International Journal of Number Theory
Vol. 5, No. 2 (2009) 303–310

A CHARACTERIZATION OF THE MODULAR UNITS

AMANDA FOLSOM

Max-Planck-Institut für Mathematik
Vivatsgasse 7, 53111 Bonn, Germany
alfolsom@mpim-bonn.mpg.de

Received 20 October 2007
Accepted 1 November 2007

We provide an exact formula for the complex exponents in the modular product expansion of the modular units in terms of the Kubert–Lang structure theory, and deduce a characterization of the modular units in terms of the growth of these exponents, answering a question posed by Kohnen.

Keywords: Modular units; modular functions; modular forms; Borcherds; Borcherds product; Kubert; Lang; weakly holomorphic; Heegner; Heegner divisor.

Mathematics Subject Classification 2000: 11F11

1. Introduction

Let $\Phi(\tau)$ be the modular product defined by

$$\Phi(\tau) = \kappa q^{\beta} \prod_{n=1}^{\infty} (1-q^n)^{c(n)} \tag{1}$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$, $\beta \in \mathbb{Q}$, and $c(n), \kappa \in \mathbb{C}$. In [1], Borcherds shows a duality between the exponents $c(n)$ of the modular products $\Phi(\tau)$ and the coefficients $a(j)$ in the Fourier expansions $f(\tau) = \sum_{k\geq m} a(k)q^k$ of modular forms of prescribed weight and level with divisor supported at Heegner points and cusps. More precisely, this is obtained by lifting a (vector-valued) weakly holomorphic form of weight 1/2, whose constant term dictates the weight of the corresponding Borcherds product. (Thus, in order to obtain a modular unit as a Borcherds product, the constant coefficient of the weight 1/2 form must vanish.) In [2], the authors provide an exact formula for the exponents $c(n)$ when $\Phi(\tau)$ is a weight k meromorphic modular form on $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ whose first Fourier coefficient is 1, in terms of the unique modular functions $j_d(\tau)$, $d \in \mathbb{N}$, holomorphic on the upper half plane $\mathcal{H}$ with Fourier expansion

$$j_d(\tau) = q^{-d} + \sum_{m=1}^{\infty} a_d(m)q^m.$$

The exact expansion in [2] in this setting is given by

$$c(n) = 2k + n^{-1} \sum_{\tau \in \Gamma(1)\backslash\mathcal{H}^*} e_\tau \mathrm{ord}_\tau(\varPhi) \sum_{d|n} \mu(n/d) j_d(\tau), \tag{2}$$

where the numbers e_τ are defined by

$$e_\tau = \begin{cases} \frac{1}{2} & \tau = i \\ \frac{1}{3} & \tau = \frac{1+i\sqrt{3}}{2} \\ 1 & \text{otherwise,} \end{cases}$$

$\mathrm{ord}_\tau(\varPhi)$ refers to the order of $\varPhi$ at τ, $\mu(n)$ is the usual Möbius function, and * denotes the compactification of the quotient space $\Gamma(1)\backslash\mathcal{H}$. The modular units may be characterized as those meromorphic modular functions (meromorphic modular forms of weight 0) with divisors supported in the cusps. We consider the product expansions (1) of the modular units, and provide an exact formula for the exponents $c(n)$, and from this deduce a characterization of the modular units in terms of the growth of the exponents, answering a question of Kohnen's.

In what follows, for a ring R, we let R^* denote the multiplicative group of R, we let $\mathbb{Z}_\ell = \mathbb{Z}/\ell\mathbb{Z}$, $\mathbb{Z}_\ell^C = \mathbb{Z}_\ell\backslash(Z_\ell^* \cup \{0\})$, and for an integer n let $\overline{n}$ denote the equivalence class of n modulo ℓ. We let $q_\ell := q^{1/\ell}$, $\ell \geq 1$, and consider modular units of level ℓ, that is, modular units with respect to the principal congruence subgroups $\Gamma(\ell) := \{\gamma \in \Gamma(1) \mid \gamma \equiv 1 \mod \ell\}$.

Theorem 1. *Let $u(\tau) = \varPhi(\tau/\ell)$ be a modular unit of level $\ell = p^f$, p prime, $p \neq 2, 3$, $f \in \mathbb{N}$. Then*

$$c(n) = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{k|\frac{n}{d}} t_k\left(\frac{n}{dk}\right) \tag{3}$$

where

$$t_m(n) = \begin{cases} n \sum_{s \in \mathbb{Z}_\ell} m(\overline{n}, s) e(\epsilon(n)ms/\ell) & (n,p) = 1 \\ n \sum_{s \in \mathbb{Z}_\ell^*} m(\overline{n}, s) e(\epsilon(n)ms/\ell) & p|n, \ell \nmid n \\ 2n \sum_{s \in \mathbb{Z}_\ell^*/\{\pm 1\}} m(\overline{0}, s) \cos(ms/\ell) & \ell | n, \end{cases}$$

$$e(z) = e^{2\pi i z},$$

$$\epsilon(n) = \begin{cases} 1 & \text{if } n \equiv j \mod \ell \quad \text{for some } j,\ 1 \leq j \leq (\ell-1)/2 \\ -1 & \text{if } n \equiv j \mod \ell \quad \text{for some } j,\ (\ell-1)/2 < j \leq \ell - 1, \end{cases} \tag{4}$$

and $\{m_a = m(r,s)\}_{a\in\overline{T_\ell^*}}$ *is a set of integers indexed by*

$$\overline{T_\ell^*} = \left\{a = (a_1, a_2) = (r/\ell, s/\ell) \in \frac{1}{\ell}\mathbb{Z}^2/\mathbb{Z}^2 \mid \mathrm{ord}(a) = \ell\right\} \tag{5}$$

satisfying

$$\sum_{a\in\overline{T_\ell^*}} m_a r^2 \equiv \sum_{a\in\overline{T_\ell^*}} m_a s^2 \equiv \sum_{a\in\overline{T_\ell^*}} m_a rs \equiv 0 \mod \ell \tag{6}$$

and

$$\sum_{a\in\overline{T_\ell^*}} m_a \equiv 0 \mod 12 \tag{7}$$

where $\mathrm{ord}(a) = \min\{n \in \mathbb{Z}_{\geq 0} \mid n\cdot a \in \mathbb{Z}^2\}$.

Theorem 2. *A meromorphic modular form* $u(\tau) = \Phi(\tau/\ell)$ *of weight zero on* $\Gamma(\ell)$ *is a modular unit if and only if*

$$c(n) \ll_u (\log\log n)^2$$

for all $n \geq 1$, *where the implied constant depends only on* $u(\tau)$.

Theorem 2 is proved more generally by Kohnen in [3] for modular forms of weight k on congruence subgroups $\Gamma \subseteq \Gamma(1)$. Following the statement of the theorem in [3, Theorem 1, p. 66] the author remarks

> "It might be interesting to investigate if [3, Theorem 1, p. 66] could also be proved [in the case weight $k = 0$ using the theory of the modular units]."

Indeed we respond to the above remark of Kohnen and apply the theory of the modular units to give an exact formula for the modular exponents $c(n)$ in Theorem 1 (not given in [3]) which allows us to prove Theorem 2.

2. Modular Units

Much of the theory of the modular units has been developed by Kubert and Lang [4], who provide a description of the modular units in terms of Siegel functions. The Siegel functions are defined using Klein forms $t_a(\tau)$, $a \in \mathbb{R}^2$, $\tau \in \mathcal{H}$ and are given by

$$\mathfrak{t}_a(\tau) = e^{-\eta_a(\tau)a\cdot(\tau,1)/2}\sigma_a(\tau)$$

where σ and η are the usual Weierstrass functions. The Siegel functions $g_a(\tau)$ are defined by

$$g_a(\tau) = \mathfrak{t}_a(\tau)\Delta(\tau)^{1/12}$$

where $\Delta(\tau)$ is the discriminant function. The modular units of a particular level ℓ form a group, and a major result of Kubert and Lang provides a description of the modular unit groups of level $\ell = p^f$, p prime, $p \neq 2, 3$, $f \in \mathbb{N}$, in terms of the Siegel functions.

Theorem 3 ([4]). *For prime power $\ell = p^f$, $p \neq 2, 3$ prime, $f \in \mathbb{N}$, the modular units of level ℓ consist of products*

$$\prod_{a \in \overline{T_\ell^*}} g_a^{m_a} \tag{8}$$

of Siegel functions g_a, where $\{m_a\}_{a \in \overline{T_\ell^*}}$ is a set of integers satisfying the quadratic relations (6) *and* (7).

We remark that choosing different representatives in $\overline{T_\ell^*}$ changes the Siegel function by a root of unity, so it is understood that the theorem of Kubert and Lang is stated modulo constants. From the q-product expansion for the function $\sigma_a(\tau)$, one may obtain the q-product expansion for the Siegel functions

$$g_a(\tau) = -q^{\frac{1}{2}\mathbf{B}_2(a_1)} e(a_2(a_1 - 1)/2) \prod_{n=1}^{\infty} (1 - q^{n-1+a_1} e(a_2))(1 - q^{n-a_1} e(-a_2)) \tag{9}$$

where $\mathbf{B}_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. We will use this theory to prove Theorems 1 and 2.

3. Proofs

We let $\widetilde{\mathbb{Z}_\ell^C}$ be the set of equivalence classes $[z]$ in $\mathbb{Z}_\ell^C$, defined by

$$\widetilde{\mathbb{Z}_\ell^C} = \{[z] \mid z \in \mathbb{Z}_\ell^C, [z] = [w] \Leftrightarrow z + w \equiv 0 \bmod \ell\}$$

and let $\overline{T_\ell^*}$ be represented by

$$\overline{T_\ell^*} \cong \frac{1}{\ell}\left(\mathbb{Z}_\ell^*/\{\pm 1\} \times \mathbb{Z}_\ell\right) \cup \frac{1}{\ell}\left(\widetilde{\mathbb{Z}_\ell^C} \times \mathbb{Z}_\ell^*\right) \cup \frac{1}{\ell}\left(\{0\} \times \mathbb{Z}_\ell^*/\{\pm 1\}\right).$$

Let $u(\tau)$ be a modular unit of level ℓ, $\ell = p^f$, p prime, $p \neq 2, 3$, $f \in \mathbb{N}$. Then there exist $\{m_a\}_{a \in \overline{T_\ell^*}}$ satisfying (6) and (7) such that $u(\tau)$ has an expression as given in (8). By applying the product expansion (9) for the Siegel functions, we see that $u(\tau)$

$$u(\tau) = \xi q^{\alpha} \prod_a \prod_{n=1}^{\infty} [(1 - q^{n-1+a_1} e(a_2))(1 - q^{n-a_1} e(-a_2))]^{m_a}$$

where $\alpha \in \mathbb{Q}$, $\xi \in \mathbb{C}$. We compute

$$\begin{aligned}
&\log(\xi q_\ell^\alpha u(\tau)^{-1}) \\
&\quad = \sum_{a\in \overline{T_\ell^*}} \sum_{n\geq 1} \sum_{m\geq 1} m_a/m(q_\ell^{\ell m(n-1+a_1)} e(ma_2) + q^{\ell m(n-a_1)} e(-ma_2)) \\
&\quad = \sum_{n\geq 1} \sum_{m\geq 1} \sum_{(r,s)\in \mathbb{Z}_\ell^*/\{\pm 1\}\times \mathbb{Z}_\ell} m(r,s)/m(q_\ell^{m(\ell(n-1)+r)} e(sm/\ell) + q_\ell^{m(\ell n-r)} e(-sm/\ell)) \\
&\qquad + \sum_{n\geq 1} \sum_{m\geq 1} \sum_{(r,s)\in \widetilde{\mathbb{Z}_\ell^C}\times \mathbb{Z}_\ell^*} m(r,s)/m(q_\ell^{m(\ell(n-1)+r)} e(sm/\ell) + q_\ell^{m(\ell n-r)} e(-sm/\ell)) \\
&\qquad + \sum_{n\geq 1} \sum_{m\geq 1} \sum_{s\in \mathbb{Z}_\ell^*/\{\pm 1\}} m(0,s)/m(q_\ell^{m(n-1)\ell} e(ms/\ell) + q_\ell^{m\ell n} e(-ms/\ell)) \\
&\quad = \sum_{m\geq 1} \sum_{\substack{n\geq 1\\ (n,p)=1}} \sum_{s\in \mathbb{Z}_\ell} m(\overline{n},s)/m\; e(\epsilon(n)ms/\ell) q_\ell^{mn} \\
&\qquad + \sum_{m\geq 1} \sum_{\substack{n\geq 1\\ p|n,\ \ell\nmid n}} \sum_{s\in \mathbb{Z}_\ell^*} m(\overline{n},s)/m\; e(\epsilon(n)ms/\ell) q_\ell^{mn} \\
&\qquad + \sum_{m\geq 1} \sum_{n\geq 1} \sum_{s\in \mathbb{Z}_\ell^*/\{\pm 1\}} 2m(\overline{0},s)/m \cos(ms/\ell) q_\ell^{\ell mn} \\
&\qquad + \sum_{m\geq 1} \sum_{s\in \mathbb{Z}_\ell^*/\{\pm 1\}} m(\overline{0},s)/m\; e(ms/\ell)
\end{aligned} \tag{10}$$

where $\epsilon(n)$ defined as in (4). We will apply the theta operator Θ_ℓ with respect to the parameter q_ℓ, defined by

$$\Theta_\ell(f) = q_\ell \frac{df}{dq_\ell}.$$

Using (10) we find

$$\begin{aligned}
\frac{\Theta_\ell(u)}{u} &= \alpha\ell - \sum_{m\geq 1} \sum_{\substack{n\geq 1\\ (n,p)=1}} n \sum_{s\in \mathbb{Z}_\ell} m(\overline{n},s) e(\epsilon(n)ms/\ell) q_\ell^{mn} \\
&\quad - \sum_{m\geq 1} \sum_{\substack{n\geq 1\\ p|n,\ \ell\nmid n}} n \sum_{s\in \mathbb{Z}_\ell^*} m(\overline{n},s) e(\epsilon(n)ms/\ell) q_\ell^{mn} \\
&\quad - \sum_{m\geq 1} \sum_{n\geq 1} 2\ell n \sum_{s\in \mathbb{Z}_\ell^*/\{\pm 1\}} m(\overline{0},s) \cos(ms/\ell) q_\ell^{\ell mn} \\
&= \alpha\ell - \sum_{m\geq 1} \sum_{n\geq 1} t_m(n) q^{mn} \\
&= \alpha\ell - \sum_{n\geq 1} \sum_{d|n} t_{n/d}(d) q^n
\end{aligned} \tag{11}$$

where

$$t_m(n) = \begin{cases} n \displaystyle\sum_{s \in \mathbb{Z}_\ell} m(\overline{n}, s) e(\epsilon(n) ms/\ell) & (n,p) = 1 \\ n \displaystyle\sum_{s \in \mathbb{Z}_\ell^*} m(\overline{n}, s) e(\epsilon(n) ms/\ell) & p|n, \ell \nmid n \\ 2n \displaystyle\sum_{s \in \mathbb{Z}_\ell^*/\{\pm 1\}} m(\overline{0}, s) \cos(ms/\ell) & \ell | n. \end{cases}$$

On the other hand, $u(\tau)$ has a modular product expansion of the form given in (1). We compute

$$\log\left(\prod_{n \geq 1} (1 - q_\ell^n)^{-c(n)}\right) = \sum_{n \geq 1} \sum_{m \geq 1} c(n)/m \; q_\ell^{mn}$$

so that

$$\begin{aligned} \frac{\Theta_\ell(u)}{u} &= \beta - \sum_{n \geq 1} \sum_{m \geq 1} nc(n) q_\ell^{mn} \\ &= \beta - \sum_{n \geq 1} \sum_{d|n} dc(d) q_\ell^n. \end{aligned} \tag{12}$$

Comparing (11) and (12), we find $\alpha\ell = \beta$ and

$$\sum_{d|n} t_d(n/d) = \sum_{d|n} dc(d).$$

By Möbius inversion, we find

$$c(n) = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{k|\frac{n}{d}} t_k\left(\frac{n}{dk}\right).$$

This proves Theorem 1.

To prove Theorem 2, suppose first $u(\tau) = \Phi(\tau/\ell)$ is a modular unit of level $\ell = p^f$, p prime, $p \neq 2, 3$, $f \in \mathbb{N}$. Then by Theorem 1, the modular exponents $c(n)$ are of the form given in (3). Thus,

$$\begin{aligned} |c(n)| &\leq \frac{1}{n} \sum_{d|n} \sum_{k|\frac{n}{d}} |t_{n/dk}(k)| \\ &= \frac{1}{n} \sum_{d|n} \sum_{\substack{k|\frac{n}{d} \\ (k,p)=1}} |t_{n/dk}(k)| \\ &\quad + \frac{1}{n} \sum_{d|n} \sum_{\substack{k|\frac{n}{d} \\ p|k, \ell\nmid k}} |t_{n/dk}(k)| \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n}\sum_{d|n}\sum_{\substack{k|\frac{n}{d}\\ \ell|k}}|t_{n/dk}(k)|\\
&\le \sum_{d|n}\sum_{\substack{k|\frac{n}{d}\\ (k,p)=1}} k\sum_{s\in\mathbb{Z}_\ell}|m(\overline{k},s)|\\
&\quad+\sum_{d|n}\sum_{\substack{k|\frac{n}{d}\\ p|k,\,\ell\nmid k}} k\sum_{s\in\mathbb{Z}_\ell^*}|m(\overline{k},s)|\\
&\quad+\sum_{d|n}\sum_{\substack{k|\frac{n}{d}\\ \ell|k}} 2k\sum_{s\in\mathbb{Z}_\ell^*/\{\pm1\}}|m(\overline{0},s)|\\
&\le \ell\mathcal{M}_u\sum_{d|n}\sum_{\substack{k|\frac{n}{d}\\ (k,p)=1}} k+\phi(\ell)\mathcal{M}_u\sum_{d|n}\sum_{\substack{k|\frac{n}{d}\\ p|k,\,\ell\nmid k}} k\\
&\quad+\phi(\ell)\mathcal{M}_u/2\sum_{d|n}\sum_{\substack{k|\frac{n}{d}\\ \ell|k}} 2k
\end{aligned}
$$

where $\mathcal{M}_u=\max_{a\in\overline{T_\ell^*}}\{|m_a|\}$, so that

$$
\begin{aligned}
|c(n)| &\le \ell\mathcal{M}_u\sum_{d|n}\sum_{k|\frac{n}{d}} k\\
&= \ell\mathcal{M}_u\sum_{d|n}\sigma_1(n/d)\\
&\le \ell\mathcal{M}_u\sum_{d|n}2d\log\log d\\
&\le 2\ell\mathcal{M}_u\log\log n\cdot\sigma_1(n)\\
&\le 4\ell\mathcal{M}_u(\log\log n)^2,
\end{aligned}
$$

where we use the fact that $\sigma_1(n)\le 2n\log\log n$ for sufficiently large n. Hence $c(n)\ll_u(\log\log n)^2$. Conversely, suppose $\Phi(\tau/\ell)$ is a weight 0 modular form of level $\ell=p^f$, ℓ prime, $\ell\ne 2,3$, $f\in\mathbb{N}$, with exponents $c(n)$ satisfying $c(n)\ll_u(\log\log n)^2$. Then for all $\tau\in\mathcal{H}$, by (12) we see that $\Theta_\ell(\Phi(\tau/\ell))$ converges, hence has no zeros or poles in $\mathcal{H}$.

References

[1] R. E. Borcherds, Automorphic forms on $O_{s+2,2}(\mathbf{R})$ and infinite products, *Invent. Math.* **120**(1) (1995) 161–213.

[2] J. H. Bruinier, W. Kohnen and K. Ono, The arithmetic of the values of modular functions and the divisors of modular forms, *Compos. Math.* **140** (2004) 552–566.

[3] W. Kohnen, On a certain class of modular functions, *Proc. Amer. Math. Soc.* **133**(1) (2005) 65–70.

[4] D. S. Kubert and S. Lang, *Modular Units*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Science), Vol. 244 (Springer-Verlag, New York, Berlin, 1981).

[5] K. Ono, Hecke Operators and the q-expansion of modular forms, in *Number Theory*, eds. H. Kisilevsky, CRM Proc. Lecture Notes, Vol. 36 (Amer. Math. Soc., Providence, 2004), pp. 229–235.