau}{2} - a_1^{\pm}(a_2 + \tfrac{1}{2})\right) \int_0^{i\infty} \frac{g_{a_1^{\pm}+\frac{1}{2}, a_2+\frac{1}{2}}(u)}{\sqrt{-i(u+4\tau)}} \mathrm{d}u. \tag{5.19}$$

In the integral in (5.18), we let $u = \beta/2 - 1/\rho$ so that the right-hand side of (5.18) becomes

$$
\begin{aligned}
&- e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^{\pm}+\tfrac{1}{2})\right)\int_{\frac{2}{\beta}}^{0}\frac{g_{a_2+\frac{1}{2},b_1^{\pm}+\frac{1}{2}}\left(\frac{\beta}{2}-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}}\frac{\sqrt{4\rho\tau}\,\mathrm{d}\rho}{\rho^2}\\
&\quad = -e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^{\pm}+\tfrac{1}{2})\right)e\left(-\frac{\frac{\beta}{2}(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right)\\
&\quad\times\int_{\frac{2}{\beta}}^{0}\frac{g_{a_2+\frac{1}{2},\frac{\beta}{2}(a_2+\frac{1}{2})+b_1^{\pm}+\frac{1}{2}+\frac{\beta}{4}}\left(-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}}\frac{\sqrt{4\rho\tau}\,\mathrm{d}\rho}{\rho^2},
\end{aligned}
\tag{5.20}
$$

where we used that for $n \in \mathbb{N}_0$,

$$
g_{A,B}(\tau+n) = e\left(-\frac{nA(A+1)}{2}\right)g_{A,nA+B+\frac{n}{2}}(\tau),
$$

which is easily deduced from Lemma 8. We rewrite

$$
\frac{\beta}{2}\left(a_2+\frac{1}{2}\right)+b_1^{\pm}+\frac{1}{2}+\frac{\beta}{4} = -a_1^{\pm}+\frac{1}{2}+\frac{\beta}{4}
$$

and obtain, using Lemma 8, that (5.20) is equal to

$$
\begin{aligned}
&- e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^{\pm}+\tfrac{1}{2})\right)e\left(-\frac{\frac{\beta}{2}(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right)\\
&\quad\times\int_{\frac{2}{\beta}}^{0}\frac{g_{a_2+\frac{1}{2},-a_1^{\pm}+\frac{1}{2}+\frac{\beta}{4}}\left(-\frac{1}{\rho}\right)}{\sqrt{(-i)(-1)(4\tau+\rho)}}\frac{\sqrt{4\rho\tau}\,\mathrm{d}\rho}{\rho^2}\\
&= e\left(\frac{a_2^2\tau_1}{2} - a_2(b_1^{\pm}+\tfrac{1}{2})\right)e\left(-\frac{\frac{\beta}{2}(a_2+\frac{1}{2})(a_2+\frac{3}{2})}{2}\right)\\
&\quad\times\sqrt{\frac{4\tau}{-1}}\,i(-i)^{\frac{3}{2}}e\left(\left(a_2+\frac{1}{2}\right)\left(-a_1^{\pm}+\frac{1}{2}+\frac{\beta}{4}\right)\right)\\
&\quad\times\int_{\frac{2}{\beta}}^{0}\frac{g_{a_1^{\pm}+\frac{1}{2},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}\mathrm{d}\rho.
\end{aligned}
$$

After some additional simplifications, we find that

$$
\begin{aligned}
&a_{\pm}(z;\tau)h(z_1^{\pm}\tau_1;\tau_1) = a_{\pm}(z;\tau)\sqrt{\frac{4\tau}{-1}}\,i(-i)^{\frac{3}{2}}\zeta_{4\beta}^{\alpha}\zeta_{16}^{-\beta}\mathrm{e}^{\pi i\frac{(1\pm1)}{4}}\\
&e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\int_{\frac{2}{\beta}}^{0}\frac{g_{a_1^{\pm}+\frac{1}{2},a_2+\frac{1}{2}}(\rho)}{\sqrt{-i(4\tau+\rho)}}\mathrm{d}\rho.
\end{aligned}
\tag{5.21}
$$

Using (5.19) and simplifying, we also have that

$$
\begin{aligned}
-b_{\pm}(z;\tau)h(z_2^{\pm};4\tau) &= a_{\pm}(z;\tau)(-1)^{\beta/4}\zeta_8^{\beta/2}\zeta_{4\beta}^{\alpha}\mathrm{e}^{-\pi i\frac{1\mp1}{4}}e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\sqrt{-4i\tau}\\
&\quad\times\int_0^{i\infty}\frac{g_{a_1^{\pm}+\frac{1}{2},a_2+\frac{1}{2}}(u)}{\sqrt{-i(u+4\tau)}}\mathrm{d}u.
\end{aligned}
\tag{5.22}
$$

After simplifying the constants in (5.21) and (5.22), we have that

$$G_{\pm}(z;\tau) = \zeta_4^{-1-\frac{\beta}{4}}\zeta_8^{\pm1}a_{\pm}(z;\tau)\sqrt{4\tau}\zeta_{4\beta}^{\alpha}e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\int_{\frac{2}{\beta}}^{i\infty}\frac{g_{a_1^{\pm}+\frac{1}{2},a_2+\frac{1}{2}}(u)}{\sqrt{-i(u+4\tau)}}\mathrm{d}u$$
$$= \zeta_4^{-1-\frac{\beta}{4}}\zeta_8^{\pm1}\sqrt{i}\sqrt{1+2\beta\tau}e\left(\frac{-(z_1^{\pm})^2\tau_1}{2}\right)$$
$$\times\zeta_{4\beta}^{\alpha}e\left(-\frac{1}{32\tau}-\frac{z}{8\tau}-\frac{z^2}{8\tau}\right)\int_{\frac{2}{\beta}}^{i\infty}\frac{g_{-\frac{\alpha}{2\beta}+\frac{3\mp1}{4},-z}(u)}{\sqrt{-i(u+4\tau)}}\mathrm{d}u.$$

Thus, we have after some further simplifications that (5.15) equals

$$\frac{-i}{2}q^{\frac{1}{8}}\int_{\frac{2}{\beta}}^{i\infty}\frac{\sum_{\pm}g_{-\frac{\alpha}{2\beta}+\frac{3\mp1}{4},-z}(u)}{\sqrt{-i(u+4\tau)}}\mathrm{d}u = \frac{1}{2}q^{\frac{1}{8}}\int_0^{\infty}\frac{\sum_{\pm}g_{-\frac{\alpha}{2\beta}+\frac{3\mp1}{4},-z}\left(\frac{2}{\beta}+it\right)}{\sqrt{-i(\frac{2}{\beta}+it+4\tau)}}\mathrm{d}t$$

where we integrate from $2/\beta \to 2/\beta + i\infty$ then $2/\beta + i\infty \to i\infty$ (the latter vanishes), and then let $u = 2/\beta + it$ where t runs from $0 \to \infty$. Multiplying by $-q^{-\frac{1}{8}}$ proves (5.9) in Proposition 3 and, hence, concludes the proof of Proposition 3. □

5.2.2 C^{∞} properties

We now establish Theorem 1 (1)'s C^{∞} properties of the error to Jacobi transformation in $Q_{\alpha,\beta}$. The proof follows in a similar manner to the proof of a related result in [3], and we refer the reader there for additional details. We begin with the hypotheses given on z preceding (5.9) in Proposition 3 and will prove that the expression in (5.9) (with $\tau \mapsto -\tau$) is C^{∞}.

Since $0 < \alpha < \beta$ we have that $0 < \alpha/(2\beta) < 1/2$ and $0 < 1/2 - \alpha/(2\beta) < 1/2$. Thus, in studying $g_{-\alpha/(2\beta)+(3\mp1)/4,-z}$, we seek to minimize $(n\mp\rho)^2$ for $0<\rho<\frac{1}{2}$. This is minimized at $n=0$, so as calculated in [3], we obtain

$$\frac{\partial}{\partial z^{\ell}}g_{\rho,-z}(\tau) = (-2\pi i)^{\ell}\sum_n (n+\rho)^{\ell+1}\,\mathrm{e}^{\pi i(n+\rho)^2\tau+2\pi i(n+\rho)(-z)} \ll \mathrm{e}^{-\pi\rho^2\nu}$$

where $\nu = \mathrm{Im}(\tau)$. Thus, the function in (5.9) (with $\tau\mapsto-\tau$) is C^{∞} in $\left(-\frac{\alpha}{\beta^2}, -\frac{\alpha}{\beta^2}+\frac{1}{\beta}-\epsilon\right)\times\left(\mathbb{R}\setminus\{\frac{1}{2\beta}\}\right)$ using the Leibniz rule as in [3].

To finish the proof of Theorem 1 (1), we are left to establish the C^{∞} nature of the error to Jacobi transformation in $\mathbb{Q}\times\mathbb{Q}$ in the larger region in $\mathbb{R}\times\mathbb{R}$ given in Theorem 1 (1). If it is not the case that $z \in \left(-\frac{\alpha}{\beta^2}, -\frac{\alpha}{\beta^2}+\frac{1}{\beta}-\epsilon\right)$, equivalently, $b_1^+ \in \left(0, \frac{1}{2}-\frac{\epsilon\beta}{2}\right)$, then we appeal to the following facts:

$$h(u+1;\tau) = -h(u;\tau) + \frac{2}{\sqrt{-i\tau}}e\left(\frac{(u+\frac{1}{2})^2}{2\tau}\right), \tag{5.23}$$

$$h(u+\tau;\tau) = -e\left(u+\frac{\tau}{2}\right)h(u;\tau) + 2e\left(\frac{u}{2}+\frac{3\tau}{8}\right), \tag{5.24}$$

which are proved in [42]. We divide the proof into cases below and follow an argument similar to one given in [3].

Case 1 Suppose $-\frac{1}{2}+\epsilon\frac{\beta}{2}<b_1^+<0$. Then $-\frac{1}{2}<-a_2<\frac{1}{2}$, and we shift $b_1^-\mapsto b_1^-+1=:\tilde{b}_1^-$, which satisfies $0<\tilde{b}_1^-<\frac{1}{2}$. Then appealing to (5.23), we have that

$$\begin{aligned} h(a_2\tau_1-b_1^-;\tau_1)&=h(a_2\tau_1-\widetilde{b}_1^-+1;\tau_1)\\ &=-h(a_2\tau_1-\widetilde{b}_1^-;\tau_1)+\frac{2}{\sqrt{-i\tau_1}}e\left(\frac{(a_2\tau_1-\widetilde{b}_1^-+\frac{1}{2})^2}{2\tau_1}\right). \end{aligned}$$

The proof now follows as above with b_1^- replaced by $\widetilde{b}_1^-$, together with the observation that the error term $2(-i\tau_1)^{-\frac{1}{2}}e((a_2\tau_1-\widetilde{b}_1^-+\frac{1}{2})^2/(2\tau_1))$ is suitably analytic.
Case 2 If $\frac{1}{2}-\epsilon\frac{\beta}{2}<b_1^+<\frac{1}{2}$ then $-\frac{1}{2}<b_1^-<0$, and $\frac{1}{4}<-a_2<\frac{3}{4}$

- *Case 2a* If $\frac{1}{4}<-a_2<1/2$, we may proceed as above.
- *Case 2b* If instead $\frac{1}{2}<-a_2<\frac{3}{4}$, we shift $-a_2\mapsto -a_2-1$, which satisfies $\frac{-1}{2}<-a_2-1<\frac{-1}{4}$. We argue as in Case 1 using (5.23) and (5.24).
- *Case 2c* The final case $-a_2=1/2$, equivalently $z=0$ and $b_1^+=\frac{\alpha}{2\beta}$, is removed from consideration, as it is not included in the statement of the theorem.

Case 3 If $-\frac{1}{2}<b_1^+<-\frac{1}{2}+\epsilon\frac{\beta}{2}$ then $0<b_1^-+1<\frac{1}{2}$ and $-\frac{1}{2}<-a_2<\frac{1}{2}$, so we work with the shifted $b_1^-\mapsto b_1^-+1$ and argue as in Case 1 and Case 2.

Of course, Cases 1–3 above do not exhaust all possibilities, but these cases suffice: There must be some $n\in\mathbb{Z}$ such that

$$\frac{2n-1}{2}<b_1^+<\frac{2n+1}{2},$$

excluding b_1^+ in

$$\frac{\alpha}{2\beta}+\mathbb{Z},\mathbb{Z},\ \frac{1}{2}+\mathbb{Z},\ \frac{1}{2}\pm\epsilon\frac{\beta}{2}+\mathbb{Z}.$$

This is equivalent to excluding z in

$$\frac{2}{\beta}\mathbb{Z},-\frac{\alpha}{\beta^2}+\frac{2}{\beta}\mathbb{Z},\ \frac{1}{\beta}-\frac{\alpha}{\beta^2}+\frac{2}{\beta}\mathbb{Z},\ \pm\epsilon+\frac{1}{\beta}-\frac{\alpha}{\beta^2}+\frac{2}{\beta}\mathbb{Z}.$$

We shift $b_1^+\mapsto b_1^+-n=:\widetilde{b}_1^+$ so that $\widetilde{b}_1^+\in(-1/2,1/2)$, appealing to (5.23) above. Working with $\widetilde{b}_1^+$, we argue as in Cases 1, 2, or 3 above.

5.2.3 Quantum Jacobi elliptic transformations

We provide a sketch of proof, as similar arguments are given in [1–3,11]. From Proposition 2 and (5.5), we have that $C_{\alpha,\beta}(z;-\tau)=-N(\tau)B_{\alpha,\beta}(z;\tau)$. The elliptic transformation properties of $C_{\alpha,\beta}$ on $Q_{\alpha,\beta}$ (and subsequently the required analytic properties in $\mathbb{R}\times\mathbb{R}$) may therefore be deduced from those of $B_{\alpha,\beta}$. These may be obtained using the definition of $B_{\alpha,\beta}$, from (2.10), and the elliptic transformation properties of μ and ϑ in [42] and Lemma 4, respectively. In particular, the desired elliptic transformation properties of $C_{\alpha,\beta}$ hold with respect to the sublattice $4\mathbb{Z}\times 2\mathbb{Z}$.

6 Proof of Theorem 2

We define

$$c(n)=c_{\alpha,\beta,a,b,h,k}(n):=\begin{cases}\zeta_{2b}^{as}\zeta_{2k\beta^2}^{hn^2}, & n\equiv\alpha+s\beta \pmod{2b\beta},\ \text{some } s \pmod{2b},\\ 0, & \text{else}.\end{cases} \tag{6.1}$$

Then for $t \in \mathbb{R}^+$ we have that

$$\begin{aligned}\sum_{n=0}^{\infty} c(n)\mathrm{e}^{-\frac{n^2t}{\beta^2}} &= \sum_{s \pmod{2b}} \zeta_{2b}^{sa} \sum_{n=0}^{\infty} \mathrm{e}^{-t\frac{(\alpha+\beta s+2b\beta n)^2}{\beta^2}} \zeta_{2k\beta^2}^{h(\alpha+\beta s+2b\beta n)^2} \\ &= \sum_{s \pmod{2b}} \sum_{n=0}^{\infty} \zeta_{2b}^{a(s+2bn)} \mathrm{e}^{-t(\frac{\alpha}{\beta}+s+2bn)^2} \zeta_{2k}^{h(\frac{\alpha}{\beta}+s+2bn)^2} \\ &= \sum_{n=0}^{\infty} \zeta_{2b}^{an} \mathrm{e}^{\pi i\left(\frac{\alpha}{\beta}+n\right)^2\left(\frac{h}{k}+\frac{it}{\pi}\right)} = \zeta_{2\beta b}^{-\alpha a} C_{\alpha,\beta}(\tfrac{a}{b}; \tfrac{h}{k}+\tfrac{it}{\pi}). \end{aligned} \tag{6.2}$$

With this, we will ultimately apply [30, Proposition, p. 98], but first establish a number of technical lemmas to justify its use.

Lemma 15 *For $\left(\frac{a}{b}, \frac{h}{k}\right) \in Q_{\alpha,\beta}$, we have that $b|\beta k$. Moreover, if $bc = \beta k$, then c is even, and $c/2$ is odd. Finally, for any $r \in \mathbb{N}$ such that $2^r \mid \beta$, we have that $2^r \mid b$.*

Proof of Lemma 15 Let $\ell := \mathrm{lcm}(2b, k\beta)$. Then $2bc_1 = k\beta c_2 = \ell$ for some $c_1, c_2 \in \mathbb{Z}$. By hypothesis, there is some $m \in \mathbb{Z}$ such that

$$\frac{a}{2b} + \frac{h}{k}\left(\frac{\alpha}{\beta} + \frac{1}{2} + 2m\right) \in \mathbb{Z},$$

which implies

$$-ac_1 \equiv 0 \pmod{c_2}.$$

But $c_1/c_2 = k\beta/(2b)$ so this means there is some integer x such that $-ak\beta/2 = xb$ (recalling that $4 \mid \beta$). Since $\gcd(a, b) = 1$, we have that $b \mid k\beta/2$ which means there is some d such that $bd = k\beta/2$, or, $bc = k\beta$ where $c = 2d$. We have that

$$\frac{ac}{2} + h\left(\alpha + \frac{\beta}{2} + 2m\beta\right) \in \beta k\mathbb{Z},$$

which implies

$$-\frac{ac}{2} \equiv h\alpha \equiv 1 \pmod{2}$$

(using that h and α are odd). Thus, a and $c/2$ must both be odd.

To prove the divisibility condition on b, suppose $2^r \mid \beta$ for some $r \in \mathbb{N}$ (and note that r is at least 2). Then

$$\frac{bc}{2^{r+1}} = \frac{\beta k}{2^{r+1}} \in \mathbb{Z}$$

since $2^r \mid \beta$ and $2 \mid k$. But $c/2$ is odd so we must have that $2^r \mid b$. □

Lemma 16 *The function $c(n)$ is periodic $\pmod{\beta^2 k}$ with mean value 0.*

Proof of Lemma 16 Using notation as in Lemma 15, by definition, we see that the coefficients $c(n)$ are periodic $\pmod{L}$ where $L := \mathrm{lcm}(k\beta^2, 2b\beta)$. But since $bc = \beta k$, we have that

$$L = \mathrm{lcm}(bc\beta, 2b\beta) = b\beta\mathrm{lcm}(c, 2) = b\beta c = \beta^2 k,$$

since c is even by Lemma 15.

Now for each $s \pmod{2b}$ there are exactly $c/2$ solutions $n \pmod{L = bc\beta}$ to the congruence condition in the definition of $c(n)$: They are $\alpha+s\beta+2b\beta j$ where j runs $\pmod{c/2}$. These are all distinct—suppose for contradiction's sake that for some $s \pmod{2b}$,

$$\alpha + s\beta + 2b\beta j_1 = \alpha + s\beta + 2b\beta j_2 \pmod{bc\beta}.$$

Then $j_1 \equiv j_2 \pmod{c/2}$. Moreover, if

$$\alpha + s_1\beta + 2b\beta j_1 = \alpha + s_2\beta + 2b\beta j_2 \pmod{bc\beta}$$

then $s_1 \equiv s_2 \pmod{2b}$, since c is even.

Thus,

$$\sum_{n \pmod{bc\beta}} c(n) = \sum_{s \pmod{2b}} \zeta_{2b}^{as} \sum_{j \pmod{c/2}} \zeta_{2k\beta^2}^{hY_{s,j}^2}, \tag{6.3}$$

where

$$Y_{s,j} = Y_{s,j}(\alpha, \beta, b) := \alpha + \beta s + 2b\beta j.$$

Suppose $\nu \in \mathbb{N}$ is such that $2^{\nu+1} \mid \beta$, but $2^{\nu+2} \nmid \beta$. (Note that $4 \mid \beta$ so $\nu \geq 1$.) Then the sum in (6.3) equals

$$\sum_{r=1}^{2^{\nu+1}} \sum_{s=(r-1)\frac{b}{2^\nu}}^{r\frac{b}{2^\nu}-1} \sum_{j \pmod{c/2}} \zeta_{2b}^{as}\zeta_{2k\beta^2}^{hY_{s,j}^2} = \sum_{t=1}^{2^\nu} \sum_{r=2t-1}^{2t} \sum_{s=(r-1)\frac{b}{2^\nu}}^{r\frac{b}{2^\nu}-1} \sum_{j \pmod{c/2}} \zeta_{2b}^{as}\zeta_{2k\beta^2}^{hY_{s,j}^2}.$$

For each $1 \leq t \leq 2^\nu$, we claim that

$$\sum_{r=2t-1}^{2t} \sum_{s=(r-1)\frac{b}{2^\nu}}^{r\frac{b}{2^\nu}-1} \sum_{j \pmod{c/2}} \zeta_{2b}^{as}\zeta_{2k\beta^2}^{hY_{s,j}^2} = 0.$$

To show this, let

$$\Sigma_1 := \sum_{s=(2t-2)\frac{b}{2^\nu}}^{(2t-1)\frac{b}{2^\nu}-1} \sum_{j \pmod{c/2}} \zeta_{2b}^{as}\zeta_{2k\beta^2}^{hY_{s,j}^2}, \qquad \Sigma_2 := \sum_{s=(2t-1)\frac{b}{2^\nu}}^{2t\frac{b}{2^\nu}-1} \sum_{j \pmod{c/2}} \zeta_{2b}^{as}\zeta_{2k\beta^2}^{hY_{s,j}^2}.$$

Then

$$\Sigma_2 = \sum_{s=(2t-2)\frac{b}{2^\nu}}^{(2t-1)\frac{b}{2^\nu}-1} \sum_{j \pmod{c/2}} \zeta_{2b}^{a(s+\frac{b}{2^\nu})}\zeta_{2k\beta^2}^{hY_{s+\frac{b}{2^\nu},j}^2}.$$

Because $c/2$ is odd, there are integers c', ρ such that $c'c/2 = 1+2^{\nu+1}\rho$. We shift $j \mapsto j+\rho$. The term

$$\begin{aligned} Y_{s+\frac{b}{2^\nu},j+\rho} &= \alpha + \beta\left(s + \frac{b}{2^\nu}\right) + 2b\beta(j+\rho) \\ &= Y_{s,j} + \beta\frac{b}{2^\nu} + 2b\beta\rho = Y_{s,j} + \frac{c}{2}\cdot c'b\cdot\frac{\beta}{2^\nu}. \end{aligned}$$

Thus,

$$\zeta_{2k\beta^2}^{hY^2_{s+\frac{b}{2^\nu},j+\rho}} = \zeta_{2k\beta^2}^{h(Y^2_{s,j}+Y_{s,j}cc'b\frac{\beta}{2^\nu}+\left(\frac{c}{2}c'b\frac{\beta}{2^\nu}\right)^2)} = \zeta_{2k\beta^2}^{hY^2_{s,j}}\zeta_{2^{\nu+1}}^{hc'\alpha}$$

where we have also used that $bc = \beta k$, $2 \mid k$, and $2^{\nu+1} \mid \beta$.

Thus far, we have shown that

$$\Sigma_2 = \zeta_{2^{\nu+1}}^{a+hc'\alpha}\Sigma_1.$$

We now prove that $\zeta_{2^{\nu+1}}^{a+hc'\alpha} = -1$, which holds if and only if

$$a + hc'\alpha \equiv 2^\nu \pmod{2^{\nu+1}}. \tag{6.4}$$

By hypothesis we have that

$$-\frac{ac}{2} - h\left(\alpha + \frac{\beta}{2} + 2m\beta\right) \in \beta k\mathbb{Z},$$

which implies that

$$-a - h\alpha c' - h\cdot\frac{\beta}{2}\cdot c' \equiv 0 \pmod{2^{\nu+1}}. \tag{6.5}$$

But since each of h, $\beta/2^{\nu+1}$, and c' are odd, (6.4) follows from (6.5). □

Resuming the proof of Theorem 2, we use Lemma 15 and (6.2) and apply [30, Proposition, p. 98]. We find that

$$\begin{aligned}\zeta_{2\beta b}^{-\alpha a}C_{\alpha,\beta}\left(\frac{a}{b},\frac{h}{k}+\frac{it}{\pi}\right) &= \sum_{n=0}^{\infty}\zeta_{2b}^{an}e^{\pi i\left(\frac{\alpha}{\beta}+n\right)^2\left(\frac{h}{k}+\frac{it}{\pi}\right)} = \sum_{n=0}^{\infty}c(n)e^{-n^2t/\beta^2}\\ &\sim \sum_{r=0}^{\infty}\frac{L(-2r,c)}{r!}\left(\frac{-t}{\beta^2}\right)^r\end{aligned}$$

as $t\to 0^+$, where $L(-2r,c)$ is as given in Theorem 2. In particular, this also shows that

$$\zeta_{2\beta b}^{-\alpha a}C_{\alpha,\beta}\left(\frac{a}{b},\frac{h}{k}\right) = L(0,c) = -\sum_{n=1}^{\beta^2 k}c(n)B_1\left(\frac{n}{\beta^2 k}\right).$$

The second (q-hypergeometric) expression for $C_{\alpha,\beta}\left(\frac{a}{b},\frac{h}{k}\right)$ given in Theorem 2 follows from (5.5) and Proposition 2.

7 Proof of Theorem 3

This proof extends an interesting observation given in [26] in the case $q^N = 1$, and we attribute the idea to Hikami and Lovejoy. (The two-variable $F_t(-w;q)$ was not defined in [26].) First we note that by definition, it is not difficult to verify that $F_t(-q^N;q^{-1}) \in \mathbb{Z}[q]$ and $U_t(-q^N;q) \in \mathbb{Z}[q]$, in particular, that they are polynomials when specialized in this way, as opposed to infinite sums as initially defined. The colored Jones polynomial for $T_{(2,2t+1)}$ is given in [26, (3.2)], and using its definition there, combined with the definition of $F_t(w;q)$ in (1.5), we see that for $N \in \mathbb{N}$, $F_t(-q^{-N};q) = J_N(T_{(2,2t+1)};q)$. We also see from [26, (3.22)] and the fact that $U_t(w;q) = U_t(w^{-1};q)$, that $U_t(-q^{-N};q) = J_N(T^*_{(2,2t+1)};q)$, where K^* denotes the mirror of K. It is known that $J_N(K;q) = J_N(K^*;q^{-1})$. Hence, $F_t(-q^{-N};q) = U_t(-q^{-N};q^{-1})$ (and this is a polynomial in $\mathbb{Z}[q^{-1}]$). Letting $q \mapsto q^{-1}$ proves part (1) of Theorem 3.

We now prove part (2). Since $b|k$, there is some $b' \in \mathbb{N}$ such that $bb' = k$. Let $N \in \mathbb{N}$ satisfy $N \equiv ab'\overline{h} \pmod{k}$, where $h\overline{h} \equiv 1 \pmod{k}$. This implies that $Nh \equiv ab' \equiv ak/b \pmod{k}$. Then by applying Theorem 3 (1) to such N, specializing $q = \zeta_k^h$, and using the conditions on N, we have that

$$F_t(-\zeta_b^a;\zeta_k^{-h}) = F_t(-\zeta_k^{hN};\zeta_k^{-h}) = U_t(-\zeta_k^{hN};\zeta_k^{h}) = U_t(-\zeta_b^a;\zeta_k^h),$$

as claimed.

8 Proof of Theorem 4

We first establish the following lemma.

Lemma 17 *Let $\beta_t = 4(2t+1)$, and let $\alpha_t^{(j)}$ $(1 \le j \le 4)$ be as in Sect. 1.2. We have that*

$$\mathcal{F}_t(z;\tau) = \sum_{j=1}^{4} \chi_{8t+4}(\alpha_t^{(j)}) C_{\alpha_t^{(j)},\beta_t}(\beta_t z;\beta_t\tau). \tag{8.1}$$

Proof of Lemma 16 Consider the function

$$T_t(w;q) := \sum_{n=0}^{\infty} \chi_{8t+4}(n) q^{\frac{n^2}{8(2t+1)}} w^{\frac{n}{2}},$$

where χ_{8t+4} is the periodic function (mod $8t+4$) defined by

$$\chi_{8t+4}(n) := \begin{cases} 1, & n \equiv \alpha_t^{(1)}, \alpha_t^{(4)} \pmod{\beta_t}, \\ -1, & n \equiv \alpha_t^{(2)}, \alpha_t^{(3)} \pmod{\beta_t}, \\ 0, & \text{else.} \end{cases}$$

We rewrite

$$\begin{aligned} T_t(w;q) &= \sum_{j=1}^{4} \chi_{8t+4}(\alpha_t^{(j)}) \sum_{\substack{n \equiv \alpha_t^{(j)} \pmod{\beta_t} \\ n \ge 0}} q^{\frac{n^2}{2\beta_t}} w^{\frac{n}{2}} \\ &= \sum_{j=1}^{4} \chi_{8t+4}(\alpha_t^{(j)}) q^{\frac{(\alpha_t^{(j)})^2}{2\beta_t}} w^{\frac{\alpha_t^{(j)}}{2}} \sum_{n \ge 0} q^{\frac{n^2\beta_t}{2} + n\alpha_t^{(j)}} w^{\frac{\beta_t n}{2}} \\ &= \sum_{j=1}^{4} \chi_{8t+4}(\alpha_t^{(j)}) C_{\alpha_t^{(j)},\beta_t}(\beta_t z;\beta_t\tau). \end{aligned}$$

We now apply an earlier result due to Hikami on certain difference equations, namely [22, Theorem 8]; combined with the above, the result follows. □

With Lemma 17, Theorem 4 follows from Theorem 1. Namely, by Theorem 1 (1), each $C_{\alpha_t^{(j)},\beta_t}(z;\tau)$ is a quantum Jacobi form of weight $1/2$ and index $-1/8$, with respect to the group G_{β_t}. Therefore, each $C_{\alpha_t^{(j)},\beta_t}(\beta_t z;\beta_t\tau)$ is a quantum Jacobi form of weight $1/2$ and index $-\beta_t/8 = -t - \frac{1}{2}$, with respect to the group K_t. Note that the group, index, weight, and character are all independent of $\alpha_t^{(j)}$, so that the sum in (8.1) transforms appropriately. In the explicit expression given in (1.6), which is also ultimately deduced from Theorem 1 (1), we have used that $A_t = \max_{1\leq j\leq 4}\{\alpha_t^{(j)}\}$, and $\alpha_t = \min_{1\leq j\leq 4}\{\alpha_t^{(j)}\}$. As argued in the proof of Theorem 3, $\mathcal{F}_t(z;\tau)$ is defined on Q_2, and from Lemma 10, we have that Q_2 is closed under $K_t \ltimes (\mathbb{Z}\times\mathbb{Z})$. This proves (1) of Theorem 4. The proof of Theorem 4 (2) follows similarly using Lemma 17 and Theorem 1 (2).

9 Proof of Theorem 5

Let $a/b \in \mathbb{Q}$ be reduced, and let $\ell_{b,\beta} = \mathrm{lcm}(b,\beta)$. Recall that $\beta_t = 4(2t+1)$. Define the function

$$\Theta_t(\tfrac{a}{b};\tau) = \sum_{n\in\mathbb{Z}} n\chi_{8t+4}(n)\zeta_{2b}^{an} q^{\frac{n^2}{2\beta_t}},$$

where $q = e(\tau)$ and χ_{8t+4} is as defined in Sect. 8. We view Θ_t as a one-variable function of τ with all other parameters fixed. Then it is not difficult to show that $\zeta_b^{-a/2}\Theta_t\left(\frac{a}{b};\frac{\beta_t\tau}{\ell_{b,\beta_t}}\right)$ is a cusp form on $\Gamma(2\ell_{b,\beta_t})$ of weight $3/2$ with character $\left(\frac{\ell_{b,\beta_t}}{D}\right)\left(\frac{2C}{D}\right)\rho_D^{-1}$ (defined for a matrix $\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \in \Gamma(2\ell_{b,\beta_t})$, where ρ_D is either 1 or i depending on whether D is 1 or 3 $\pmod 4$), using results from [40]. (See also [20, Section 3.2] for a similar argument.) In turn, this implies that $\Theta_t\left(\frac{a}{b};\tau\right)$ transforms of weight $3/2$ with character $\left(\frac{\ell_{b,\beta_t}}{D}\right)\left(\frac{2C\beta/\ell}{D}\right)\rho_D^{-1}$ with respect to the group $X_{b,\beta_t} \subseteq \Gamma_0(2\ell_{b,\beta}^2/\beta)$. We now compute the Eichler integral of this function. (See [8], and the narrative in Sect. 1.1.) We have that

$$\Theta_t(\tfrac{a}{b};\tau) = \sum_{n>0} a(n) q^{\frac{n}{2\beta}},$$

where $a(n) := \sqrt{n}\chi_{8t+4}(\sqrt{n})\left(\zeta_{2b}^{a\sqrt{n}} - \zeta_{2b}^{-a\sqrt{n}}\right)$ if n is a square, and otherwise, $a(n) = 0$. Here, we have used that χ_{8t+4} is an even function. Thus, the Eichler integral of $\Theta_t(\frac{a}{b};\tau)$ is

$$\sum_{n>0} a(n) n^{1-\frac{3}{2}} q^{\frac{n}{2\beta}} = \mathcal{F}_t(\tfrac{a}{b};\tau) - \mathcal{F}_t(-\tfrac{a}{b};\tau),$$

where we have also used Lemma 17. The claimed weight $1/2$ quantum modularity of this function now follows by work of Bringmann–Rolen [8].

10 Proof of Theorem 6

Write $p = 2m^2$ for some $m \in \mathbb{N}$, and recall that $1 \leq s \leq p-1$. We may rewrite

$$\eta(\tau)\mathrm{ch}[M_{1,s}^{iz}](\tau) = \sum_{\pm} \pm C_{p\mp s,2p}(2\sqrt{2p}z;p\tau). \tag{10.1}$$

Observe: With $\beta := 2p = 4m^2$, we have that $4 \mid \beta$ and $\beta > 0$. Let $\alpha_1 := p - s$ and $\alpha_2 := p + s$. Then since $1 \leq s \leq p-1$ we have that $0 < \alpha_j < \beta$ for $j \in \{1,2\}$. Thus, the conditions required to apply Theorem 1 hold.

The claimed transformation properties of these functions follow from Theorem 1, noting the stated changes in index, group, and character (as well as the fact that $2\sqrt{2p} = 4m \in \mathbb{N}$). That is, due to the change of variables in the arguments of the functions $C_{\alpha_j,\beta}$, the index changes from $-1/8$ to -1, the groups and characters change accordingly as stated in parts (1) and (2) of the theorem, and the functions exhibit appropriate elliptic properties with respect to the sublattice $\sqrt{2p}\,\mathbb{Z}\times\mathbb{Z}$. Moreover, it is not difficult to verify using the fact that $C_{\alpha,\beta}$ is defined on $Q_{\alpha,\beta}$ that the sum in (10.1) is defined on $\mathcal{Q}_{s,p}$. The set $\mathcal{Q}_{s,p}$ is closed under $M_p \ltimes (\sqrt{2p}\,\mathbb{Z}\times\mathbb{Z})$ by Lemma 12.

Index

Acknowledgements

The author thanks Kazuhiro Hikami and Jeremy Lovejoy for their helpful comments on an earlier version of this manuscript, and she is also thankful for the careful reading and helpful comments made by the referees of this paper. The author is grateful for the partial support of NSF Grant DMS-1901791 and NSF CAREER Grant DMS-1449679, and thanks the Simons Foundation Fellows in Mathematics Program, and the Institute for Advanced Study (Charles Simonyi Endowment), Princeton, for partial support provided during her sabbatical in 2018–2019.

Received: 15 February 2019 Accepted: 4 June 2019
Published online: 25 June 2019

References

1. Barnett, M., Folsom, A., Ukogu, O., Wesley, W.J., Xu, H.: Quantum Jacobi forms and balanced unimodal sequences. J. Number Theory **186**, 16–34 (2018)
2. Barnett, M., Folsom, A., Wesley, W.J.: Rank generating functions for odd-balanced unimodal sequences, quantum Jacobi forms and mock Jacobi forms. Submitted for publication (2018)
3. Bringmann, K., Folsom, A.: Quantum Jacobi forms and finite evaluations of unimodal rank generating functions. Arch. Math. (Basel) **107**(4), 367–378 (2016)
4. Bringmann, K., Folsom, A., Milas, A.: Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters. J. Math. Phys. **58**, 011702 (2017)
5. Bringmann, K., Folsom, A., Ono, K., Rolen, L.: Harmonic Maass Forms and Mock Modular Forms: Theory and Applications. American Mathematical Society. Colloquium Publications. (2017)
6. Bringmann, K., Folsom, A., Rhoades, R.C.: Unimodal sequences and "strange" functions: a family of quantum modular forms. Pac. J. Math. **274**(1), 1–25 (2015)
7. Bringmann, K., Milas, A.: $\mathcal{W}$-algebras, false theta functions and quantum modular forms. I. Int. Math. Res. Not. IMRN **21**, 11351–11387 (2015)
8. Bringmann, K., Rolen, L.: Half-integral weight Eichler integrals and quantum modular forms. J. Number Theory **161**, 240–254 (2016)
9. Bringmann, K., Rolen, L.: Radial limits of mock theta functions. Res. Math. Sci. **2**, 17 (2015)
10. Bryson, J., Ono, K., Pitman, S., Rhoades, R.C.: Unimodal sequences and quantum and mock modular forms. Proc. Natl. Acad. Sci. USA **109**, 16063–16067 (2012)
11. Carroll, G., Corbett, J., Folsom, A., Thieu, E.: Universal mock theta functions as quantum Jacobi forms. Res. Math. Sci. **6**, 6 (2019)
12. Choi, D., Lim, S., Rhoades, R.C.: Mock modular forms and quantum modular forms. Proc. Am. Math. Soc. **144**(6), 2337–2349 (2016)
13. Creutzig, T., Milas, A.: False theta functions and the Verlinde formula. Adv. Math. **262**, 520–554 (2014)
14. Creutzig, T., Milas, A., Wood, S.: On regularised quantum dimensions of the singlet vertex operator algebra and false theta functions. Int. Math. Res. Not. IMRN **5**, 1390–1432 (2017)
15. Eichler, M.: Eine Verallgemeinerung der Abelschen Integrale. Math. Z. **67**, 267–298 (1957)
16. Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Progress in Mathematics, vol. 55. Birkhäuser Boston Inc., Boston (1985)
17. Fine, N.J.: Basic Hypergeometric Series and Applications. Mathematical Surveys and Monographs, vol. 27. American Mathematical Society, Providence (1988)
18. Folsom, A., Garthwaite, S., Kang, S.-Y., Swisher, H., Treneer, S.: Quantum mock modular forms arising from eta-theta functions. Res. Number Theory **2**, 14 (2016)
19. Folsom, A., Jang, M.-J., Kimport, S., Swisher, H.: Quantum modular forms and singular combinatorial series with distinct roots of unity. Springer Research Directions in Number Theory: Women in Numbers IV. Accepted for publication
20. Folsom, A., Ki, C., Truong Vu, Y.N., Yang, B.: "Strange" combinatorial quantum modular forms. J. Number Theory **170**, 315–346 (2017)
21. Folsom, A., Ono, K., Rhoades, R.C.: Mock theta functions and quantum modular forms. Forum Math. **1**, e2 (2013)
22. Hikami, K.: Difference equation of the colored Jones polynomial for torus knot. Int. J. Math. **15**(9), 959–965 (2004)
23. Hikami, K.: Quantum invariant for torus link and modular forms. Commun. Math. Phys. **246**(2), 403–426 (2004)

24. Hikami, K.: q-series and L-functions related to half-derivatives of the Andrews–Gordon identity. Ramanujan J. **11**(2), 175–197 (2006)
25. Hikami, K., Lovejoy, J.: Hecke-type formulas for families of unified Witten–Reshetikhin–Turaev invariants. Commun. Number Theory Phys. **11**(2), 249–272 (2017)
26. Hikami, K., Lovejoy, J.: Torus knots and quantum modular forms. Res. Math. Sci. **2**, 2 (2015)
27. Jang, M.-J., Löbrich, S.: Radial limits of the universal mock theta function g_3. Proc. Am. Math. Soc. **145**(3), 925–935 (2017)
28. Kim, B., Lim, S., Lovejoy, J.: Odd-balanced unimodal sequences and related functions: parity, mock modularity and quantum modularity. Proc. Am. Math. Soc. **144**(9), 3687–3700 (2016)
29. Kimport, S.: Quantum modular forms, mock modular forms, and partial theta functions. Ph.D. Thesis, Yale University (2015)
30. Lawrence, R., Zagier, D.: Modular forms and quantum invariants of 3-manifolds. Asian J. Math. **3**(1), 93–108 (1999)
31. Lee, S.: Maass wave forms, quantum modular forms and Hecke operators. Res. Math. Sci. **6**(1), 7 (2019)
32. Lee, S.: Quantum modular forms and Hecke operators. Res. Number Theory **4**(2), 18 (2018)
33. Lewis, J., Zagier, D.: Cotangent sums, quantum modular forms, and the generalized Riemann hypothesis. Res. Math. Sci. **6**(1), 4 (2019)
34. McIntosh, R.J.: The H and K family of mock theta functions. Can. J. Math. **64**(4), 935–960 (2012)
35. "Modular Forms and Quantum Knot Invariants" workshop. Banff International Research Station (BIRS), Banff, Alberta, Canada, 11–16 March 2018. https://www.birs.ca/events/2018/5-day-workshops/18w5007
36. "Modularity and 3-manifolds" workshop. The Institute for Computational and Experimental Research in Mathematics (ICERM), Providence, RI, USA. 8–10 March 2019. https://icerm.brown.edu/events/ht19-2-m3m
37. Rademacher, H.: Topics in analytic number theory. In: Grosswald, E., Lehner, J., Newman, M. (eds.) Die Grundlehren der Mathematischen Wissenschaften, vol. 169. Springer, New York (1973)
38. Rogers, L.: On two theorems of combinatory analysis and some allied identities. Proc. Lond. Math. Soc. (2) **16**, 315–336 (1917)
39. Shimura, G.: Sur les intégrales attacheées aux formes automorphe. J. Math. Soc. Jpn. **11**, 291–311 (1959)
40. Shimura, G.: On modular forms of half integral weight. Ann. Math. (2) **97**, 440–481 (1973)
41. Zagier, D.: Quantum modular forms, Quanta of maths. In: Clay Mathematics Proceedings, vol. 11, pp. 659–675. American Mathematical Society, Providence, RI (2010)
42. Zwegers, S.P.: Mock Theta Functions, Ph.D. Thesis, Universiteit Utrecht (2002)

Publisher's Note